The Banach contraction mapping principle and cohomology

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Abstract. By a dynamical system (X,T) we mean the action of the semigroup $(\mathbb{Z}^+,+)$ on a metrizable topological space X induced by a continuous selfmap $T : X \to X$. Let $M(X)$ denote the set of all compatible metrics on the space X. Our main objective is to show that a selfmap T of a compact space X is a Banach contraction relative to some $d_1 \in M(X)$ if and only if there exists some $d_2 \in M(X)$ which, regarded as a 1-cocycle of the system $(X,T) \times (X,T)$, is a coboundary.

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1. Introduction and notation

The object of our study is the discrete dynamical system (X,T) by which we mean a continuous selfmap $T : X \to X$ of a metrizable topological space X. By $C(X)$ and $M(X)$ we denote the set of all continuous real-valued functions on X and the set of all compatible metrics on X, respectively. The action of T on X induces the action of the semigroup $(\mathbb{Z}^+)$ on $C(X)$ by setting $(nf)(x) = f(T^n x)$ for $f \in C(X)$, $n \in \mathbb{Z}^+$ and $x \in X$, where by T^n we mean the n-th iteration of T and T^0 is the identity map on X. For every $f \in C(X)$ the expression $0f + 1f + \cdots (n-1)f$, regarded as a map from $\mathbb{Z}^+$ to $C(X)$ is a 1-cocycle of the $\mathbb{Z}^+$-module $C(X)$, and it is a 1-coboundary if f can be presented in the form $f = g - 1g$ for some $g \in C(X)$. These facts justify calling any element $f \in C(X)$ a cocycle, and if there is a $g \in C(X)$ such that $f(x) = g(x) - g(Tx)$, we say that f is a coboundary of the system, (X,T). We denote by $B(X) \subseteq C(X)$ the submodule of all coboundaries of the system (X,T). Given two systems (X_i,T_i), $i = 1,2$, we define their product by $(X_1,T_1) \times (X_2,T_2) = (X_1 \times X_2, T)$ where $T : X_1 \times X_2 \to X_1 \times X_2$ is defined by $T(x_1,x_2) = (T_1x_1, T_2x_2)$ for $x_1 \in X_1$ and $x_2 \in X_2$.

Observing that the set $M(X)$ is a subset of $C(X \times X)$, one may ask what dynamical properties of the system (X,T) are implied by the existence of a metric $d \in M(X)$ which, regarded as a cocycle of the system $(X,T) \times (X,T)$, is a coboundary, i.e. $d \in B(X \times X)$. The main purpose of this note is to show that this condition completely characterizes the celebrated Banach contraction mapping principle for compact spaces. We call a system (X,T) Banach, or a B-system for short, if T has a unique fixed point and there is a metric $d \in M(X)$

relative to which T is a Banach contraction. Our main result is the following statement.

Theorem 1. *A system (X, T) on a compact space X is a B-system if and only if there is a metric $d \in M(X)$ which, regarded as a cocycle of the system $(X, T) \times (X, T)$, is a coboundary.*

2. Definitions

Given a system (X, T) we say that a metric $d \in M(X)$ is an E-metric (due to M. Edelstein, see [5]) if $x \neq y$ implies $d(Tx, Ty) < d(x, y)$ for $x, y \in X$, and d is called a B-metric if for some $c \in (0, 1)$ we have $d(Tx, Ty) \leq cd(x, y)$ for $x, y \in X$. In order to formulate clearly our results and facilitate their proofs, we introduce the following seven properties A_n, $n = 1, \ldots, 7$, which a system (X, T) may possess.

A1. The selfmap T has a unique fixed point.
A2. T has a unique fixed point to which every T-orbit $\{T^n x : n \in \mathbb{N}\}$, $x \in X$, converges.
A3. (X, T) is a Lagrange system, i.e. every T-orbit has compact closure.
A4. T has a unique fixed point $x^* \in X$ and there exists an open neighborhood U of x^* such that if V is any open neighborhood of x^* there exists $n \in \mathbb{N}$ such that $m \geq n$ implies that $T^m U \subseteq V$.
A5. T admits an E-metric, i.e. there is $d \in M(X)$ which is an E-metric.
A6. T admits a B-metric.
A7. T admits a metric $d \in M(X)$ which is a coboundary of the system $(X, T) \times (X, T)$, i.e. the intersection $M(X) \cap B(X, X)$ is not empty.

Along with the B-system which we introduced as a system satisfying A1, and A6, we shall study also an E-system defined as a system satisfying A2, and A5. The E-system is a generalization of a B-system and admits a purely topological characterization using the elegant concept of "attractor for compact sets" due to R. Nussbaum (see [8]). A nonempty subset $A \subseteq X$ of a system (X, T) is called an attractor for compact sets if A is compact, T-invariant, i.e. $TA \subseteq A$, and if for every compact set $K \subseteq X$ and any open set U containing A there exists $n \in \mathbb{N}$ such that $m \geq n$ implies that $T^m(K) \subseteq U$. The topological characterization of the E-system in terms of our notation is given by

Theorem A. *A system (X, T) is an E-system if and only if T has a fixed point x^* such that the singleton $\{x^*\}$ is an attractor for compact sets. For proof see* [5].

We shall need two more previous results:

Theorem B. *A system (X, T) is a B-system if and only if it satisfies* A2, *and* A4. *For proof see* [6] *or* [9].

Theorem C. *A system (X, T) on a compact space X is a B-system if and only if $\bigcap\{T^n X : n \in \mathbb{N}\}$ is a singleton. For proof see* [4].

3. Proof of the Theorem

Lemma 1. *A Lagrange system (X,T) satisfying* A7 *is an E-system.*

PROOF: There is a metric $d \in M(X)$ which satisfies $d(x,y) = f(x,y) - f(Tx,Ty)$, $x, y \in X$, where $f \in C(X \times X)$. From this it follows that T cannot have two distinct fixed points. Considering the sum $\sum_{n=0}^{k} d(T^n x, T^n y) = f(x,y) - f(T^{k+1}x, T^{k+1}y)$ we observe that the right hand side is bounded if $k \to \infty$ since both T-orbits $\{T^n x\}$ and $\{T^n y\}$ have compact closure. Setting $y = Tx$ we see that the orbit $\{T^n x\}$ is a Cauchy sequence so that it converges to a fixed point x^* of T. Since, as observed above, there cannot be more than one fixed point we conclude that the system (X,T) satisfies A2. If we let $k \to \infty$ in the above equation, we obtain that the infinite sum $\sum_{n=0}^{\infty} d(T^n x, T^n y)$, denoted by $d^*(x,y)$, equals to $f(x,y) - f(x^*, x^*)$ so that $d^*(x,y)$ is a continuous function in x and y and therefore $d^* \in C(X \times X)$. On the other hand, it is easy to check that d^* is a metric, and since it is continuous we see that $d^* \in M(X)$. From the very definition, i.e. $d^*(x,y) = \sum_{n=0}^{\infty} d(T^n x, T^n y)$, it follows that $d^*(x,y) = d^*(Tx,Ty) + d(x,y)$ showing that d^* is an E-metric and completing the proof that (X,T) is an E-system.

□

Lemma 2. *Every B-system (X,T) is an E-system, and in the case that X is compact, the converse is also true.*

PROOF: Given a B-system (X,T), we take the fixed point $x^* \in X$ of T guaranteed by A1, and a B-metric $d \in M(X)$ guaranteed by A6. Choosing any $x \in X$ we observe that the infinite sum $\sum_{n=0}^{\infty} d(T^n x, x^*)$ converges, implying that the T-orbit $\{T^n x\}$ converges to x^*, hence the system satisfies A2. Since every B-metric is also an E-metric it follows that (X,T) is an E-system.

Now suppose (X,T) is an E-system on a compact space X, and let x^* be the fixed point of T. From Theorem A it follows that $\{x^*\}$ is an attractor for compact sets. Applying this fact to the whole space X, we see that the sets $T^n X$ are arbitrarily small as $n \to \infty$ so that $\bigcap\{T^n X : n \in \mathbb{N}\}$ is the singleton $\{x^*\}$, and Theorem C implies that the system (X,T) is a B-system. □

Now we are ready to prove our main result.

Theorem 1. *A system (X,T) on a compact space X is a B-system if and only if there is a metric $d \in M(X)$ which is a coboundary of the system $(X,T) \times (X,T)$.*

PROOF: If the system (X,T) is a B-system, there is a B-metric $d \in M(X)$, and we observe that $\sum_{n=0}^{\infty} d(T^n x, T^n y)$ converges uniformly to a continuous function $f(x,y)$. Thus, we have that $d(x,y) = f(x,y) - f(Tx,Ty)$ showing that d is a coboundary of $(X,T) \times (X,T)$. Conversely, if (X,T) satisfies A7, Lemma 1 implies that the system (X,T) is an E-system since every system (X,T) on a compact space X is Lagrange. Finally, Lemma 2 shows that (X,T) is a B-system which completes our proof. □

4. Examples and remarks

Going back to Lemma 2, we construct an E-system (X,T) which is not a B-system.

Example 1. Let the space X be the real Hilbert space $\ell_2 = \{(x_n)_1^\infty : \sum_{n=1}^\infty x_n^2 < \infty\}$ and let $T : \ell_2 \to \ell_2$ be defined by setting $T(x_n)_1^\infty = (a_n x_n)_1^\infty$ where $\{a_n\}$ is a sequence of numbers from the interval $(0,1)$ satisfying $\lim a_n = 1$ as $n \to \infty$. We first prove that (ℓ_2, T) is not a B-system by showing that it fails the condition A4 of Theorem B. This condition would imply in our case that there exists a number $m \in \mathbb{N}$ such that for every $x \in \ell_2$ and every $k \geq m$ we have $\|T^k x\| \leq \frac{1}{2}\|x\|$. Applying this to the basis vectors e_n and choosing $k = m$ we obtain $\|T^m e_n\| = a_n^m \leq \frac{1}{2}$ for every $n \in \mathbb{N}$ which is evidently impossible since $\lim a_n = 1$. To verify that (ℓ_2, T) is an E-system we first observe that the metric $\|x-y\|$ is an E-metric since $\|Tx\|^2 = \sum_{n=1}^\infty a_n^2 x_n^2 < \sum_{n=1}^\infty x_n^2 = \|x\|^2$ for every $x \neq 0$. Finally, we must show that for every $x \in X$ the orbit $\{T^k x\}$ converges to zero. For a given $x = (x_i)$ and arbitrary $\varepsilon > 0$ let $n \in \mathbb{N}$ be such that $\sum_{n=1}^\infty x_i^2 \leq \varepsilon$. Then we can write $\|T^k x\|^2 = \sum_{i=1}^\infty a_i^{2k} x_i^2 = \sum_{i=1}^n a_i^{2k} x_i^2 + \sum_{n+1}^\infty a_i^{2k} x_i^2 \leq \sum_{i=1}^n a_i^{2k} x_i^2 + \varepsilon$. The right hand side of this inequality converges to ε as $k \to \infty$ and since ε is arbitrary this concludes our proof.

Realizing that every B-metric d in a B-system is a coboundary, one may ask whether also the converse is true; namely, whether every metric d which is a coboundary is a B-metric. In our Example 2 we show that it is not so. We shall construct a noncompact B-system (X,T) and exhibit a metric $d \in M(X)$, which fails to be an E-metric and which is a coboundary.

Example 2. Let $X = [0,1)$ and let $T : X \to X$ be the quadratic function $Tx = x^2$. It is readily seen that our system satisfies A2. It satisfies also A4 since if we choose $U - [0,a)$ with $0 < a < 1$ we see that for any b with $0 < b < 1$ we have that $T^n U \subseteq [0,b)$ for sufficiently large $n \in \mathbb{N}$. From Theorem B it follows that (X,T) is a B-system. It is clear that the usual metric $d(x,y) = |x-y|$ is not an E-metric since $|x^2 - y^2| > |x-y|$ if $x \neq y$ and $x > \frac{1}{2}$ and $y > \frac{1}{2}$. Defining the function $\varphi : [0,1) \to [0,\infty)$ by $\varphi(x) = \sum_{n=0}^\infty x^{(2^n)}$ (the function φ is known to be analytic in the open unit disc of the complex plane), we see that $\sum_{n=0}^\infty d(T^n x, T^n y) = |\varphi(x) - \varphi(y)|$ is continuous which implies that $|x-y|$ is a coboundary.

Finally, we give an example of a system (X,T) on a compact space X which is not a B-system but satisfies A2, showing that A4 cannot be removed from the hypotheses of Theorem B even if the space X is compact.

Example 3. Let $X = [0,1]$ and $T : X \to X$ be the quadratic function $Tx = x^2$ as in the previous example. Identifying the two fixed points 0 and 1 to a point, say 0, we obtain a system (S^1, T) on the circle S^1 satisfying A2, but it is not a B-system since T maps S^1 onto itself so that T^n cannot shrink S^1 to a point as required by Theorem C. It should be also noted that none of the metrics $d \in M(S^1)$ is a coboundary, since otherwise the system (S^1, T) would be an E-system according to Lemma 1 and therefore a B-system according to Lemma 2. Thus, this system has the property that $M(S^1) \cap B(S^1 S^1)$ is empty.

Remark 1. If we introduce a stronger property, say A8, than A6, claiming the existence of a complete B-metric, then A8 alone would imply the conjunction of A1, and A6. We prefer the weaker property A6, so as to make our results applicable also to spaces which are not topologically complete.

Remark 2. In the last two decades several authors (see e.g. [2], [3], [7], [10]) demonstrated the relevance of the cohomology theory to the study of dynamical systems by showing that a system satisfies certain conditions if and only if a certain one-cocycle is a coboundary. The use of cohomology in our investigation is motivated by the fact that if the infinite sum $\sum_{n=0}^{\infty} d(T^n x, T^n y)$ exists as an element of the $\mathbb{Z}^+$-module $C(X \times X)$ then the metric $d \in M(X)$ is a coboundary. We do not feel very comfortable with the sets $M(X)$ since they do not possess a nice algebraical structure. If the space X is separable, then there are metrics $d \in M(X)$ expressible in the form $d(x, y) = \sum_{n=1}^{\infty} |f_n(x) - f_n(y)|$ where the f_n are elements of $C(X)$. This fact support our belief that at least in separable cases the properties A5, A6 and A7, involving the set $M(X)$ can be reformulated in terms of the $\mathbb{Z}^+$-modules $C(X)$ and $B(X)$.

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