On (transfinite) small inductive dimension of products*

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Abstract. In this paper we study the behavior of the (transfinite) small inductive dimension (*trind*) *ind* on finite products of topological spaces. In particular we essentially improve Toulmin's estimation [T] of *trind* for Cartesian products.

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In this paper we study the behavior of the (transfinite) small inductive dimension (*trind*) *ind* on finite products of topological spaces. It is known that if the finite sum theorem for *ind* holds in the factors X, Y then the inequality

$$ind\,(X \times Y) \leq ind\,X + ind\,Y \tag{1}$$

is true (Pasynkov [9] for completely regular spaces, see also [1] for regular T_1-spaces). Similar statements for the transfinite small inductive dimension *trind* one can find in [11] (the case of regular T_1-spaces) and in [2] (the case of normal T_1-spaces).

But if the finite sum theorem for *ind* fails even in one factor then the inequality (1) is not valid for two compact spaces. Filippov [5] has constructed compact spaces X, Y such that $ind\,X = Ind\,X = dim\,X = 1$, $ind\,Y = Ind\,Y = dim\,Y = 2$ but $ind\,(X \times Y) = 4$ (see also [8]).

In the sequel, $\alpha = \lambda(\alpha) + n(\alpha)$ is the natural decomposition of the ordinal number α into the sum of the limit ordinal number $\lambda(\alpha)$ and the non-negative integer $n(\alpha) \geq 0$.

In [10] Toulmin has given the following estimation of the transfinite small inductive dimension for the product of two spaces X, Y ($X \times Y$ is hereditarily normal). Namely,

$$trind\,(X \times Y) \leq trind\,X (+) trind\,Y + \psi(n(trind\,X), n(trind\,Y)) \tag{2}$$

where $(+)$ is the natural sum of Hessenberg [6], $\psi(0, m) = \psi(m, 0) = 0$ if m is a non-negative integer and $\psi(n, m) = n + m - 1 + \max\{\psi(n-1, m), \psi(n, m-1)\} + \psi(n-1, m-1)$ if n, m are positive integers.

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In particular for finite dimensional spaces X, Y the inequality

(3) $$ind\,(X \times Y) \leq \varphi_T(ind\,X, ind\,Y)$$

is valid, where $\varphi_T(n,m) = n + m + \psi(n,m), n, m$ are non-negative integers (see Tab. 1).

Observe that formula (2) can be written as follows

(2′) $trind\,(X \times Y) \leq \lambda(trind\,X)(+)\lambda(trind\,Y) + \varphi_T(n(trind\,X), n(trind\,Y))$.

In [9] another estimation of the small inductive dimension ind has been proved. Namely,

(4) $$ind\,(X \times Y) \leq \varphi_P(ind\,X, ind\,Y),$$

where $\varphi_P(0,m) = \varphi_P(m,0) = m$ if m is a non-negative integer and $\varphi_P(n,m) = \varphi_P(n-1,m) + \varphi_P(n,m-1) + 2$ if n, m are positive integers (see Tab. 2) (X, Y are regular).

In this paper we essentially improve the inequalities (2)–(4).

By a space we mean a regular T_1-space. We let BdU denote the boundary of the set U. Our terminology follows [E].

The following lemma is evident.

Lemma 1. *Let $X = X_1 \cup X_2$, where X_i is a subset of X. If $Int\,X_1 \cup Int\,X_2 = X$ and $trind\,X_i \leq \alpha_i$, $i = 1,2$, then $trind\,X \leq \max\{\alpha_1, \alpha_2\}$.*

Theorem 2. *Let $X = X_1 \cup X_2$, where X_i is closed in X, and $trind\,X_i \leq \alpha_i$, $i = 1,2$. Then*

$$trind\,X \leq \begin{cases} \max\{\alpha_1, \alpha_2\} & \text{if } \lambda(\alpha_1) \neq \lambda(\alpha_2) \\ \max\{\alpha_1, \alpha_2\} + 1 & \text{if } \lambda(\alpha_1) = \lambda(\alpha_2). \end{cases}$$

In particular, in the finite-dimensional case we have

$$ind\,X \leq \max\{ind\,X_1, ind\,X_2\} + 1.$$

Proof: If $\lambda(\alpha_1) \neq \lambda(\alpha_2)$ then the inequality is valid due to [4, Theorem 7.2.6]. Let $\lambda(\alpha_1) = \lambda(\alpha_2)$. If $x \in X_1 \setminus X_2$ or $x \in X_2 \setminus X_1$ then $trind_x X \leq \max\{\alpha_1, \alpha_2\}$. Let now $x \in X_1 \cap X_2$ and A be a closed subset of X such that $x \notin A$ and $A \cap X_i \neq \emptyset$, $i = 1,2$. Choose a partition C_1 in X_1 between the point x and the set $A \cap X_1$. Obviously one can choose the partition C_1 with $trind\,C_1 < \alpha_1$. Let $X_1 \setminus C_1 = U_1 \cup V_1$, where U_1, V_1 are open in X_1 and disjoint, and $x \in U_1$, $A \cap X_1 \subset V_1$. Choose a partition C_2 in X_2 between the point x and the closed set $((C_1 \cup V_1) \cup A) \cap X_2$. Obviously one can choose the partition C_2 with $trind\,C_2 < \alpha_2$. Let $X_2 \setminus C_2 = U_2 \cup V_2$, where U_2, V_2 are open in X_2 and disjoint, and $x \in U_2$, $((C_1 \cup V_1) \cup A) \cap X_2 \subset V_2$. Observe that the space $Y = C_1 \cup C_2 \cup (X_1 \cap X_2)$ is equal to the union $Y_1 \cup Y_2$, where $Y_i = C_i \cup (X_1 \cap X_2)$ is a subset of Y. Moreover $Int\,Y_1 \cup Int\,Y_2 = Y$, $trind\,Y_i \leq \alpha_i$ (recall that $Y_i \subset X_i$). So by Lemma 1 we have the inequality $trind\,Y \leq \max\{\alpha_1, \alpha_2\}$. The set $C = X \setminus (((U_1 \setminus X_2) \cup U_2) \cup (V_1 \cup (V_2 \setminus X_1)))$ is a partition between the point x and the set A. Besides $C \subset Y$. Hence $trind\,C \leq \max\{\alpha_1, \alpha_2\}$. $\square$

Remark 3. a) Theorem 2 is similar to [3, Theorem 3.9] in the case of regular T_1-spaces. The analog of [3, Corollary 3.10] (the finite sum theorem for closed subspaces) in the case of regular T_1-spaces is also valid.
b) Recall that there exists a compact space L with $ind\,Y = 2$ which can be represented as the union of two closed subspaces L_1 and L_2 such that $ind\,L_1 = ind\,L_2 = 1$ [4, Lokucievskij's example 2.2.14].
c) Recall also that van Douwen and Przymusinski [4, Problem 4.1.B] defined even a metrizable space Y with $ind\,Y = 1$ which can be represented as the union of two closed subspaces Y_1 and Y_2 such that $ind\,Y_1 = ind\,Y_2 = 0$.

Let $P = X \times Y$. Note that for a rectangular open subset $U \times V$ of P we have

$$Bd(U \times V) = (Bd(U) \times [V]) \cup ([U] \times Bd(V)). \tag{$*$}$$

The following lemma is evident.

Lemma 4. *Let $trind\,X = 0$. Then $trind\,(X \times Y) = trind\,Y$ for any space Y.*

Observe that in particular Lemma 4 is also valid for *ind*.

Now let us consider the finite-dimensional case.

Theorem 5. *Let $P = X \times Y$. Then*

$$ind\,P \le \varphi_1(ind\,X, ind\,Y) \tag{5}$$

where $\varphi_1(0, m) = \varphi_1(m, 0) = m$ if m is a non-negative integer, $\varphi_1(n, m) = 2(n + m) - 1$ if n, m are positive integers (see Tab. 3, observe that $\varphi_1(n, m) = \max\{\varphi_1(n - 1, m), \varphi_1(n, m - 1)\} + 2$ if $n, m \ge 1$).

Proof: If at least one of the factors is zero-dimensional in the sense of *ind* then the inequality holds due to Lemma 4. Suppose that $ind\,X, ind\,Y \ge 1$. Apply an induction on the sum $ind\,X + ind\,Y = k$, $k \ge 2$.
Let $k = 2$. Then for any point $p \in P$ and its any neighborhood W there is a rectangular neighborhood $U \times V \subset W$ of this point with $ind\,BdU \le 0$, $ind\,BdV \le 0$.
By Lemma 4 each element from the right part of equality $(*)$ is not more than one-dimensional. From Theorem 2 it follows that $ind\,Bd(U \times V) \le 2$. Hence formula (5) is valid.
Let the theorem hold for $k < n, n \ge 3$. Put $k = n$. For any point $p \in P$ and its any neighborhood W there is a rectangular neighborhood $U \times V \subset W$ of this point with $ind\,BdU \le ind\,X - 1$, $ind\,BdV \le ind\,Y - 1$. By induction assumption the small inductive dimension of each element from the right part of equality $(*)$ is not more than $2(n-1)-1$. From Theorem 2 it follows that $ind\,Bd(U \times V) \le 2(n-1)$. Hence $ind\,P \le 2(n - 1) + 1 = 2(ind\,X + ind\,Y) - 1$. □

Using induction one can easily obtain the following

Estimations.

(a) $\psi(n,m) \leq \psi(n+1,m), \psi(n,m) \leq \psi(n,m+1)$;
(b) $\varphi_1(n,m) \leq \varphi_T(n,m) \leq \varphi_P(n,m)$, if $n,m \geq 1$ and if at least one of the numbers is > 1 then both inequalities are strict.

Remark 6. It is easy to see that $\psi(n,n) \geq 2n - 1 + 2\psi(n-1,n-1)$, $n \geq 1$. Moreover, if $n > k$ then $\psi(n,n) \geq 2(2^k - 1)n + 2^k\psi(n-k,n-k) + f(k)$. Hence, for every natural number m the inequality $\varphi_T(n,n) \geq mn$ holds for large n.

Estimation from Theorem 5 can be improved for the class of completely paracompact spaces.

Let us recall [12] that a topological space X is *completely paracompact* if, for any open cover λ of X, there exist open star-finite covers μ_i of X, $i \in \mathbb{N}$, such that, for any $x \in X$ there exist $O \in \lambda$, $i \in \mathbb{N}$ and $V \in \mu_i$ for which $x \in V \subset O$.

It is known ([12]) that:

(a) any F_σ subset of a completely paracompact space is completely paracompact;
(b) any regular completely paracompact space is paracompact and any strongly paracompact space is completely paracompact;
(c) $dim\, X \leq ind\, X$ for any completely paracompact space.

Lemma 7. *Let Z be a completely paracompact space and $Z = Z_1 \cup Z_2$, where Z_i is closed, $ind\, Z_i \leq 1$, $i = 1,2$, and $ind\,(Z_1 \cap Z_2) \leq 0$. Then $ind\, Z \leq 1$.*

PROOF: If $x \in Z_1 \setminus Z_2$ or $x \in Z_2 \setminus Z_1$ then $ind_x Z \leq 1$. Let now $x \in Z_1 \cap Z_2$ and A be a closed subset of Z such that $x \notin A$. Then from the proof of Theorem 2 it follows that there exists a partition C between x and A such that $C \subset Y = (Z_1 \cap Z_2) \cup C_1 \cup C_2$, where $ind\, C_i \leq 0$, $i = 1,2$. By property **(c)** and the finite sum theorem for dim it follows that dim $Y \leq 0$. From **(b)** it follows that $ind Y \leq 0$. Hence $ind\, Z \leq 1$. □

Theorem 8. *Let $P = X \times Y$ be completely paracompact. Then*

$$ind\, P \leq \varphi_2(ind\, X, ind\, Y), \tag{6}$$

where $\varphi_2(0,m) = \varphi_2(m,0) = m$ if m is a non-negative integer, $\varphi_2(n,m) = 2(n+m) - 2$ if n,m are positive integers (see Tab. 4, observe that $\varphi_2(n,m) = \max\{\varphi_2(n-1,m), \varphi_2(n,m-1)\} + 2$ if $n,m \geq 1$ and $(n,m) \neq (1,1)$).

PROOF: If at least one of the factors is zero-dimensional in the sense of ind then the inequality holds due to Lemma 4. Suppose that $ind\, X, ind\, Y \geq 1$. Apply an induction on the sum $ind\, X + ind\, Y = k$, $k \geq 2$.

Let $k = 2$. Then for any point $p \in P$ and its any neighborhood W there is a rectangular neighborhood $U \times V \subset W$ of this point with $ind\, BdU \leq 0$, $ind\, BdV \leq 0$.

Put $Z = Bd(U \times V), Z_1 = Bd(U) \times [V], Z_2 = [U] \times Bd(V)$ then $Z = Z_1 \cup Z_2, Z_1 \cap Z_2 = Bd(U) \times Bd(V)$. By Lemma 7 and property **(a)** we have $ind\, Z \leq 1$. Hence

formula (6) is valid.
Let the theorem hold for $k < n$, $n \geq 3$. Put $k = n$. For any point $p \in P$ and any its neighborhood W there is a rectangular neighborhood $U \times V \subset W$ of this point with $ind\, BdU \leq ind\, X - 1$, $ind\, BdV \leq ind\, Y - 1$. By induction assumption the small inductive dimension of each element from the right part of equality $(*)$ is not more than $2(n-1)-2$. From Theorem 2 it follows that $ind\, Bd(U \times V) \leq 2(n-1)-1$. Hence $ind\, P \leq 2(n-1) = 2(ind\, X + ind\, Y) - 2$.
□

Corollary 9. *Let $P = X \times Y$, where X, Y are compact spaces, and $ind\, X, ind\, Y \geq 1$. Then*

$$ind\, P \leq 2(ind\, X + ind\, Y) - 2. \tag{7}$$

Observe that estimation (7) is exact (i.e. it cannot be improved) for $ind\, X = ind\, Y = 1$ (it is evident) and for $ind\, X = 1$, $ind\, Y = 2$ (the named earlier Filippov's result [5]).

Question A. Is estimation (7) exact for all situations?

Question B. Are there spaces X, Y such that $ind\, X = ind\, Y = 1$ and $ind\, X \times Y = 3$?

Remark 10. Let $P = \prod_{i=1}^{n} X_i$, where X_i is a compact space with $ind\, X_i \geq 1$, $i = 1, \ldots, n$. Then $ind\, P \leq n(\sum_{i=1}^{n} ind\, X_i - n + 1)$. In the case when all spaces are one-dimensional in the sense of ind the formula coincides with Lifanov's result [7].

Now let us consider the transfinite case.

Theorem 11. *Let $P = X \times Y$ and $trind\, X \leq \alpha$, $trind\, Y \leq \beta$. Then*

$$trind\, P \leq \begin{cases} \alpha(+)\beta + n(\alpha) + n(\beta) - 1 & \text{if } n(\alpha), n(\beta) \geq 1; \\ \alpha(+)\beta & \text{otherwise.} \end{cases} \tag{8}$$

(*Observe that formula* (8) *can be written as follows*

$$trind\,(X \times Y) \leq \lambda(\alpha)(+)\lambda(\beta) + \varphi_1(n(\alpha), n(\beta)). \qquad) \tag{8'}$$

Proof: Use induction on $\alpha(+)\beta = \gamma$. If $\gamma < \omega$ then the inequality holds due to Theorem 5.
Let the theorem be valid for $\gamma < \nu \geq \omega$. Put $\gamma = \nu$. Then for any point $p \in P$ and its any neighborhood W there is a rectangular neighbourhood $U \times V \subset W$ of this point with $trind\, BdU < \alpha$, $trind\, BdV < \beta$.
If ν is limit then $\nu = \lambda(\nu)$ and $\lambda(\alpha) = \alpha, \lambda(\beta) = \beta$. We can assume that $\lambda(\alpha) \geq \omega$ and $\lambda(\beta) \geq \omega$ (otherwise apply Lemma 4). By induction assumption

the transfinite small inductive dimension of each element from the right part of equality $(*)$ is less than ν. From Theorem 2 it follows that $trind\, Bd(U \times V) < \nu$. So the theorem holds in this case.
Let now $n(\nu) \geq 1$. Observe that $\lambda(\nu) = \lambda(\alpha)(+)\lambda(\beta)$ and $n(\nu) = n(\alpha) + n(\beta)$. Let $n(\alpha) = 0$ (analogously with $n(\beta) = 0$). Then $trind\, BdU = \alpha' < \lambda(\alpha)$ and $trind\, BdV \leq \lambda(\beta) + n(\beta) - 1$. By induction assumption we have $trind\, Bd(U) \times [V] \leq \lambda(\alpha')(+)\lambda(\beta) + \varphi_1(n(\alpha'), n(\beta))$ and $trind\,[U] \times Bd(V) \leq \lambda(\alpha)(+)\lambda(\beta) + n(\beta) - 1$. Observe that $\lambda(\alpha')(+)\lambda(\beta) < \lambda(\alpha)(+)\lambda(\beta)$. From Theorem 2 it follows that $trind\, Bd(U \times V) \leq \lambda(\alpha)(+)\lambda(\beta) + n(\beta) - 1$. So the theorem also holds in the case.
Let $n(\alpha) \geq 1$ and $n(\beta) \geq 1$. By induction assumption the transfinite small inductive dimension of each element from the right part of equality $(*)$ is not more than $\lambda(\alpha)(+)\lambda(\beta) + \max\{\varphi_1(n(\alpha) - 1, n(\beta)), \varphi_1(n(\alpha), n(\beta) - 1)\}$. From Theorem 2 it follows that

$$trind\, Bd(U \times V) \leq \lambda(\alpha)(+)\lambda(\beta) + \max\{\varphi_1(n(\alpha) - 1, n(\beta)), \varphi_1(n(\alpha), n(\beta) - 1)\} + 1.$$

Hence

$$\begin{aligned} trind\, P &\leq \lambda(\alpha)(+)\lambda(\beta) + \max\{\varphi_1(n(\alpha) - 1, n(\beta)), \varphi_1(n(\alpha), n(\beta) - 1)\} + 2 \\ &= \lambda(\alpha)(+)\lambda(\beta) + \varphi_1(n(\alpha), n(\beta)). \end{aligned}$$

The theorem is proved. □

Tab 1., $\varphi_T(n, m)$:

	0	1	2	3	...	n
0	0	1	2	3	...	
1	1	3	6	10	...	
2	2	6	11	19	...	
3	3	10	19	32	...	
...	...	...	...	...	...	
m						

Tab 2., $\varphi_P(n, m)$:

	0	1	2	3	...	n
0	0	1	2	3	...	
1	1	4	8	13	...	
2	2	8	18	33	...	
3	3	13	33	68	...	
...	...	...	...	...	...	
m						

Tab 3., $\varphi_1(n,m)$:

	0	1	2	3	...	n
0	0	1	2	3	...	
1	1	3	5	7	...	
2	2	5	7	9	...	
3	3	7	9	11	...	
...	...	...	...	...	...	
m						

Tab 4., $\varphi_2(n,m)$:

	0	1	2	3	...	n
0	0	1	2	3	...	
1	1	2	4	6	...	
2	2	4	6	8	...	
3	3	6	8	10	...	
...	...	...	...	...	...	
m						

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