Racks and orbits of dressing transformations

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Abstract. A new algebraic structure on the orbits of dressing transformations of the quasitriangular Poisson Lie groups is provided. This gives the topological interpretation of the link invariants associated with the Weinstein-Xu classical solutions of the quantum Yang-Baxter equation. Some applications to the three-dimensional topological quantum field theories are discussed.

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Introduction

Three-dimensional topological quantum field theories and especially Chern-Simons type theory (see [1], [2], [3]) have been attracting interest of mathematicians and physicists. Many of them give us new invariants of links and 3-manifolds. The article [11] gives an excellent account of main developments in knot theory which followed upon the discovery of the Jones polynomials [12] in 1984. From our point of view, nevertheless, we are far from the understanding of the topological meaning of the new invariants.

In the very deep paper [7], A. Weinstein and P. Xu defined a broad class of knot and link invariants using a kind of the classical solutions of the quantum Yang-Baxter equation. In [7] the classical analogue is developed for the part of the standard construction in which generalized Jones invariants are produced from representations of quantum groups. As far as I know their article is the first attempt to understand the topological meaning of the *quantum group invariants* on the quasi-classical level. It was established in [10] that in the case of a factorizable Poisson Lie group G the Weinstein-Xu link invariant coincides with the space of link group representations in G. The general case of an arbitrary quasitriangular Poisson Lie group is connected with Joyce's theory of knot quandles or fundamental racks.

Any codimension two link has a fundamental rack which contains more information than the fundamental group and which is a complete invariant for irreducible links in any 3-manifold.

The rack provides a complete algebraic and topological framework in which to study links and knots in 3-manifolds. The search of rack structures inside of the quantum link invariants and of the three-dimensional topological quantum field theories looks like the search of hidden symmetries in the integrable equations.

It was done in [8] for topological quantum field theories associated with finite groups. I think that this is a very perspective area for investigation and that the concept of rack gives us powerful tools for description of the topological quantum field theories.

Our main goal in this paper is to give the rack structure for the Poisson Lie group. This is the main ingredient in our interpretation of the Weinstein-Xu link invariant in the general case of an arbitrary quasitriangular Poisson Lie group.

1. Racks and quandles

In this section we state some properties of racks and quandles which will be used in this paper. For more details on this subject, see [4], [5], [6]. To simplify reading, we keep the notations of [4] wherever possible, on one hand, but give all necessary definitions, on the other.

Recall that a set with product is a pair $(\Delta, *)$ where Δ is a set and $*$ is a map $\Delta \times \Delta \to \Delta$. The value of this map for (a, b) will be denoted by a^b or by $a * b$. The reasons for writing the operation exponentially are explained in [4]. A morphism of sets with product $(\Delta, *) \to (\Delta', *')$ is a map $\phi : \Delta \to \Delta'$ such that $\phi(a * b) = \phi(a) *' \phi(b)$. For any set with product $(\Delta, *)$ and $b \in \Delta$, the right translation r_b is the map $r_b : \Delta \to \Delta$ defined by $r_b(a) = a * b = a^b$.

Definition 1.1 ([4]). *A* **rack** *is a non-empty set Δ with a product satisfying the following two axioms.*

- *Given $a, b \in \Delta$ there is a unique $c \in \Delta$ such that $a = c^b$.*
- *Given $a, b, c \in \Delta$ the formula*

$$a^{bc} = a^{cb^c}$$

holds.

Here a^{bc} means $(a^b)^c$ and a^{b^c} means $a^{(b^c)}$.

In other words, a set with product $(\Delta, *)$ is a rack, iff all right translations are automorphisms:

- $\forall\, a, b \in \Delta \quad \exists! \quad c \in \Delta \quad a = c * b$
- $\forall\, a, b, c \in \Delta \qquad (a * c) * (b * c) = (a * b) * c.$

One can find many examples of automorphic sets in [4], [6].

The rack axioms are the algebraic distillation of two of the Reidemeister moves (the second and third moves).

Definition 1.2 ([7], [13]). *A map $R : S \times S \to S \times S$, where S is any set, is called a solution to the set-theoretic quantum Yang-Baxter equation if*

$$R_{13}R_{23}R_{12} = R_{12}R_{23}R_{13},$$

where $R_{ij} : S \times S \times S \to S \times S \times S$ is R on the i^{th} and j^{th} factors of the cartesian product and Id on the third one.

The following fact is crucial for using racks in low-dimensional topology.

Lemma 1.3. *If Δ is a rack then the map*

$$R : \Delta \times \Delta \to \Delta \times \Delta \qquad (a, b) \mapsto (a, b^a)$$

is a solution to the set-theoretic Yang-Baxter equation.

Given a rack Δ, we can get an action of the braid group B_n on $(\Delta)^n$. More precisely, suppose that $R : \Delta \times \Delta \to \Delta \times \Delta$ is a solution to the set-theoretic Yang-Baxter equation from the lemma above. Let $\hat{R} = R \circ \sigma$ with $\sigma : \Delta \times \Delta \to \Delta \times \Delta$ being the exchange of components, and let $\hat{R}_i(n)$ be the endomorphism of the cartesian power Δ^n defined by:

$$\hat{R}_i(n)((x_1, \dots, x_n)) = (x_1, \dots, x_{i-1}, \hat{R}(x_i, x_{i+1}), x_{i+2}, \dots, x_n).$$

Then by the assignment of $\hat{R}_i(n)$ to the i^{th} generator b_i of the braid group B_n we obtain an action of B_n on Δ^n for each n.

In what follows, all examples will be satisfied by the identity

$$a^a = a \qquad \text{for all} \qquad a \in \Delta,$$

which we call the **quandle** condition. This condition is quarantine the first Reidemeister move. We shall call a rack satisfying the quandle condition a **quandle rack** or **quandle**. The term quandle is due to Joyce [5].

Finally, we recall a definition of Freyd and Yetter [14] (see also [4]).

Definition 1.4. *A* (*right*) *crossed G-set for a group G is a set, X, with a right action of the group G, which we write as*

$$(x, g) \mapsto x \cdot g \qquad \textit{where} \qquad x, x \cdot g \in X \quad \textit{and} \quad g \in G,$$

and a function $\delta : X \to G$ satisfying the **augmentation identity***:*

$$\delta(a \cdot g) = g^{-1}(\delta(a))g \quad \textit{for all} \quad a \in X, g \in G,$$

which is precisely the same as saying that δ is G-map when G is regarded as a right G-set under right conjugation.

Given a crossed G-set X, we can define an operation of X on itself by defining a^b to be $a \cdot \delta(b)$. One can easily check that the operation $(a, b) \mapsto a^b$ gives us the rack structure on X, which we call *the augmented rack with augmentation δ*. For more details on the theory of augmented rack see [4].

2. Quasitriangular Poisson Lie groups

As usual, the *Poisson bracket* $\{\cdot, \cdot\}$ on a smooth manifold M is understood as a Lie algebra structure on the space of smooth functions $C^\infty(M)$ satisfying the Leibnitz identity, i.e. a bilinear operation $\{\cdot, \cdot\}$ such that

i. $\{f, g\} = -\{g, f\}$,
ii. $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ (Jacobi identity), and
iii. $\{fg, h\} = f\{g, h\} + \{f, h\}g$ (Leibnitz identity).

One can easily see that in local coordinates x^i on the manifold M, an arbitrary Poisson bracket looks like

$$\{f,g\}(x) = \pi^{ij}(x)\partial f(x)/\partial x^i \cdot \partial g(x)/\partial x^j, \tag{1}$$

for some *Poisson tensor* $\pi^{ij}(x)$. Note that summation over repeated indices is always understood. Bracket (1) is Poisson if and only if the Poisson tensor satisfies the equation:

$$\pi^{lk}(x)\frac{\partial \pi^{ij}(x)}{\partial x^l} + \pi^{li}(x)\frac{\partial \pi^{jk}(x)}{\partial x^l} + \pi^{lj}(x)\frac{\partial \pi^{ki}(x)}{\partial x^l} = 0.$$

The Poisson tensor $\pi^{ij}(x)$ is called a *Poisson structure* on the manifold M.

A product $M_1 \times M_2$ of two manifolds equipped with Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$, respectively, may be given a *product* Poisson bracket. The latter is a Poisson bracket whose Poisson tensor $\pi_p^{ij}(x,y)$ in the point $(x,y) \in M_1 \times M_2$ is

$$\pi_p(x,y) = \begin{pmatrix} \pi_1^{ij}(x) & 0 \\ 0 & \pi_2^{ij}(y) \end{pmatrix},$$

where π_1^{ij} and π_2^{ij} are Poisson tensors of $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$, respectively. If $p_1 : M_1 \times M_2 \to M_1$ and $p_2 : M_1 \times M_2 \to M_2$ are natural projections, and $p_1^* : C^\infty(M_1) \to C^\infty(M_1 \times M_2)$ and $p_2^* : C^\infty(M_2) \to C^\infty(M_1 \times M_2)$ are the corresponding pullbacks, then it is easy to see that the product Poisson bracket may be defined as the only Poisson bracket on $M_1 \times M_2$ satisfying the conditions:

i. $\{p_1^*(f), p_1^*(g)\} = \{f,g\}_1$, and $\{p_2^*(f), p_2^*(g)\} = \{f,g\}_2$,
ii. $\{p_1^*(f), p_2^*(g)\} = 0$.

A mapping $F : M_1 \to M_2$ of two manifolds equipped with Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$, respectively, is called *Poisson* if the pullback $F^* : C^\infty(M_2) \to C^\infty(M_1)$ is a Lie algebra homomorphism, i.e.

$$\{f \circ F, g \circ F\}_1 = \{f,g\}_2 \circ F.$$

Let G be a Poisson Lie group. This means that G is a Lie group equipped with a Poisson structure π such that the multiplication in G viewed as a map $G \times G \to G$ is a Poisson mapping, where $G \times G$ carries the product Poisson structure. The theory of Poisson Lie groups is a quasiclassical version of the theory of quantum groups.

One can easily check that the Poisson structure π must vanish at the identity $e \in G$, so that its linearization $d_e\pi : \mathcal{G} \to \mathcal{G} \wedge \mathcal{G}$ at e is well defined (here $\mathcal{G}$ is the Lie algebra of G). It turns out that this linear homomorphism is a 1–cocycle with respect to the adjoint action. Moreover, the dual homomorphism $\mathcal{G}^* \wedge \mathcal{G}^* \to \mathcal{G}^*$ satisfies the Jacobi identity; i.e. $\mathcal{G}^*$ becomes a Lie algebra. Such a pair $(\mathcal{G}, \mathcal{G}^*)$ is called a Lie bialgebra ([15]). Each Lie bialgebra corresponds to a unique connected, simply connected Poisson Lie group. It is easy to show that the pair $(\mathcal{G}^*, \mathcal{G})$ is a Lie bialgebra as soon as $(\mathcal{G}, \mathcal{G}^*)$ is one. The Poisson Lie group (G^*, π^*) corresponding to $(\mathcal{G}^*, \mathcal{G})$ will be called dual to (G, π). Thus (connected, simply connected) Poisson Lie groups come in dual pairs.

$\mathcal{G}$ and $\mathcal{G}^*$ may be put as Lie subalgebras into the greater Lie algebra $\widetilde{\mathcal{G}}$ which is called the double Lie algebra. A vector space $\widetilde{\mathcal{G}}$ equals $\mathcal{G} \oplus \mathcal{G}^*$, with Lie bracket

$$[X + \xi, Y + \eta] = [X, Y] + [\xi, \eta] + ad^*_X \eta - ad^*_Y \xi + ad^*_\xi Y - ad^*_\eta Y.$$

Here $X, Y \in \mathcal{G}$, $\xi, \eta \in \mathcal{G}^*$ and ad^* denotes the coadjoint representations of $\mathcal{G}$ on $\mathcal{G}^*$ and of $\mathcal{G}^*$ on $\mathcal{G} = (\mathcal{G}^*)^*$. We use $[\cdot, \cdot]$ to denote both the bracket on $\mathcal{G}$ and $\mathcal{G}^*$.

With respect to the ad-invariant non-degenerate canonical bilinear form

$$(X + \xi, Y + \eta) = \langle X, \eta \rangle + \langle Y, \xi \rangle,$$

$\mathcal{G}$ and $\mathcal{G}^*$ form maximal isotropic subspaces of $\widetilde{\mathcal{G}}$.

The simply connected group $\widetilde{G}$ corresponding to $\widetilde{\mathcal{G}}$ is the classical Drinfeld double of the Poisson Lie group (G, π).

Conversely, any Lie algebra $\widetilde{\mathcal{G}}$ with a non-degenerate symmetric ad-invariant bilinear form and a pair of maximal isotropic subalgebras (a Manin triple) gives a pair of dual Lie bialgebras by identifying one of the subalgebras with the dual of the other by means of this bilinear form.

Let $r = \sum_i a_i \otimes b^i$ be an element of $\mathcal{G} \otimes \mathcal{G}$; we say that r satisfies the classical Yang-Baxter equation if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

Here, for instance, $[r_{12}, r_{13}] = \sum_{i,j} [a_i, a_j] \otimes b^i \otimes b^j$. A quasitriangular Lie bialgebra is a pair $(\mathcal{G}, r)$, where $\mathcal{G}$ is Lie bialgebra, $r \in \mathcal{G} \otimes \mathcal{G}$, the coboundary of r is the cobracket $d_e\pi : \mathcal{G} \to \mathcal{G} \wedge \mathcal{G}$ and r satisfies the classical Yang-Baxter equation.

Let us associate with r a linear operator

$$r_+ : \mathcal{G}^* \to \mathcal{G}, \qquad \xi \mapsto \langle r, \xi \otimes id \rangle.$$

Its adjoint is given by

$$-r_- = r_+^* : \mathcal{G}^* \to \mathcal{G}, \qquad \xi \mapsto \langle r, id \otimes \xi \rangle = \langle P(r), \xi \otimes id \rangle,$$

where P is the permutation operator in $\mathcal{G} \times \mathcal{G}$, $P(X \otimes Y) = Y \otimes X$.

The Lie bracket $[\cdot, \cdot]$ in $\mathcal{G}^*$ is given by

$$[\xi, \eta] = ad^*_{r_+(\xi)} \eta - ad^*_{r_-(\eta)} \xi.$$

Lemma 2.1 ([7], [16]). *For any quasitriangular Lie bialgebra $(\mathcal{G}, r)$, the linear maps*

$$r_+, r_- : \mathcal{G}^* \to \mathcal{G}$$

defined above are both Lie algebra homomorphisms.

We now turn our attention to groups.

Definition 2.2 ([7]). *A Poisson Lie group G is called quasitriangular if its corresponding Lie bialgebra $(\mathcal{G}, \mathcal{G}^*)$ is quasitriangular and if the Lie algebra homomorphisms r_+ and r_- from $\mathcal{G}^*$ to $\mathcal{G}$ lift to Lie group homomorphisms R_+ and R_- from G^* to G.*

It turns out that if G is quasitriangular, the maps ϕ and ψ from G^* to G are Poisson morphisms, where $\phi(x) = R_+(x^{-1})$ and $\psi(x) = R_-(x^{-1})$ for any $x \in G^*$. For every Poisson Lie group G there are naturally defined left and right "dressing" actions of G on G^* ([17]), whose orbits are exactly the symplectic leaves of G^*. When G has the zero Poisson structure, its dual Poisson Lie group is simply $\mathcal{G}^*$ with the abelian Lie group structure and ordinary Lie-Poisson bracket. The left and right dressing actions in this case are simply the left and right coadjoint actions of G on $\mathcal{G}^*$.

Given a Poisson Lie group (G, π), we can consider the Lie algebra homomorphism from $\mathcal{G}$ to the Lie algebra of vector fields on G^*. To describe this homomorphism, we pick an element $v \in \mathcal{G} = (\mathcal{G}^*)^*$. It may be identified with an element $\alpha_v \in T_e^* G^*$. Let α be an extension of α_v to a right-invariant 1-form on G^*. Then the vector field v_* corresponding to v is obtained from $-\alpha$ by means of the Poisson structure π^*. It turns out that in this way we get a Lie algebra homomorphism from $\mathcal{G}$ to the Lie algebra of vector fields on $\mathcal{G}^*$ ([18]). Hence, if all the vector fields v_* are complete (G is a complete Poisson Lie group), we obtain a G-action λ on G^* called the left dressing action. If in this construction we replace the right-invariant 1-form by the left-invariant 1-form on G^* we obtain the right dressing action ρ of G on G^*.

3. Poisson Lie rack

Let G be a quasitriangular Poisson Lie group with Lie bialgebra $\mathcal{G}$, and let G^* be its simply connected dual. We can lift the Lie algebra homomorphisms $r_\pm : \mathcal{G}^* \to \mathcal{G}$ to the group homomorphisms $R_\pm : G^* \to G$, and define the map $J : G^* \to G$ by $J(x) = R_+(x)(R_-(x))^{-1}$. The group G is *factorizable* if J is a global diffeomorphism. In this case, for each element $x \in G$ we have a factorization $x = x_+ x_-^{-1}$, where $x_\pm = R_\pm(J^{-1}(x))$.

The following theorem gives us an augmented rack structure for G^*.

Theorem 3.1. *Let G be a quasitriangular Poisson Lie group and let G^* be its simply connected dual. Then G^* has a structure of crossed G-set with (right) action*

$$(x, g) \mapsto \lambda_{g^{-1}} x$$

and augmentation δ:

$$\delta(x) = \phi(x^{-1})\psi(x).$$

This theorem is the generalization for quasitriangular Poisson Lie groups of the well-known fact that the left dressing action of G on G^*, for a factorizable Poisson Lie group G, coincides with the conjugation action Ad_x under the identification of G with G^* by J.

Proof: It follows from Lemma 2.1 that the Lie algebra homomorphisms $r_\pm$ naturally extend to Lie algebra homomorphisms $f_\pm$ from the double Lie algebra $\widetilde{\mathcal{G}}$ onto $\mathcal{G}$ defined by: $f_\pm(X+\xi) = X + r_\pm\xi$. By $F_\pm$ we denote the Lie group homomorphism from the classical Drinfeld double $\widetilde{G}$ of the Poisson Lie group (G,π). For any $d = gu = \bar{u}\bar{g} \in \widetilde{G}$ with $g, \bar{g} \in G$ and $u, \bar{u} \in G^*$ we have

$$F_+ = g\phi(u^{-1}) = \phi(\bar{u}^{-1})\bar{g}$$

and

$$F_- = g\psi(u^{-1}) = \psi(\bar{u}^{-1})\bar{g}.$$

These imply that

$$\phi(\bar{u}^{-1})\psi(\bar{u}) = g\phi(u^{-1})\psi(u)g^{-1}.$$

Finally, a straightforward calculation based on the identity $\lambda_g u = \bar{u}$ shows that G^* has the structure of a crossed G-set. □

It follows from a result of Weinstein-Xu (Lemma 8.5 from [7]) that the rack from Theorem 3.1 is the quandle.

Definition 3.2. *Let G be a quasitriangular Poisson Lie group and let G^* be its simply connected dual. The* **Poisson Lie quandle** *is G^* with the following rack operation:*

$$b^a = \lambda_{\psi(a^{-1})\phi(a)} b.$$

Symplectic Poisson Lie quandles are the symplectic leaves of G^* (orbits of "dressing" actions of G on G^*) with the same rack operation.

For any rack Δ we have the braid group B_n action on Δ^n. Recall that two braids give rise to equivalent links if and only if they are equivalent under Markov moves. There are two types of Markov moves: one is conjugation $A \to BAB^{-1}$; the other is the increase of the number of strings in braid by a simple twist: $A \to Ab_n^\pm$, for $A \in B_n$, where b_n is n^{th} generator of B_{n+1}. After an elementary calculation we get

Lemma 3.3. *Suppose that Δ is a quandle. If $A \in B_n$ and $B \in B_m$ define equivalent links, then the fixed point sets of A on Δ^n and of B on Δ^m are isomorphic.*

This implies that for the Poisson Lie quandle G^* we have the link invariant as the fixed point set of the corresponding braid action on $(G^*)^n$. This invariant is the space of representations of the fundamental augmented rack of a link into Poisson Lie quandle (see Proposition 7.6 from [4]).

It turns out that this space of representations of the fundamental augmented rack of a link into the Poisson Lie quandle is equal to the Weinstein-Xu link invariant, associated with the quasitriangular Poisson Lie group G ([7]).

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