Existence of mild solutions on semiinfinite interval for first order differential equation with nonlocal condition

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Abstract. In this paper we investigate the existence of mild solutions defined on a semiinfinite interval for initial value problems for a differential equation with a nonlocal condition. The results is based on the Schauder-Tychonoff fixed point theorem and rely on a priori bounds on solutions.

Keywords: initial value problems, mild solution, semiinfinite interval, nonlocal condition, fixed point

Classification: 34A60, 34G20, 35R10, 47H20

1. Introduction

In this paper we study the existence of mild solutions, defined on a semiinfinite interval $J = [0,\infty)$, for an initial value problem (IVP) for a semilinear evolution equation, with a nonlocal condition, of the form

$$y' - A(t)y(t) = f(t,y), \quad t \in J := [0,\infty), \tag{1.1}$$

$$y(0) + g(y) = y_0, \tag{1.2}$$

where $f : J \times E \longrightarrow E$, $g \in C(J,E)$ are given functions, $y_0 \in E$, $A(t), t \in J$ is a linear operator from a dense subspace $D(A(t))$ of E into E and E is a real Banach space with norm $\|\cdot\|$.

The paper by Byszewski [6] was the first about the nonlocal problems for evolution equations in Banach spaces. The nonlocal conditions were motivated by physical problems. For the importance of nonlocal conditions in different fields we refer to [6] and the references cited therein. In fact, more authors have paid attention to the research of IVP with nonlocal conditions, in the few past years. We refer to Balachandran and Chandrasekaran [4], Byszewski [5], [6], Ntouyas and Tsamatos [11], [12].

The method we are going to use is to reduce the existence of mild solutions to problem (1.1)–(1.2) to the search for fixed points of a suitable map on a Fréchet space $C(J,E)$. In order to prove the existence of fixed points, we shall rely on the theorem of Schauder-Tychonoff.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

$C(J,E)$ is the linear metric Fréchet space of continuous functions from J into E with the metric (see Dugundji and Granas [9], Corduneanu [7]):

$$d(y,z) := \sum_{m=0}^{\infty} \frac{2^{-m}\|y-z\|_m}{1+\|y-z\|_m} \quad \text{for each} \quad y,z \in C(J,E),$$

where

$$\|y\|_m := \sup\{\|y(t)\| : t \in [0,t_m]\}$$

and

$$t_1 < t_2 < \cdots < t_m \longrightarrow \infty.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable. (For properties of Bochner integral see Yosida [13].)

$L^1(J,E)$ denotes the Banach space of continuous functions $y : J \longrightarrow E$ which are Bochner integrable, normed by

$$\|y\|_{L^1} := \int_0^{\infty} \|y(t)\|\,dt \quad \text{for all} \quad y \in L^1(J,E).$$

The convergence in $C(J,E)$ is the uniform convergence on compact intervals, i.e. $y_j \longrightarrow y$ in $C(J,E)$ if and only if for each $m \in \mathbb{N}$, $\|y_j - y\|_m \longrightarrow 0$ in $C([0,t_m],E)$ as $j \longrightarrow \infty$.

$M \subseteq C(J,E)$ is a bounded set if and only if there exists a positive function $\phi \in C(J,\mathbb{R}_+)$ such that

$$\|y(t)\| \le \phi(t) \ \text{ for all } \ t \in J \ \text{ and all } \ y \in M.$$

The Ascoli-Arzela theorem says that a set $M \subseteq C(J,E)$ is compact if and only if for each $m \in \mathbb{N}$, M is a compact set in the Banach space $(C([0,t_m],E), \|\cdot\|_m)$.

The operator $G : E \longrightarrow E$ is said to be completely continuous if $G(B)$ is relatively compact in E, for every bounded subset $B \subseteq E$.

A function $f : J \times E \longrightarrow E[(t,y) \longmapsto f(t,y)]$ is said to be an L^1- Carathéodory function if:

(i) for each $t \in J$ the function $f(t,.) : E \longrightarrow E$ is continuous;
(ii) for each $y \in E$ the function $f(.,y) : J \longrightarrow E$ is strongly measurable;
(iii) for every positive integer k there exists $h_k \in L^1(J,\mathbb{R})$ such that for a.e. $t \in J$

$$\sup_{\|y\| \le k} \|f(t,y)\| \le h_k(t).$$

We will need the following assumptions.

(H1) $A(t)$, $t \in J$, is the infinitesimal generator of the linear semigroup $T(t,s)$, $(t,s) \in \gamma := \{(t,s) : 0 \le s \le t < \infty\}$, that is

$$A(t)y = \lim_{h \to 0^+} \frac{T(t+h,t)y - y}{h}, \quad y \in D(A(t)),$$

where $T(t,s) \in B(E)$ for each $(t,s) \in \gamma$, satisfying:

(i) $T(t,t) = I$ (I is the identity operator in E),
(ii) $T(t,s)T(s,r) = T(t,r)$ for $0 \le r \le s \le t < \infty$,
(iii) the mapping $(t,s) \longmapsto T(t,s)y$ is strongly continuous in γ for each $y \in E$.

(H2) $f : J \times E \longrightarrow E[(t,y) \longmapsto f(t,y)]$ is an L^1- Carathéodory function.

(H3) There exists a constant $G > 0$ such that $\|g(y)\| \le G$ for each $y \in E$.

(H4) $\|f(t,y)\| \le p(t)\psi(\|y\|)$ for almost all $t \in J$ and all $y \in E$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0,\infty)$ is continuous and increasing with

$$M \int_0^a p(s)\,ds < \int_c^\infty \frac{du}{\psi(u)} \quad \text{for each} \quad a > 0,$$

where $M = \sup\{\|T(t,s)\|; (t,s) \in \gamma\}$ and $c = M\|y_0\| + MG$.

(H5) For each bounded set $B \subset C(J,E)$, $y \in B$ and $t \in J$, the set

$$\left\{T(t,0)y_0 - T(t,0)g(y) + \int_0^t T(t,s)f(s,y(s))\,ds\right\}$$

is relatively compact.

The following lemma is crucial in the proof of our main theorem:

Lemma 2.1 (Schauder-Tychonoff [9], [7]). *Let Ω be a closed convex subset of a locally convex Hausdorff space E. Assume that $N : \Omega \longrightarrow \Omega$ is continuous and that $N(\Omega)$ is relatively compact in E. Then N has at least one fixed point in Ω.*

3. Main result

A continuous solution $t \longrightarrow y(t)$ of the integral equation

$$y(t) = T(t,0)y_0 - T(t,0)g(y) + \int_0^t T(t,s)f(s,y(s))\,ds, \quad t \in y,$$

is called a *mild solution* of (1.1)–(1.2) on J.

Now, we are able to state and prove our main theorem.

Theorem 3.1. *Let $g : C(J,E) \longrightarrow E$ be a continuous function. Assume that hypotheses* (H1)–(H5) *are satisfied. Then problem* (1.1)–(1.2) *has at least one mild solution on J.*

PROOF: We transform problem (1.1)–(1.2) into a fixed point problem. Consider the map $N : C(J,E) \longrightarrow C(J,E)$ defined by

$$(Ny)(t) = T(t,0)y_0 - T(t,0)g(y) + \int_0^t T(t,s)f(s,y(s))\,ds, \quad t \in J.$$

Let

$$\Omega := \{y \in C(J,E) : \|y(t)\| \le a(t), \quad t \in J\},$$

where

$$a(t) := I^{-1}\Big(M\int_0^t p(s)\,ds\Big)$$

and

$$I(z) := \int_0^z \frac{du}{\psi(u)}.$$

Clearly Ω is a convex subset of $C(J,E)$.

We shall show that Ω is closed and the operator N defined on Ω has values in Ω and it is compact. The proof will be given in five steps.

Step 1. Ω *is closed.*

Let $y_n \in \Omega$ with $\|y_n\|_m \longrightarrow \|y\|_m$ (i.e. y_n converges uniformly to y on $[0,t_m]$) for each $m \in \{1,2,\dots\}$. Then, for each fixed $t \in [0,t_m]$, we have

$$\|y_n(t)\| \le a(t),$$

which implies

$$\|y(t)\| \le a(t).$$

So, $y \in \Omega$.

Step 2. $N(\Omega) \subseteq \Omega$.

Let $y \in \Omega$ and fix $t \in J$. We must show that $Ny \in \Omega$.
Let $x < t$. Then

$$\begin{aligned}
\|(Ny)(x)\| &\le M\|y_0\| + MG + M\int_0^x p(s)\psi(\|y(s)\|)\,ds \\
&\le M\|y_0\| + MG + M\int_0^x p(s)(\psi(a(s)))\,ds \\
&= M\|y_0\| + MG + \int_0^x a'(s)\,ds \\
&= a(x)
\end{aligned}$$

since

$$\int_c^{a(s)} \frac{du}{\psi(u)} = M \int_0^s p(\tau)\, d\tau.$$

Thus $Ny \in \Omega$. So, $N : \Omega \longrightarrow \Omega$.

Step 3. *N is continuous.*

Let $y_n \longrightarrow y$ in $C(J, E)$. We will show that

$$Ny_n \longrightarrow Ny \ \text{ in } \ C(J, E).$$

Now, $\|y_n\|_m \longrightarrow \|y\|_m$ implies that there exists $r > 0$ such that

$$\|y_n\|_m \le r \ \text{ and } \ \|y\|_m \le r.$$

Also, there exists $h_r \in L^1_{loc}(J, \mathbb{R})$ with $\|f(s, y)\| \le h_r(s)$ for a.e. $s \in [0, t_m]$ and all $\|y\| \le r$. For each $t \in [0, t_m]$, we have

$$f(s, y_n(s)) \longrightarrow f(s, y(s)) \ \text{ for a.e. } \ s \in [0, t_m].$$

The above formula together with the Lebesgue dominated convergence theorem implies that

$$\begin{aligned}
\|Ny_n - Ny\|_m &= \sup_{t\in[0,t_m]} \Big\| -T(t,0)[g(y_n) - g(y)] \\
&\quad + \int_0^t T(t,s)[f(s, y_n(s)) - f(s, y(s))]\, ds \Big\| \\
&\le \| -T(t,0)\| \cdot \|g(y_n) - g(y)\| \\
&\quad + \int_0^m T(t,s) \|f(s, y_n(s)) - f(s, y(s))\|\, ds \to 0.
\end{aligned}$$

Thus, N is continuous.

Step 4. *N maps bounded sets in $C(J, E)$ into uniformly bounded sets.*

Let $B_r = \{y \in C(J, E) : \|y\| \le r\}$ be a bounded set in $C(J, E)$. Then, there exists $h_r \in L^1_{loc}(J, \mathbb{R})$ with $\|f(s, y)\| \le h_r(s)$ for a.e. $s \in [0, t_m]$ and all $y \in B_r$. Thus,

$$\begin{aligned}
\|Ny(t)\|_m &\le M\|y_0\| + MG + \int_0^t T(t,s) f(s, y(s))\, ds \\
&\le M\|y_0\| + MG + M \int_0^{t_m} h_r(s)\, ds.
\end{aligned}$$

Step 5. *N maps bounded sets in $C(J,E)$ into equicontinuous family.*

Let $\tau_1, \tau_2 \in [0, t_m]$, $\tau_1 < \tau_2$ and $B_r = \{y \in C(J,E) : \|y\| \leq r\}$ be a bounded set in $C(J,E)$. Thus,

$$\begin{aligned}
\|Ny(\tau_2) - Ny(\tau_1)\| &\leq \|(T(\tau_2,0) - T(\tau_1,0))y_0\| + \|(T(\tau_2,0) - T(\tau_1,0))\|G \\
&\quad + \left\|\int_0^{\tau_2} [T(\tau_2,s) - T(\tau_1,s)] f(s,y(s))\,ds\right\| \\
&\quad + \left\|\int_{\tau_1}^{\tau_2} T(\tau_1,s) f(s,y(s))\,ds\right\| \\
&\leq \|(T(\tau_2,0) - T(\tau_1,0))y_0\| + \|(T(\tau_2,0) - T(\tau_1,0))\|G \\
&\quad + \left\|\int_0^{\tau_2} [T(\tau_2,s) - T(\tau_1,s)] h_r(s)\,ds\right\| + M\int_{\tau_1}^{\tau_2} \|h_r(s)\|\,ds.
\end{aligned}$$

As $\tau_2 \longrightarrow \tau_1$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 3–5 and assumption (H5) together with the Ascoli-Arzela theorem we can conclude that $N(B_r)$ is relatively compact in $C(J,E)$.

Moreover, as a consequence of the Schauder-Tychonoff theorem (Theorem 2.1) we can conclude that N has a fixed point y in Ω which is a solution of (1.1)–(1.2). $\square$

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