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A duality between infinitary varieties and algebraic theories

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Abstract. A duality between λ-ary varieties and λ-ary algebraic theories is proved as a direct generalization of the finitary case studied by the first author, F.W. Lawvere and J. Rosický. We also prove that for every uncountable cardinal λ, whenever λ-small products commute with $\mathcal{D}$-colimits in Set, then $\mathcal{D}$ must be a λ-filtered category. We nevertheless introduce the concept of λ-sifted colimits so that morphisms between λ-ary varieties (defined to be λ-ary, regular right adjoints) are precisely the functors preserving limits and λ-sifted colimits.

Keywords: variety, Lawvere theory, sifted colimit, filtered colimit

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1. Introduction

Varieties of finitary algebras can be described via algebraic theories (i.e., small categories with finite products): given an algebraic theory $\mathcal{T}$, then the category $\mathsf{Mod}(\mathcal{T})$ of models, i.e., set-valued functors on $\mathcal{T}$ preserving finite products, is a variety. And every variety has an algebraic theory, i.e., is equivalent to $\mathsf{Mod}(\mathcal{T})$ for some $\mathcal{T}$. This has been shown in the by now classical dissertation of F.W. Lawvere [L]. However, a variety has typically many (non-equivalent) algebraic theories. In [GU] it has been proved that all algebraic theories of a given variety $\mathcal{V}$ have the same Cauchy completion (where a category is Cauchy complete if it has split idempotents, and a Cauchy completion of a category is a reflection in the quasicategory of all Cauchy complete categories). This Cauchy completion is called the canonical algebraic theory, $\mathsf{Th}(\mathcal{V})$, of the variety $\mathcal{V}$. And we obtain a duality between finitary varieties and algebraic theories by $\mathsf{Mod}(\_)$ in one direction and $\mathsf{Th}(\_)$ in the opposite one, see [ALR].

The aim of this note is two-fold. First, we prove that (as expected), the above duality can analogously be formulated between varieties of λ-ary algebras and λ-ary theories, for any infinite cardinal λ. Secondly, we show that, unexpectedly, one important feature of the finitary duality has no infinitary generalization. It concerns the concept of sifted colimit used in [ALR] to characterize morphisms of varieties. These morphisms, called algebraically exact functors, are precisely the functors induced by morphisms of algebraic theories (which, of course, are just

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the functors preserving finite products). Or, equivalently, morphisms of finitary varieties are the regular right adjoints preserving filtered colimits (where *regular functors* are those preserving regular epimorphisms). Now a third way of describing these morphisms between varieties uses the concept of sifted colimits, i.e., colimits of diagrams whose scheme (=domain) is a small *sifted category* $\mathcal{D}$ which means that $\mathcal{D}$-colimits commute in Set with finite products. Examples: filtered colimits are, of course, sifted; and reflexive coequalizers are sifted colimits. In contrast, we prove the following below:

Proposition. *If λ is an uncountable cardinal then there exist no small categories $\mathcal{D}$ such that $\mathcal{D}$ is not λ-filtered but $\mathcal{D}$-colimits commute in* Set *with products of less than λ objects.*

However, we are able to define a λ-*sifted diagram* $D : \mathcal{D} \longrightarrow \mathcal{V}$ in a variety so that morphisms between λ-ary varieties are precisely the functors preserving limits and λ-sifted colimits. The concept of λ-sifted diagram depends on the functor D, not only on the domain $\mathcal{D}$, of course.

During the collaboration of the first author with F. Borceux, S. Lack and J. Rosický in Louvain-la-Neuve in April 1999 the problem of colimits commuting with λ-ary products in Set has been discussed, and the hypothesis that the above proposition holds has been formulated; the first author is grateful for the fruitful atmosphere of that collaboration.

2. The duality between VAR_λ and TH_λ

2.1. We work below with λ-*ary varieties* where λ is an infinite cardinal. This means that a signature Σ of λ-ary operation symbols is given. In the classical one-sorted case, Σ is a set together with an arity function *ar* assigning to every symbol $\sigma \in \Sigma$ a cardinal $ar(\sigma) < \lambda$. In the many-sorted case we consider here, a set S of sorts is given, and *ar* assigns to every symbol $\sigma \in \Sigma$ a pair $ar(\sigma) = (n, s)$, where $n = (n_t)_{t\in S}$ is a collection of cardinals with $\sum\limits_{t\in S} n_t < \lambda$ and $s \in S$ is the "result" sort of σ. A Σ-algebra is an S-indexed collection of (underlying) sets $(A_s)_{s\in S}$ together with a collection of operations $\sigma_A : \prod\limits_{t\in S} A_t^{n_t} \longrightarrow A_s$. We denote by

$$\mathsf{Alg}(\Sigma)$$

the category of S-sorted algebras of signature Σ and homomorphisms. A λ-*ary variety* is a full subcategory of $\mathsf{Alg}(\Sigma)$ closed under regular quotients (or homomorphic images), subobjects and products; or, equivalently, presentable by equations in Σ-terms.

2.2. By a λ-*ary algebraic theory* is meant a small category $\mathcal{T}$ with λ-small products, i.e., products of families indexed by sets of cardinalities smaller than λ. We denote by

$$\mathsf{Mod}(\mathcal{T})$$

the category of all *models* of $\mathcal{T}$, i.e., set valued functors preserving λ-small products; this is a full subcategory of $[\mathcal{T}, \mathsf{Set}]$. Then $\mathsf{Mod}(\mathcal{T})$ is a λ-ary variety, and conversely, every λ-ary variety has a λ-ary theory, i.e., is equivalent to a category $\mathsf{Mod}(\mathcal{T})$. In the finitary, one-sorted case this has been proved by F.W. Lawvere, for the general case the reader can consult e.g. [AR].

However, a variety does not determine a λ-ary theory uniquely. Even in the simplest case of Set (no operations, no equations) there exist non-equivalent finitary theories $\mathcal{T}$, $\mathcal{T}'$ with $\mathsf{Mod}(\mathcal{T}) \simeq \mathsf{Set} \simeq \mathsf{Mod}(\mathcal{T}')$.

Proposition 2.3. *Let $\mathcal{V}$ be a λ-ary variety. Then all λ-ary algebraic theories of $\mathcal{V}$ have the same Cauchy completion, called the* canonical theory *of $\mathcal{V}$. It is equivalent to the full subcategory of $\mathcal{V}^{op}$ formed by all λ-presentable regular projectives in $\mathcal{V}$.*

PROOF: This is analogous to 2.6 in [ALR]. $\square$

Remark 2.4. In every variety $\mathcal{V}$ the λ-presentable regular projectives in $\mathcal{V}$ are precisely the retracts of all $\mathcal{V}$-free algebras on less than λ generators. That is, if $\mathcal{V}$ is S-sorted and

$$U : \mathcal{V} \longrightarrow \mathsf{Set}^S$$

denotes the natural forgetful functor, then U has a left adjoint,

$$F \dashv U.$$

An algebra in $\mathcal{V}$ is regularly projective and λ-presentable iff it is a retract of FI for some I in Set^S, such that $\sum_{s \in S} \operatorname{card}(I_s) < \lambda$. This is easy to prove (compare 2.1 in [ALR]).

2.5. We now want to introduce morphisms between λ-ary varieties. For finitary varieties, they have been chosen to be the regular right adjoint functors preserving filtered colimits, see [ALR], and they have been called algebraically exact. Algebraically exact functors are characterized as precisely all functors preserving limits and sifted colimits. The choice has been motivated by the fact that algebraically exact functors are precisely those induced by theory morphisms of algebraic theories. The situation with λ-ary varieties is completely analogous:

By a morphism of λ-ary algebraic theories there is, as expected, meant a functor preserving products of less than λ objects. For every such morphism

$$H : \mathcal{T} \longrightarrow \mathcal{T}'$$

we obtain a functor

$$(\_) \cdot H : \mathsf{Mod}(\mathcal{T}') \longrightarrow \mathsf{Mod}(\mathcal{T})$$

of composition with H. We say that the last functor (between λ-ary varieties) is *induced* by the theory morphism H. This concept is then extended by equivalence as follows: let $\mathcal{V}_1$, $\mathcal{V}_2$ be λ-ary varieties, then a functor

$$F : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$$

is said to be *induced* by a theory morphism $H : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$ provided that $\mathcal{T}_i$ is the canonical algebraic theory for $\mathcal{V}_i$ (i.e., we have an equivalence $E_i : \mathcal{V}_i \longrightarrow \mathsf{Mod}(\mathcal{T}_i)$) for $i = 1, 2$ and the following square

$$\begin{tikzcd} \mathcal{V}_1 \arrow[rr, "F"] \arrow[d, "E_1"'] \arrow[drr, phantom, "\cong"] & & \mathcal{V}_2 \arrow[d, "E_2"] \\ \mathsf{Mod}(\mathcal{T}_1) \arrow[rr, "(\_)\cdot H"'] & & \mathsf{Mod}(\mathcal{T}_2) \end{tikzcd}$$

commutes up to natural isomorphism.

Proposition 2.6. *A functor between λ-ary varieties is induced by a morphism of the corresponding λ-ary theories iff it is a regular right adjoint preserving λ-filtered colimits.*

Proof: Necessity. It is sufficient to show that for every morphism $H : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$ of λ-ary theories the functor $(\_) \cdot H : \mathsf{Mod}(\mathcal{T}_1) \longrightarrow \mathsf{Mod}(\mathcal{T}_2)$ has the three properties above. Observe that the functor

$$(\_) \cdot H : [\mathcal{T}_1, \mathsf{Set}] \longrightarrow [\mathcal{T}_2, \mathsf{Set}]$$

preserves all limits and colimits, since they are formed object-wise in presheaf categories. Thus, it is sufficient to observe that $\mathsf{Mod}(\mathcal{T}_i)$ is closed under limits, λ-filtered colimits, and regular epimorphisms in $[\mathcal{T}_i, \mathsf{Set}]$. It follows then immediately that the functor $(\_) \cdot H : \mathsf{Mod}(\mathcal{T}_1) \longrightarrow \mathsf{Mod}(\mathcal{T}_2)$ preserves limits, λ-filtered colimits, and regular epimorphisms. Finally, preservation of limits and λ-filtered colimits implies right adjointness (since λ-ary varieties are locally λ-presentable), see Theorem 3.28 in [AR].

Sufficiency. This is analogous to 3.9 in [ALR]: given a right adjoint $F : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ preserving λ-filtered colimits and regular epimorphisms, it follows that a corresponding left adjoint $L : \mathcal{V}_2 \longrightarrow \mathcal{V}_1$ preserves λ-presentability and regular projectives. Thus, L^{op} has a domain-codomain restriction to the canonical λ-ary theories of $\mathcal{V}_2$ and $\mathcal{V}_1$, respectively. If H denotes that restriction (which preserves λ-small products because L^{op} preserves limits, being a right adjoint of F^{op}, and the canonical theories are closed under λ-small products), then one proves that F is induced by H precisely as in the finitary case. □

Definition 2.7. A functor between λ-ary varieties is called λ-*algebraically exact* provided that it is a right adjoint preserving λ-filtered colimits and regular epimorphisms.

Notation 2.8. We denote by

$$\mathsf{VAR}_\lambda$$

the 2-category of all λ-ary varieties (as objects), all λ-algebraically exact functors (as morphisms, i.e., 1-cells) and natural transformations (as 2-cells).

Examples 2.9.

1. Every concrete functor between λ-ary varieties is λ-algebraically exact. That is, given two S-sorted λ-ary varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ with the forgetful functors $U_i : \mathcal{V}_i \longrightarrow \mathsf{Set}^S$, then every functor $F : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ with

$$\begin{tikzcd} \mathcal{V}_1 \arrow[rr, "F"] \arrow[dr, "U_1"'] & \arrow[d, phantom, "\cong"] & \mathcal{V}_2 \arrow[dl, "U_2"] \\ & \mathsf{Set}^S & \end{tikzcd}$$

is λ-algebraically exact. (In case when $\lambda = \aleph_0$, concrete functors between varieties are called algebraic, see e.g. [Bo].) The reason for this fact is that the forgetful functors U_i preserve and reflect (a) limits, (b) λ-filtered colimits and (c) coequalizers of equivalence relations, thus, regular epimorphisms.
2. Representable functors of λ-presentable regularly projective objects are λ-algebraically exact. They are typically not concrete.

Notation 2.10. We denote by

$$\mathsf{TH}_\lambda$$

the 2-category of all Cauchy complete λ-ary algebraic theories, all theory morphisms, and all natural transformations.

Remark 2.11. The objects of TH_λ are small categories with λ-small products and split idempotents. Consequently, TH_λ is indeed a "legitimate", locally small category — a 2-subcategory of CAT (the 2-category of all small categories, functors and natural transformations).

In contrast, VAR_λ should in fact be called a 2-quasicategory rather than 2-category. The reason is that hom's of VAR_λ are indeed very large: consider all functors $\mathsf{Set} \longrightarrow \mathsf{Set}$ naturally isomorphic to the identity functor, then they all are morphisms of VAR_λ, and the collection of these functors is as large as 2^{Ord}, see [AP].

However, there are no more "substantial" difficulties with the size of VAR_λ than of the type indicated above. This follows from the next result comparing VAR_λ with the dual of TH_λ. Our argument above shows that, unfortunately, VAR_λ

is not equivalent (as a category) to $\mathsf{TH}_\lambda{}^{op}$, in fact, it is not equivalent to any locally small category. But we claim that the 2-categories VAR_λ and $\mathsf{TH}_\lambda{}^{op}$ are *biequivalent* (where $\mathsf{TH}_\lambda{}^{op}$ denotes the dual of TH_λ where the 1-cells are reversed but not the 2-cells). Recall from [S] that a biequivalence between 2-categories $\mathcal{K}$ and $\mathcal{L}$ is a pseudofunctor $F : \mathcal{K} \longrightarrow \mathcal{L}$ such that

(a) every object of $\mathcal{L}$ is equivalent to FK for some object K of $\mathcal{K}$

and

(b) the derived functors $\mathcal{K}(K_1, K_2) \longrightarrow \mathcal{L}(FK_1, FK_2)$ are equivalence functors for arbitrary objects K_1 and K_2 of $\mathcal{K}$.

Informally, F is "equivalence up to an equivalence".

We denote by

$$\mathsf{Mod}(\,\_\,) : \mathsf{TH}_\lambda{}^{op} \longrightarrow \mathsf{VAR}_\lambda$$

the 2-functor of forming model-categories of λ-algebraic theories: to a theory $\mathcal{T}$ it assigns the variety $\mathsf{Mod}(\mathcal{T})$; to a theory morphism $H : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ it assigns the induced λ-algebraically exact functor

$$(\,\_\,) \cdot H : \mathsf{Mod}(\mathcal{T}_2) \longrightarrow \mathsf{Mod}(\mathcal{T}_1);$$

and to every natural transformation $h : H \longrightarrow H'$ $(: \mathcal{T}_1 \longrightarrow \mathcal{T}_2)$ it assigns the natural transformation with components $Fh : FH \longrightarrow FH'$ (F in $\mathsf{Mod}(\mathcal{T}_2)$).

Duality Theorem 2.12. *For every infinite cardinal λ the 2-functor*

$$\mathsf{Mod}(\,\_\,) : \mathsf{TH}_\lambda{}^{op} \longrightarrow \mathsf{VAR}_\lambda$$

is a biequivalence.

Proof: We know that every λ-ary variety is equivalent to $\mathsf{Mod}(\mathcal{T})$, where $\mathcal{T}$ in TH_λ is its canonical λ-ary theory. Thus, to prove that $\mathsf{Mod}(\,\_\,)$ is a biequivalence, we only need to show that for every pair $\mathcal{V}_1$, $\mathcal{V}_2$ of λ-ary varieties the formation of induced functor defines an equivalence functor

$$\mathsf{TH}_\lambda(\mathcal{T}_1, \mathcal{T}_2) \longrightarrow \mathsf{VAR}_\lambda(\mathcal{V}_1, \mathcal{V}_2)$$

where $\mathcal{T}_i$ is the canonical theory of $\mathcal{V}_i$, $(i = 1, 2)$. By Proposition 2.6, this functor is isomorphism-dense. The argument why this functor is full and faithful is standard and completely analogous to the proof of Gabriel-Ulmer duality. The latter has been carefully discussed e.g. in [AP]. $\square$

3. Non-existence of λ-sifted categories

As mentioned in the introduction, a small category $\mathcal{D}$ is called sifted if $\mathcal{D}$-colimits commute with finite products in Set. Here we discuss the generalization to λ-*small* products (i.e., products indexed by sets of less than λ elements). Every λ-filtered category $\mathcal{D}$ has, of course, the property that $\mathcal{D}$-colimits commute with λ-small products (indeed, with λ-small limits) in Set. And unless $\lambda = \aleph_0$, there are no others:

Theorem 3.1. *Let λ be an uncountable cardinal. For every small category $\mathcal{D}$ the following conditions are equivalent:*

1. *$\mathcal{D}$-colimits commute with λ-small products in* Set*;*
2. *$\mathcal{D}$-colimits commute with λ-small limits in* Set*;*
3. *$\mathcal{D}$ is λ-filtered.*

Proof: We only have to prove 1. $\Rightarrow$ 3., since 3. $\Rightarrow$ 2. $\Rightarrow$ 1. are trivial.

(I) We first prove that for every span

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ B & & B \end{tikzcd}$$

in $\mathcal{D}$ there exists a commutative square

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ B \arrow[dr, "g_1"'] & & B \arrow[dl, "g_2"] \\ & C & \end{tikzcd}$$

Assuming the contrary, we form a functor

$$D = \mathcal{D}(A, \_) \times \omega / \sim : \mathcal{D} \longrightarrow \mathsf{Set}$$

as the following quotient of a coproduct of ω copies of the hom-functor of A: in D we merge

(a) the n-th copy of f_1 with the $(n+1)$-th copy of f_2 $(n \in \omega)$

and

(b) the morphisms $h \cdot f_i$ and $k \cdot f_i$ for any $i = 1, 2$ and any parallel pair $h, k : B \longrightarrow B'$.

That is,

$$DX = \mathcal{D}(A, X) \times \omega / \sim$$

where for $h, k : A \longrightarrow X$ and $n, m \in \omega$ we define

$$(h, n) \sim (k, m)$$

iff

(a) $n = m$ and $h = k$, or
(b) $n = m$ and $h = h' \cdot f_i$, $k = k' \cdot f_i$ for some $h', k' : B \longrightarrow X$, and some $i = 1, 2$, or
(c) $n = m + 1$ and $h = h' \cdot f_2$, $k = k' \cdot f_1$ for some $h', k' : B \longrightarrow X$, or
(d) $m = n + 1$ and $h = h' \cdot f_1$, $k = k' \cdot f_2$ for some $h', k' : B \longrightarrow X$.

It follows from our choice of f_1, f_2 that this is indeed an equivalence relation — and it is clearly a congruence, i.e., we can define Du for morphisms $u : X \longrightarrow Y$ by the following rule:

$$Du : [h, n] \mapsto [u \cdot h, n]$$

where $[h, n]$ denotes the equivalence class of $(h, n) \in DX$.

The category $\mathsf{Elts}(D)$ of D (whose objects are pairs (d, x) where d is an object of $\mathcal{D}$ and $x \in Dd$, and morphisms from (d, x) to (d', x') are morphisms $f : d \longrightarrow d'$ in $\mathcal{D}$ with $Df(x) = x'$), is indecomposable: given elements

$$[h, n] \in DX \quad \text{and} \quad [k, m] \in DX \quad (m \geq n)$$

we have a zig-zag in $\mathsf{Elts}(D)$ as follows:

$$\begin{tikzcd}
& {[id_A, n]} \arrow[dl, "h"'] \arrow[dr, "f_1"] & & {[id_A, n+1]} \arrow[dl, "f_2"'] \arrow[dr, "f_1"] & & & & {[id_A, m]} \arrow[dl, "f_2"'] \arrow[dr, "k"] & \\
{[h, n]} & & {[f_1, n]} & & {[f_1, n+1]} & \cdots & {[f_1, m]} & & {[k, m]}
\end{tikzcd}$$

In other words, a colimit of the diagram $D : \mathcal{D} \longrightarrow \mathsf{Set}$ is a singleton set. Since $\mathcal{D}$-colimits commute with countable products, the diagram D^ω has also a singleton set as a colimit — in other words, the category $\mathsf{Elts}(D^\omega)$ is also indecomposable. This, however, is the desired contradiction: consider the following elements of $D^\omega A$:

$$r = ([id_A, 1], [id_A, 2], [id_A, 3], \dots)$$

and

$$s = ([id_A, 1], [id_A, 1], [id_A, 1], \dots).$$

Suppose that a zig-zag (of length, say, n) exists between these objects in $\mathsf{Elts}(D^\omega)$. The i-th projection of that zig-zag is, then, a zig-zag of length n from $[id_A, i]$ to $[id_A, 1]$ in $\mathsf{Elts}(D)$. However, for $i = n + 1$, every zig-zag in $\mathsf{Elts}(D)$ has length $> n$ because the above congruence is such that

$$(h, n) \sim (k, m) \quad \text{implies} \quad |n - m| \leq 1.$$

(II) For every collection X_i $(i \in I)$ of less than λ objects of $\mathcal{D}$ there exists a (discrete) cocone $f_i : X_i \longrightarrow Y$ in $\mathcal{D}$, and the category of all these cocones (where a morphism from $(f_i : X_i \longrightarrow Y)$ to $(g_i : X_i \longrightarrow Z)$ is a morphism $h : Y \longrightarrow Z$ in $\mathcal{D}$ with $h \cdot f_i = g_i$ for all $i \in I$) is indecomposable.

In fact, consider a product

$$D = \prod_{i \in I} \mathcal{D}(X_i, \_) : \mathcal{D} \longrightarrow \mathsf{Set}.$$

Since each hom-functor has a colimit given by a singleton set and since λ-small products commute with $\mathcal{D}$-colimits, the functor D also has a colimit given by a singleton set — thus, $\mathsf{Elts}(D)$ is an indecomposable category. It is evident that $\mathsf{Elts}(D)$ is equivalent to the category of discrete cocones over X_i $(i \in I)$.

(III) $\mathcal{D}$ is λ-filtered. Due to (II) it is certainly sufficient to prove that given a collection of less than λ parallel morphisms $f_i : X \longrightarrow Y$ $(i \in I)$ in $\mathcal{D}$, there exists a morphism with domain Y which coequalizes the collection. Consider the shortest zig-zag from the cone $f_i : X \longrightarrow Y$ $(i \in I)$ to the cone obtained by I copies of the identity morphism of X. It follows from (I) above that such a zig-zag has length at most 2 because whenever two neighbor arrows of that zig-zag have a common domain, we can use (I) to modify the zig-zag so that after the modification only common codomains are possible. In other words, the zig-zag has the following form

$$\begin{tikzcd}
 & & Z & & & \\
Y \arrow[urr, "p"] & & & & X \arrow[ull, "q"'] & \\
 & X \arrow[ul, "f_i"] \arrow[urrr] & X \arrow[ull] \arrow[urr] & X \arrow[ulll] \arrow[ur, "id"'] & & \cdots
\end{tikzcd}$$

which means, of course, that p is a morphism we have been looking for. $\square$

4. λ-sifted colimits

4.1. From the negative result of the previous section we know that we cannot generalize sifted colimits by introducing a class of small categories called λ-sifted and then defining a λ-sifted diagram as a diagram whose domain is a λ-sifted category. In fact, if we would provide any such notion of a λ-sifted category $\mathcal{D}$, requesting (as we certainly would) that $\mathcal{D}$-colimits commute with λ-small products in Set, then $\mathcal{D}$ would be λ-filtered. And we would lose the basic reason why sifted colimits were introduced for VAR in [ALR], viz, that morphisms of VAR_λ are precisely the functors preserving limits and λ-sifted colimits. In fact, there are many functors preserving limits and λ-filtered colimits between varieties that are not morphisms of VAR_λ (because they do not preserve regular epimorphisms) such as hom-functors

$$\mathcal{V}(A, \_) : \mathcal{V} \longrightarrow \mathsf{Set}$$

where $\mathcal{V}$ is a variety and A is a λ-presentable algebra of $\mathcal{V}$ which is not regularly projective.

Thus, our strategy is different: our concept of a λ-sifted colimit will depend not only on the domain category but also on the functor forming the diagram. This is unfortunate, but has the fortunate consequence that morphisms of VAR_λ will, like in VAR, be precisely the functors preserving limits and λ-sifted colimits.

Definition 4.2. A diagram $D : \mathcal{D} \longrightarrow \mathcal{K}$ ($\mathcal{D}$ small) in a category $\mathcal{K}$ is called λ-*sifted*, where λ is an infinite cardinal, if

(a) D has a colimit in $\mathcal{K}$

and

(b) that colimit is preserved by all hom-functors of λ-presentable regular projectives of $\mathcal{K}$.

Example 4.3. λ-sifted colimits in Set. Observe that hom-functors of λ-presentable (and regularly projective) sets are precisely the functors equivalent to the functors $(\_)^I$ of I-th power, where I is a set of less than λ elements. Therefore, λ-sifted colimits in Set are precisely those which commute with λ-ary powers.

1. Every sifted colimit in Set is $\aleph_0$-sifted because sifted colimits commute (by definition) with finite products in Set.
2. Coequalizers of equivalence relations are λ-sifted for every λ. In fact, it is easy to see that coequalizers of equivalence relations commute with arbitrary products in Set.

An important example of sifted colimits are reflexive coequalizers; they are, however, not λ-sifted for any uncountable cardinal. For example, consider the reflexive coequalizer

$$\mathbb{N} + \mathbb{N} \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{N} \longrightarrow 1$$

where both components of f are $id_{\mathbb{N}}$, whereas g has as components $id_{\mathbb{N}}$ and the successor function. The I-th powers, f^I and g^I, have a non-singleton coequalizer for any infinite set I.

Remark 4.4. λ-sifted colimits in Set can be fully described using the category of elements of the diagram (see 3.1). Recall that two elements (d, x) and (d', x') of a diagram $D : \mathcal{D} \longrightarrow \mathsf{Set}$ lie in the same component of $\mathsf{Elts}(D)$ iff they can be connected by a zig-zag of morphisms

$$\begin{tikzcd}[column sep=small]
 & (d_1, x_1) & & & & (d_{2n-1}, x_{2n-1}) & \\
(d, x) = (d_0, x_0) \arrow[ur] & & (d_2, x_2) \arrow[ul] & \cdots & (d_{2n-2}, x_{2n-2}) \arrow[ur] & & (d_{2n}, x_{2n}) = (d', x') \arrow[ul]
\end{tikzcd}$$

of $\mathsf{Elts}(D)$.

Proposition 4.5. *A small diagram $D : \mathcal{D} \longrightarrow \mathsf{Set}$ is λ-sifted iff*

1. *given less than λ elements $x_i \in Dd_i$ $(i \in I)$ of D there exists an object d in $\mathcal{D}$ such that each (d_i, x_i) lies in the same component of $\mathsf{Elts}(D)$ as some element of Dd;*
2. *given less than λ pairs (d, x_i) and (d', x'_i) $(i \in I)$ of elements of D such that for each i the pair lies in one component of $\mathsf{Elts}(D)$, there exists a*

zig-zag Z in $\mathcal{D}$ connecting d and d' such that each of the pairs above can be connected by a zig-zag in $\mathsf{Elts}(D)$ whose underlying zig-zag is Z.

Proof: Sufficiency. Suppose that 1. and 2. hold, and denote by $(c_d : Dd \longrightarrow C)$ a colimit of D in Set. We are to prove that for every set I of cardinality smaller than λ the diagram $D^I : \mathcal{D} \longrightarrow \mathsf{Set}$ has the colimit $(c_d^I : (Dd)^I \longrightarrow C^I)$. For this, it is necessary and sufficient to prove that

(i) the cocone c_d^I is collectively surjective

and

(ii) given elements $x \in (Dd)^I$ and $x' \in (Dd')^I$ such that $c_d^I(x) = c_{d'}^I(x')$ there exists a zig-zag connecting the two elements in $\mathsf{Elts}(D^I)$.

Our condition (i) follows from (in fact, is equivalent to) the condition 1. of the proposition: an element of C^I has the form $(c_{d_i}(x_i))_{i\in I}$ where (d_i, x_i) are elements of D. Since $\operatorname{card}(I) < \lambda$, there exists d in $\mathcal{D}$ such that (d_i, x_i) lies in the same component of $\mathsf{Elts}(D)$ as (d, y_i) for some $y_i \in Dd$. All these y_i's form an element $(y_i)_{i\in I} \in D^I d$ such that $c_{d_i}(x_i) = c_d(y_i)$ for each i, thus, such that c_d^I maps it onto the given element $(c_{d_i}(x_i))_{i\in I}$ of C^I.

Likewise, our condition (ii) follows from (in fact, is equivalent to) the condition 2. in the statement of the proposition: we are given elements $x = (x_i)_{i\in I}$ and $x' = (x'_i)_{i\in I}$ such that $c_d^I(x) = c_{d'}^I(x')$, or, equivalently, (d, x_i) lies in the same component of $\mathsf{Elts}(D)$ as (d', x'_i), for each $i \in I$. Consequently, there exist zig-zags connecting (d, x_i) with (d', x'_i) in $\mathsf{Elts}(D)$ for all $i \in I$ such that these zig-zags have the same underlying zig-zag in $\mathcal{D}$. In other words, all these zig-zags yield one zig-zag connecting $(d, (x_i)_{i\in I})$ with $(d', (x'_i)_{i\in I})$ in $\mathsf{Elts}(D^I)$.

Necessity. If $D : \mathcal{D} \longrightarrow \mathsf{Set}$ is a λ-sifted diagram with a colimit $(c_d : Dd \longrightarrow C)$, then for every set I of cardinality less than I the diagram $D^I : \mathcal{D} \longrightarrow \mathsf{Set}$ has a colimit $(c_d^I : Dd^I \longrightarrow C^I)$. Thus, the cocone c_d^I is collectively surjective, which proves 1. in the statement of the proposition, and whenever $c_d^I(x) = c_{d'}^I(x')$, then (d, x) is connected by a zig-zag of $\mathsf{Elts}(D^I)$ with (d', x'), and that proves 2. in that statement. $\square$

Remark 4.6. The description of λ-sifted colimits in Set extends immediately to one-sorted λ-ary varieties $\mathcal{V}$. Let

$$U : \mathcal{V} \longrightarrow \mathsf{Set}$$

be the usual forgetful functor. Then a diagram D in $\mathcal{V}$ is λ-sifted iff UD is λ-sifted in Set, and U preserves $\operatorname{colim} D$.

In fact, let $F : \mathsf{Set} \longrightarrow \mathcal{V}$ be a left adjoint of U, then λ-presentable regular projectives of $\mathcal{V}$ are, by 2.4, precisely retracts V of free algebras FI where $\operatorname{card}(I) < \lambda$. Now any colimit preserved by $\mathcal{V}(FI, \_)$ is preserved by $\mathcal{V}(V, \_)$ (this is an easily verified fact about hom-functors of retracts in general); thus a diagram D is λ-sifted in $\mathcal{V}$ iff $\mathcal{V}(FI, \_)$ preserves $\operatorname{colim} D$ for all sets I of less than

λ elements. Observe that $U \cong \mathcal{V}(F1, \_)$ and, since FI is a coproduct $\coprod_I F1$, we have $U^I \cong \mathcal{V}(FI, \_)$. Thus, D is λ-sifted iff U preserves colim D and U^I preserves colim D, or, equivalently, UD is λ-sifted in Set.

More generally, for S-sorted varieties we use the following notation: given a set $S' \subseteq S$ of sorts then

$$U_{S'} : \mathcal{V} \longrightarrow \mathsf{Set}, \quad A \mapsto \prod_{s \in S'} A_s$$

is the functor assigning to algebras (and homomorphisms) a product of the underlying sets (and mappings, respectively) of sorts from S'.

Proposition 4.7. *A diagram D in a λ-ary variety $\mathcal{V}$ is λ-sifted iff for every set S' of less than λ sorts the diagram $U_{S'}D$ is λ-sifted in* Set, *and $U_{S'}$ preserves* colim D.

Proof: Let us call an object of Set^S uniformly λ-presentable if it is λ-presentable, and all nonempty sorts are equal sets. In the notation of 2.4 above, λ-regular projectives are precisely all retracts of free algebras FI where I is uniformly λ-presentable. In fact, given I λ-presentable in Set^S, the set $S' = \{s \in S \mid I_s \neq \emptyset\}$ has less than λ elements, and so does each I_s. By substituting I with $\hat{I}$ where

$$\hat{I}_s = \begin{cases} J = \bigcup_{t \in S'} I_t, & \text{if } \ s \in S' \\ \emptyset, & \text{otherwise,} \end{cases}$$

we see that I is a retract of $\hat{I}$.

Thus, FI is a retract of $F\hat{I}$. We conclude that a diagram $D : \mathcal{D} \longrightarrow \mathcal{V}$ is λ-sifted iff colim D exists and is preserved by $\mathcal{V}(FI, \_)$ for every uniformly λ-presentable object I of Set^S. The set $J = I_s$ (for all $s \in S$ with $I_s \neq \emptyset$) has less than λ elements, and we clearly have

$$\mathcal{V}(FI, \_) \cong (U_{S'})^J$$

where $S' = \{s \in S \mid I_s \neq \emptyset\}$.

That is, a diagram D is λ-sifted in $\mathcal{V}$ iff each $(U_{S'})^J$, where $\mathrm{card}(S') < \lambda$ and $\mathrm{card}(J) < \lambda$, preserves colim D. This holds iff $U_{S'}$ preserves colim D, and $U_{S'}D$ is λ-sifted in Set. $\square$

Corollary 4.8. *In every variety*

(a) *all λ-filtered colimits are λ-sifted*

and

(b) *coequalizers of equivalence relations are λ-sifted.*

Theorem 4.9. *A functor between λ-ary varieties is λ-algebraically exact iff it preserves limits and λ-sifted colimits.*

Proof: Necessity. By 2.12, every λ-algebraically exact functor is induced by a theory morphism $H : \mathcal{T} \longrightarrow \mathcal{T}'$ of the corresponding λ-ary theories. It suffices to prove that the functor $(\_) \cdot H : \mathsf{Mod}(\mathcal{T}') \longrightarrow \mathsf{Mod}(\mathcal{T})$ preserves λ-sifted colimits. Since $(\_) \cdot H : [\mathcal{T}', \mathsf{Set}] \longrightarrow [\mathcal{T}, \mathsf{Set}]$ preserves all colimits, we only have to observe that $\mathsf{Mod}(\mathcal{T})$ is closed under λ-sifted colimits in $[\mathcal{T}, \mathsf{Set}]$ (and analogously for $\mathsf{Mod}(\mathcal{T}')$).

For every object T in $\mathcal{T}$ the model $\mathcal{T}(T, \_)$ is a λ-presentable regular projective of $\mathsf{Mod}(\mathcal{T})$, thus, its hom-functor preserves λ-sifted colimits. The hom-functor of $\mathcal{T}(T, \_)$ is naturally isomorphic to the evaluation-at-T functor $ev_T : [\mathcal{T}, \mathsf{Set}] \longrightarrow \mathsf{Set}$. Thus, λ-sifted colimits in $\mathsf{Mod}(\mathcal{T})$ are formed object-wise, which is to say, in the same way as in $[\mathcal{T}, \mathsf{Set}]$.

Sufficiency. A functor between varieties which preserves limits and λ-sifted colimits preserves λ-filtered colimits and it is therefore a right adjoint by 1.66 in [AR]. It preserves regular epimorphisms (in fact, coequalizers of kernel pairs) by 4.8. $\square$

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References

[ALR] Adámek J., Lawvere F.W., Rosický J., *On the duality between varieties and algebraic theories*, submitted.

[AP] Adámek J., Porst H.-E., *Algebraic theories of quasivarieties*, J. Algebra **208** (1998), 379–398.

[AR] Adámek J., Rosický J., *Locally Presentable and Accessible Categories*, Cambridge University Press, 1994.

[Bo] Borceux F., *Handbook of Categorical Algebra*, Cambridge University Press, 1994, (in three volumes).

[GU] Gabriel P., Ulmer F., *Lokal präsentierbare Kategorien*, LNM 221, Springer-Verlag, Berlin, 1971.

[L] Lawvere F.W., *Functorial semantics of algebraic theories*, Dissertation, Columbia University, 1963.

[S] Street R., *Fibrations in bicategories*, Cahiers Topol. Géom. Différentielles Catégoriques **XXI** (1980), 111–160.

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