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Cardinal invariants of the lattice of partitions

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Abstract. We study cardinal coefficients related to combinatorial properties of partitions of ω with respect to the order of almost containedness.

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1. Introduction

The set (ω) of all partitions of ω has a natural structure of a lattice under the order $X \leq_1 Y \leftrightarrow$ 'any piece of Y is contained in a piece of X'. The filter of partitions with finite pieces induces the almost containedness relation $\leq_1^*$ which was studied in a number of papers after Carlson and Simpson proved in [1] a dualized version of Ramsey's theorem. We mention [10], [3], [5] and [11], where combinatorial properties of (ω) are expressed in terms of suitably defined cardinal coefficients. The coefficients and relations between them are collected in so called van Douwen's diagram.

These papers involve mainly the methods developed for the Boolean algebra $P(\omega)$. On the other hand, there is a serious difference from that case: (ω) is not a Boolean algebra and does not have any natural complementation. Then the converse order and the corresponding (converse) coefficients are still unknown. This is a motivation for our paper: we study the converse order.

The above scheme also allows us to define another pair of orders, denoted below by $\leq_2^*$ and $\preceq_2^*$ (the converse order). We prove that the first one is completely clear: the corresponding cardinal coefficients are absolute (Section 4.1). One of the cases of this result improves (and simplifies the proof of) the Spinas' result that the cardinality of a maximal non-trivial family of partitions where any two members have meet $\mathbf{0}$ is continuum (Theorem 4.1 in [11]).

The converse order is very similar to the order converse to $\leq_1^*$. Nevertheless, we show that all the orders are pairwise non-isomorphic.

1.1 Notation.

We use standard set theoretic conventions and notation. $[\omega]^\omega$ and $[\omega]^{<\omega}$ stand for all infinite and all finite subsets of ω respectively. For $k \in \omega \setminus \{0\}$, let $[\omega]^k$ be the set of all k-element subsets of ω. We often identify $m \in \omega$ with $\{0, \ldots, m-1\}$.

The ideal of all first category subsets of the real line $\mathbf{R}$ is denoted by $\mathcal{K}$. The cardinal number $cov(\mathcal{K})$ is defined by:

$$cov(\mathcal{K}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{K} \wedge \bigcup \mathcal{A} = \mathbf{R}\}.$$

By (ω) we denote the set of all partitions of ω i.e. families $X \subset P(\omega)$ consisting of pairwise disjoint sets such that $\bigcup X = \omega$. A partition is *finite* if it has finitely many pieces. The set of all finite partitions will be denoted by $(\omega)^{<\omega}$ and the set of all *infinite* partitions will be denoted by $(\omega)^{\omega}$.

If X and Y are partitions of ω then we say that Y is *coarser* than X, or X is *finer* than Y, and we write $Y \leq X$, if each piece of Y is a union of pieces of X. Notice that $((\omega), \leq)$ is a complete lattice with the least element $\mathbf{0}$ — the partition $\{\omega\}$ of ω into one piece and the greatest element $\mathbf{1}$ — the partition $\{\{n\} : n \in \omega\}$ of ω into singletons.

For $X \in (\omega)$ let

$$(X)_{\leq} = \{Y \in (\omega) : Y \leq X\},$$
$$(X)_{\geq} = \{Y \in (\omega) : X \leq Y\}$$

and

$$Pairs(X) = \{\{k, l\} \in [\omega]^2 : (\exists\, a \in X)(\{k, l\} \subset a)\}.$$

We say that X is *trivial* if $Pairs(X)$ is finite, i.e. when all but finitely many finite pieces of X are singletons. By IF we denote the set of all trivial partitions.

For $d \in [\omega]^{<\omega}$ and $X \in (\omega)$ let

$$X^d = \{\bigcup\{x \in X : x \cap d \neq \emptyset\}\} \cup \{x \in X : x \cap d = \emptyset\}$$

and

$$X_d = \{x \setminus d : x \in X\} \cup \{\{n\} : n \in d\}.$$

The following definitions describe the approach from [10] (and, in fact, from [1]). Let $X \in (\omega)$. A partition $X_* \in (\omega)$ is called a *finite modification* of X if X_* is obtained from X by gluing together a finite number of pieces of X, i.e. $(\exists\, d \in [\omega]^{<\omega})(X_* = X^d)$. Of course we always have $X_* \leq X$. This notion allows us to introduce relations of *almost containedness* and *orthogonalities* defined as follows:

1. $X \leq_1^* Y$ (X is almost contained in Y) if for some finite modification X_* of X, we have $X_* \leq Y$;
2. $X \perp_1 Y$ (X is orthogonal to Y) if $X \wedge Y \in (\omega)^{<\omega}$;
3. $X \top_1 Y$ (X is *c-orthogonal* to Y) if $X \vee Y \in IF$.

The order converse to $\leq$ will be denoted by $\preceq$ and the order converse to $\leq_1^*$ — by $\preceq_1^*$. The induced equivalence relation will be denoted by $=_1^*$. Notice that the equivalence class $[\mathbf{0}]_1$ coincides with $(\omega)^{<\omega}$ and $[\mathbf{1}]_1 = IF$,

$$X \top_1 Y \leftrightarrow |Pairs(X) \cap Pairs(Y)| < \omega,$$
$$Pairs(X) \subseteq^* Pairs(Y) \rightarrow X \preceq_1^* Y$$

(by $\subseteq^*$ we denote almost containedness of sets). The converse

$$X \preceq_1^* Y \to Pairs(X) \subseteq^* Pairs(Y)$$

is true for X and Y consisting of finite pieces. Let $(\omega)_1^c$ stand for the set of all infinite non-trivial partitions.

Finally, we say that two partitions X and Y from $(\omega)^\omega$ are *compatible* if X and Y are not orthogonal, i.e. if $X \wedge Y \in (\omega)^\omega$. Non-trivial partitions X and Y are said to be c-*compatible* if they are not c-orthogonal, i.e. if $X \vee Y \notin IF$.

The main results of the paper concern the order $\preceq_1^*$. In the following lemma we state separativity of the order.

Lemma 1.1. *If $X, Y \in (\omega)_1^c$ and $X \preceq_1^* Y$ does not hold then there is $Z \in (\omega)_1^c$ such that $Z \preceq X$ and $Z \top_1 Y$.*

Proof: Since $X \preceq_1^* Y$ does not hold, $Pairs(X)$ is not almost contained in $Pairs(Y)$. Thus there is an infinite sequence $\{x_i\}_{i\in\omega}$ of pairs which meet distinct pieces of Y and the same pieces of X. Define $Z = \{x_i : i \in \omega\} \cup \{\{i\} : i \in r\}$, where $r = \omega \setminus \bigcup\{x_i : i \in \omega\}$. Then Z is contained in X and c-orthogonal to Y. $\square$

2. The order $\preceq_1^*$

Applying the general scheme from [3] and [13] to $(\omega)_1^c$ and $\preceq_1^*$ we obtain the definitions of the cardinals $\mathbf{a}_1^c$, $\mathbf{p}_1^c$, $\mathbf{t}_1^c$, $\mathbf{s}_1^c$, $\mathbf{r}_1^c$, $\mathbf{h}_1^c$. To be more precise we describe them in the following definition.

Definition 2.1. 1. *We say that $\mathcal{A} \subseteq (\omega)_1^c$ is a* maximal c-orthogonal *family of partitions* (maco) *if $\mathcal{A}$ is a maximal family of pairwise c-orthogonal partitions.*

$$\mathbf{a}_1^c = \min\{|\mathcal{A}| : \mathcal{A} \textit{ is a } \text{maco} \textit{ in } (\omega)^c\};$$

2. *We say that $\mathcal{P} \subseteq (\omega)_1^c$ is a* c-centered family *of partitions if for each finite $\mathcal{P}_0 \subseteq \mathcal{P}$ there is some $Y \in (\omega)_1^c$ such that for each $X \in \mathcal{P}_0$ we have $Y \preceq X$. We say that $\mathcal{P} \subseteq (\omega)^\omega$ has no c-bound if there is no partition $Y \in (\omega)^c$ such that $\mathcal{P} \cup \{Y\}$ is c-centered and $Y \preceq_1^* X$ for all $X \in \mathcal{P}$.*

$$\mathbf{p}_1^c = \min\{|\mathcal{P}| : \mathcal{P} \textit{ is a c-centered family of partitions with no c-bound}\};$$

3. *We say that a c-centered family $\mathcal{T} \subseteq (\omega)_1^c$ is a* c-tower *if it is well-ordered by $\leq_1^*$ and has no c-bound.*

$$\mathbf{t}_1^c = \min\{|\mathcal{T}| : \mathcal{T} \textit{ is a c-tower}\};$$

4. *For $X, Y \in (\omega)_1^c$, we say that X* c-splits *Y if there are $Z, T \in (\omega)_1^c$, $Z, T \preceq Y$, such that $Z \preceq X$ and $T \top_1 X$. We say that $\mathcal{S} \subseteq (\omega)_1^c$*-c-splits

$\mathcal{T} \subseteq (\omega)^c_1$ if for each $A \in \mathcal{T}$ there is some $S \in \mathcal{S}$ that c-splits A. The family $\mathcal{S} \subseteq (\omega)^c_1$ is a c-splitting *family if $\mathcal{S}$ c-splits $(\omega)^c_1$.*

$$\mathbf{s}^c_1 = \min\{|\mathcal{S}| : \mathcal{S} \text{ is a c-splitting family}\};$$

5. *We say that $\mathcal{R} \subseteq (\omega)^c_1$ is a* c-reaping *family if for each $A \in (\omega)^c_1$ there is some $R \in \mathcal{R}$ such that either $R \preceq^*_1 A$ or $R \top_1 A$.*

$$\mathbf{r}^c_1 = \min\{|\mathcal{R}| : \mathcal{R} \text{ is a c-reaping family}\};$$

6. *A family $\mathbf{F}$ of maco families of partitions* c-shatters *a partition $A \in (\omega)^c_1$ if there are $\mathcal{F} \in \mathbf{F}$ and two distinct partitions $X, Y \in \mathcal{F}$ such that A is c-compatible with both X and Y. A family $\mathbf{F}$ of* maco *families of partitions is* c-shattering *if $\mathbf{F}$ c-shatters each $A \in (\omega)^c$.*

$$\mathbf{h}^c_1 = \min\{|\mathbf{F}| : \mathbf{F} \text{ is a c-shattering family of } \mathrm{macos}\}.$$

To distinguish between the classical van Douwen's diagram (compare e.g. [13]) and the dual ones we use small bold case for the classical cardinals and small bold case with indexes for the dual diagrams.

2.1 Inequalities.

Several lemmas below are counterparts to those from [3]. We start with the following fact.

Lemma 2.2. *If $X_0 \leq X_1 \leq X_2 \leq \ldots$ is a decreasing (in the sense of $\preceq$) sequence of partitions from $(\omega)^c_1$ then there exists $Y \in (\omega)^c_1$ such that $Y \preceq^*_1 X_i$ for each $i \in \omega$.*

Proof: By induction. Let y_0 be any two-element subset of ω contained in one piece of X_0. Suppose that we have already constructed the first n pieces of Y. Let y_n be an arbitrary two-element subset of ω contained in one piece of X_n which is disjoint from each y_k, for $k < n$. Put

$$Y = \{y_n : n \in \omega\} \cup \{\{n\} : n \in z\},$$

where $z = \omega \setminus \bigcup\{y_n : n \in \omega\}$. □

Remark. This lemma states that $\mathbf{t}^c_1$ is uncountable. On the other hand it does not imply that the numbers $\mathbf{p}^c_1$ and $\mathbf{t}^c_1$ are defined. Indeed, to have these numbers defined we need a c-centered family without c-bound (a c-tower respectively). Nevertheless the idea of the proof yields such a family. Take a partition X consisting of two-element pieces and any c-centered family Γ consisting of partitions $\leq$-greater than X. If Y is a c-bound then $Y \vee X$ is a non-trivial partition consisting of at most two-element pieces. Let $Y' \preceq Y \vee X$ be a partition such that its two-element pieces form an infinite and coinfinite subset of the corresponding set for $Y \vee X$. Then it is easy to verify that $\Gamma \cup \{Y'\}$ is a c-centered family. If it has a c-bound we apply this construction again. It is clear that eventually we get a c-centered family without c-bound. A similar argument works for $\mathbf{t}_1^c$. $\square$

We now consider relations between $\mathbf{p}_1^c$, $\mathbf{t}_1^c$, $\mathbf{h}_1^c$, $\mathbf{s}_1^c$ and the classical coefficients. We need the following notation:

> For $N \in [[\omega]^2]^\omega$ consisting of pairwise disjoint elements we define the partition $X_N = N \cup \{\{n\} : n \in \omega \setminus \bigcup N\}$.

Lemma 2.3. *The following inequalities hold:* $\mathbf{p}_1^c \leq \mathbf{p}$, $\mathbf{t}_1^c \leq \mathbf{t} \leq \mathbf{h}_1^c \leq \mathbf{h}$, $\mathbf{s} \leq \mathbf{s}_1^c$.

Proof: $\mathbf{p}_1^c \leq \mathbf{p}$. Let $\{A_\alpha : \alpha < \mathbf{p}\} \subseteq [\omega]^\omega$ be a $\subseteq$-centered family without lower bound. For each $\alpha < \mathbf{p}$, let $X_\alpha = \{A_\alpha\} \cup \{\{n\} : n \in \omega \setminus A_\alpha\}$. Then the family $\{X_\alpha : \alpha < \mathbf{p}\} \subset (\omega)_1^c$ is a c-centered family without a c-bound.

$\mathbf{t}_1^c \leq \mathbf{t}$ is proved in a similar way.

$\mathbf{t} \leq \mathbf{h}_1^c$. A standard argument shows that there is $\kappa \leq \mathbf{h}_1^c$ and a c-consistent family of partitions $\{X_\alpha : \alpha < \kappa\}$ consisting of at most two-element pieces such that the family is well-ordered by $\leq_1^*$ and does not have a c-bound. Then the family $\{Pairs(X_\alpha) : \alpha < \kappa\} \subset [[\omega]^2]^\omega$ is a tower without a lower bound.

$\mathbf{h}_1^c \leq \mathbf{h}$. For $X \in (\omega)_1^c$ let $M_X = \omega \setminus \{min(x) : x \in X\}$. If $X \preceq_1^* Y$ then obviously $M_X \subseteq^* M_Y$. Let $\Im$ be a m.a.d. family. Then the set $\Re = \{X \in (\omega)_1^c : (\exists\, A \in \Im)(M_X \subseteq^* A)\}$ is dense in $((\omega)_1^c, \preceq_1^*)$. So, to every m.a.d. we can assign a maximal almost c-orthogonal family of partitions $\hat{\Im} \subseteq \Re$. One can easily show that if $\{\Im_\alpha : \alpha < \mathbf{h}\}$ is a shattering family then the family $\{\hat{\Im}_\alpha : \alpha < \mathbf{h}\}$ is a c-shattering family of partitions.

$\mathbf{s} \leq \mathbf{s}_1^c$. It is easy to check that a family $S \subset (\omega)_1^c$ is c-splitting iff for every $X \in (\omega)_1^c$ the family $S_X = \{X \vee A : A \in S\}$ is a c-splitting family for $(X)_\geq$. Let $\{X_\alpha : \alpha < \kappa\} \subseteq (\omega)_1^c, \kappa < \mathbf{s}$, be an arbitrary family of partitions and $X \in (\omega)_1^c$ be any partition such that $(\forall\, x \in X)(|x| \leq 2)$. Consider the family $\{A_\alpha : \alpha < \kappa\}$, where $A_\alpha = X \vee X_\alpha$. We will show that it is not a c-splitting family for $(X)_\geq$. Since $\kappa < \mathbf{s}$, the family $\{Pairs(A_\alpha) : \alpha < \kappa\} \subseteq [Pairs(X)]^\omega$ is not splitting for $[Pairs(X)]^\omega$. So, there is $N \in [Pairs(X)]^\omega$ such that

$$(\forall\, \alpha < \kappa)(N \subseteq^* Pairs(A_\alpha) \vee |N \cap Pairs(A_\alpha)| < \omega).$$

Therefore the partition $X_N \in (X)_\geq$ has the corresponding property, i.e.,

$$(\forall\, \alpha < \kappa)(X_N \preceq_1^* A_\alpha \vee A_\alpha \top_1 X_N).$$

$\square$

In the following lemma we collect preliminary information concerning $\mathbf{r}_1^c$.

Lemma 2.4. (a) *Suppose that $\kappa < cov(\mathcal{K})$ and that $\{X_\xi\}_{\xi<\kappa}$ is a family of partitions from $(\omega)^c_1$. Then there exists $Y \in (\omega)^c_1$ such that Y is c-compatible with each X_ξ and for no $\xi < \kappa$ we have $X_\xi \preceq^*_1 Y$. In particular, $\mathbf{r}^c_1 \geq cov(\mathcal{K})$.*

(b) *Let $X \in (\omega)^c_1$ and $\Re \subset (X)_\geq$ be a c-reaping family for $(X)_\geq$. Then $\Re$ is a c-reaping family of partitions.*

(c) $\mathbf{r}^c_1 \leq \mathbf{r}$.

Proof: (a) For every $\xi < \kappa$ let

$$G_\xi = \{f \in \omega^\omega : (\forall\, m \in \omega)(\exists\, a, b \in X_\xi)(\exists\, k, l \in a)(\exists\, r, s \in b)$$
$$(k \neq l \wedge m < f(k) = f(l) \wedge m < f(r) \neq f(s) > m)\}.$$

It is easy to see that G_ξ is a dense $\mathbf{G}_\delta$ subset of the Baire space ω^ω for every $\xi < \kappa$. Since $\kappa < cov(\mathcal{K})$, there is an unbounded f such that $f \in G_\xi$, for every $\xi < \kappa$. Put $Y = \{f^{-1}(n) : n \in \omega\} \setminus \{\emptyset\}$.

(b) Let $Y \in (\omega)^c_1$. If $X \top_1 Y$ then $Y \top_1 Z$ for any $Z \in \Re$. If $X \vee Y \in (\omega)^c_1$ then either there is $Z \in \Re$ such that $Z \top_1 (X \vee Y)$ or there is $Z \in \Re$ such that $Z \preceq^*_1 (X \vee Y) \preceq Y$. Since $Z \preceq^*_1 X$, the condition $Z \top_1 (X \vee Y)$ implies that $Z \top_1 Y$. So, $\Re$ is a c-reaping family.

(c) To prove $\mathbf{r}^c_1 \leq \mathbf{r}$ let $X \in (\omega)^c_1$ be any partition of at most two-element pieces. Let $\hat{\Re} \subseteq [Pairs(X)]^\omega$ be a reaping family of sets such that $|\hat{\Re}| = \mathbf{r}$. Then the family $\Re = \{X_N : N \in \hat{\Re}\}$ is a c-reaping family for $(X)_\geq$. So, by the previous statement, this is a c-reaping family of partitions. □

We now study $\mathbf{a}^c_1$.

Lemma 2.5. *The cardinal number* $\mathbf{a}$ *is not greater than* $\mathbf{a}^c_1$.

Proof: Let $\{X_\alpha : \alpha < \kappa\} \subseteq (\omega)^c_1$, $\kappa < \mathbf{a}$, be a family of c-orthogonal partitions. Then the corresponding family $\{Pairs(X_\alpha) : \alpha < \kappa\}$ is an almost disjoint family of subsets of $[\omega]^2$. Let

$$\Im = \{M \in [\omega]^\omega : (\exists\, \alpha < \kappa)(M \in X_\alpha)\}.$$

Clearly, $\Im$ consists of pairwise almost disjoint subsets of ω. One of the following cases holds: (1) $\Im = \emptyset$; (2) $\Im$ is maximal; (3) $\Im$ is not maximal.

Case 1. Since for each $\alpha < \kappa$, X_α consists of finite pieces, for every $\alpha < \kappa$ and $i \in \omega$, $Pairs(X_\alpha)$ is almost disjoint from $N_i = \{\{i, j\} : j \in \omega\}$. So, the family $\{Pairs(X_\alpha) : \alpha < \kappa\} \cup \{N_i : i \in \omega\}$ is an almost disjoint family of subsets of $[\omega]^2$ of cardinality $\kappa + \omega < \mathbf{a}$. Thus there is $N \in [[\omega]^2]^\omega$ almost disjoint from each element of the family. It is easy to see that N contains an infinite N' consisting of pairwise disjoint elements. Then for each $\alpha < \kappa$, $X_{N'}$ is c-orthogonal to X_α.

Case 2. If $\Im$ is maximal and $|\Im| \leq \kappa < \mathbf{a}$, it must be finite. Let $\Im = \{M_1, \ldots, M_n\}$. W.l.o.g. we may assume that $\Im$ consists of pairwise disjoint elements and its union is ω. For $i \in M_k$, put $N_i = \{\{i,j\} : j \in \omega \setminus M_k\}$. One can see that for each $\alpha < \kappa$ and $i \in \omega$, $Pairs(X_\alpha)$ is almost disjoint from N_i. Thus the family $\{Pairs(X_\alpha) : \alpha < \kappa\} \cup \{N_i : i \in \omega\}$ is almost disjoint of cardinality less than $\mathbf{a}$. There is N almost disjoint from each member of the family. As in Case 1, N contains an infinite N' consisting of pairwise disjoint elements. Thus for each $\alpha < \kappa$, $X_{N'}$ is c-orthogonal to X_α.

Case 3. Since $\Im$ is not maximal there is $Q \in [\omega]^\omega$ almost disjoint from each $M \in \Im$. Now, for each $\alpha < \kappa$, the partition $Z_\alpha = \{x \cap Q : x \in X_\alpha\} \setminus \{\emptyset\}$ is a partition of Q consisting of finite pieces. So, by Case 1, we can construct the partition Z of Q that is almost c-orthogonal to Z_α for all $\alpha < \kappa$. Finally, let $Y = Z \cup \{\{n\} : n \in \omega \setminus Q\}$. It is clear, that for all $\alpha < \kappa$, Y is c-orthogonal to X_α. □

Remark. Notice that the proof of the lemma additionally implies that if $\{X_i : i \in n\} \subseteq (\omega)_1^c$ is a finite family of pairwise c-orthogonal partitions then there exists $Y \in (\omega)_1^c$ such that $Y \top_1 X_i$, for every $i \in n$. This statement does not hold in the case of sets.

Also note that if Y is chosen as above then for any $k \in \omega$ there exist $m, l \geq k$ such that $m \neq l$, $\{m,l\} \in Pairs(Y)$ and $\{m,l\} \notin \bigcup\{Pairs(X_i) : i \in n\}$. This follows from the fact that $|Pairs(Y) \setminus \bigcup_{i \in n} Pairs(X_i)| = \omega$. This will be applied in forcing arguments of Section 2.2. □

Now, we are able to draw the diagram for the converse case.

Proposition 2.6. *The following relations are provable in ZFC.*

$$\begin{tikzcd}
& & & & \mathbf{a}_1^c \arrow[dr] & \\
\omega_1 \arrow[r] & \mathbf{p}_1^c \arrow[r] & \mathbf{t}_1^c \arrow[r] & \mathbf{h}_1^c \arrow[r] & \mathbf{s}_1^c \arrow[r] & 2^\omega . \\
& & & & \mathbf{r}_1^c \arrow[ur] &
\end{tikzcd}$$

Moreover $\mathbf{r}_1^c$ *and* $\mathbf{a}_1^c$ *are uncountable.*

PROOF: The inequality $\mathbf{p}_1^c \geq \omega_1$ is a consequence of Lemma 2.2 (note that we must use c-centeredness at this point). The uncountability of $\mathbf{r}_1^c$ and $\mathbf{a}_1^c$ follows from Lemmas 2.4 and 2.5.

The remaining inequalities follow from Lemma 2.3 and the classical diagram ([13]). □

2.2 Consistency.

Unlike in the case of the order $\leq_1^*$, all the converse coefficients are equal continuum under **MA**. To show this we shall define the following notions of forcing.

Definition 2.7. (1) *Let $A \subseteq (\omega)_1^c$ be a c-orthogonal family. Let $\mathbf{P}_A = (P_A, \leq)$ be the following notion of forcing:*

P_A consists of pairs (σ, F) where σ is a finite family of pairwise disjoint 2-element subsets of ω and F is a finite subfamily of A;

$\leq$: $(\sigma, F) \leq (\tau, T)$ iff $\sigma \supseteq \tau, F \supseteq T$ and for each $a \in \sigma \setminus \tau$ and $X \in T$, a is not contained in one piece of X.

(2) Let $S = \{X_\alpha : \alpha < \kappa\}$, where $\kappa < 2^\omega$, be a c-centered family of partitions. Let $\mathbf{P}_S = (S, \leq)$ be a notion of forcing, where S is a set of pairs (σ, F) such that σ is a finite family of pairwise disjoint finite sets and F is a finite subfamily of S. The ordering $\leq$ is defined as follows: $(\sigma, F) \leq (\tau, T)$ iff $\sigma \supseteq \tau$, $F \supseteq T$ and for each $a \in \sigma \setminus \tau$ and each $X \in T$ there is $x \in X$ such that $a \subseteq x$.

(3) Let $\mathbf{P}_R = (R, \leq)$ be a notion of forcing such that R is a set of finite families of at most two-element pairwise disjoint subsets of ω and $\sigma \leq \tau$ iff $\tau \subseteq \sigma$.

It is easy to see that the notions of forcing defined above satisfy *ccc*. The lemma below follows directly from Lemmas 2.4 and 2.5.

Lemma 2.8. (1) *Let $\mathbf{M} \models$ **ZFC** and $A \in \mathbf{M}$ be a c-orthogonal family. Let G be a $\mathbf{P}_A$-generic over $\mathbf{M}$, and $Y = \bigcup\{\sigma : (\sigma, F) \in G$ for some $F \subset A\} \cup \{\{n\} : n \in R\}$, where $R = \omega \setminus \bigcup\{\sigma : (\sigma, F) \in G$ for some $F \subset A\}$. Then*

$$\mathbf{M}[G] \models (\forall X \in A)(X \top_1 Y).$$

*(2) Let $\mathbf{M} \models$ **ZFC** and N be a model obtained from M via $\mathbf{P}_R$-generic G and let $Y = \bigcup\{\sigma : \sigma \in G\} \cup \{\{n\} : n \in \omega \setminus \bigcup G\}$. Then the following is true in N:*

For every $X \in \mathbf{M} \cap (\omega)_1^c$ there are $Y_1, Y_2 \preceq Y$ such that $Y_1 \preceq X$ but $Y_2 \top_1 X$.

Theorem 2.9. *Assume* **MA**. *Then $\mathbf{a}_1^c = \mathbf{p}_1^c = \mathbf{r}_1^c = 2^\omega$.*

Proof: $\mathbf{a}_1^c = \mathbf{r}_1^c = 2^\omega$ easily follows from the previous lemma.

In a similar way, using $\mathbf{P}_S$, one can show that under **MA** every c-centered family of partitions without a c-bound is of cardinality continuum. □

On the other side it is consistent that all the converse coefficients equal ω_1.

Theorem 2.10. *It is consistent with* **ZFC** $+ \neg$*CH that $\mathbf{a}_1^c = \mathbf{s}_1^c = \mathbf{r}_1^c = \omega_1$.*

Proof: Let $M \models$ **ZFC** + **CH**. Consider a sequence

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \subseteq M_{\omega_1}$$

of ccc-extensions of M such that

1. $M_0 = \mathbf{M}$;
2. $M_{\alpha+1} \models \mathbf{MA} + 2^\omega = \omega_{\alpha+1}$;
3. if $\lambda \leq \omega_1$ is a limit cardinal then M_λ is obtained via a direct limit (with finite support) of smaller generic extensions.

Then in the final model M_{ω_1} the required equalities hold.

We will give details only for $M_{\omega_1} \models \mathbf{a}_1^c = \omega_1$. Start with an arbitrary countable c-orthogonal family $A_0 \in M$. Assume that we have already constructed a sequence $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\xi \subseteq \ldots, \xi < \alpha$, such that for each $\xi < \alpha$, $A_\xi \in M_\xi$ is an c-orthogonal family. Then put

(1) $A_\alpha := \bigcup\{A_\xi : \xi < \alpha\}$, for limit α or

(2) $A_\alpha := A_\lambda \cup \{Y_{\lambda+1}\}$, where $\alpha = \lambda+1$ and $Y_{\lambda+1}$ is defined as in Lemma 2.8 (1) for $\mathbf{P}_{A_\lambda} \in M_\lambda$ and generic $G \in M_{\lambda+1}$.

Then $A = \bigcup\{A_\xi : \xi < \omega_1\}$ is a c-orthogonal family of cardinality ω_1. We claim that it is maximal. Let $X \in (\omega)_1^c \cap M_{\omega_1}$ and $\zeta_0 = min\{\zeta < \omega_1 : X \in M_\zeta\}$. Suppose that X is c-orthogonal to each $Z \in A_{\zeta_0}$. Then for each $n \in \omega$ the set $D_n = \{(\sigma, F) \in P_{A_{\zeta_0}} : |\sigma \cap Pairs(X)| \geq n\}$ is dense in $P_{A_{\zeta_0}}$. Thus X is c-compatible with Y_{ζ_0+1}. So A is maximal.

In a similar way we show $M_{\omega_1} \models \mathbf{s}_1^c = \omega_1$.

Applying Cohen notion of forcing one can easily show that $M_{\omega_1} \models \mathbf{r} = \omega_1$. Now the statement $M_{\omega_1} \models \mathbf{r}_1^c = \omega_1$ is a consequence of this fact and Lemma 2.4. □

3. Orders

Let

$Y \leq_2^* X$ iff $(\exists\, d \in [\omega]^{<\omega})(Y \leq X_d)$.

It is easily seen that IF is the set of all partitions $=_2^*$-equivalent to $\mathbf{1}$. Observe also that the class $[\mathbf{0}]_2$ consists of all partitions having a cofinite piece.

As in the case of $\leq_1^*$ we introduce $(\omega)_2^c$ by

$$(\omega)_2^c = \{X \in (\omega) : \mathbf{0} <_2^* X <_2^* \mathbf{1}\}.$$

Notice that the ordering $\leq_1^*$ is a proper extension of $\leq_2^*$. Indeed, let $X \leq_2^* Y$, for some $X, Y \in (\omega)$. Then there is $d \in [\omega]^{<\omega}$ such that $X \leq Y_d$. Put $\hat{d} = \{min(x) : x \in X \wedge ((x \cap d \neq \emptyset) \vee (\exists\, y \in Y)(x \cap y \neq \emptyset \neq d \cap y))$. It is easy to see that $\hat{d}$ is finite and $X^{\hat{d}} \leq Y$.

It is also worth noting that $\leq_1^*$ and $\leq_2^*$ are equal on the set of partitions having only finite pieces. In particular $=_1^*$- and $=_2^*$-classes of $\mathbf{1}$ are the same.

Remark. The equivalence relations $=_1^*$ and $=_2^*$ are not congruences of the lattice of partitions. Indeed, for the first case let $\{A, B, C\}$ be any partition of ω into three infinite pieces and let e_A, e_B be some enumerations of A and B respectively. Put

$$X = \{A, B\} \cup \{\{c\} : c \in C\},$$
$$Y = \{A \cup B\} \cup \{\{c\} : c \in C\},$$
$$Z = \{\{e_A(i), e_B(i)\} : i \in \omega\} \cup \{\{c\} : c \in C\}.$$

Obviously, $X =^*_1 Y$ but $Y \vee Z = Z \neq^*_1 \mathbf{1} = X \vee Z$.

To show that $=^*_2$ is not a congruence take a partition X of ω into two infinite sets A, B. Let a partition Y consist of $A \setminus \{a\}$ and $\{a\} \cup B$ where $a \in A$. Then $X =^*_2 Y$ but $X \wedge Y \neq^*_2 X \wedge X$. □

Let $\preceq^*_2$ denote the order converse to $\leq^*_2$. We are going to show that all posets introduced above ($\leq^*_1, \preceq^*_1, \leq^*_2, \preceq^*_2$) are not pairwise isomorphic. First note that $(\omega)/ =^*_2$ under $\leq^*_2$ has atoms. Indeed, they are induced by partitions with exactly two infinite classes. On the other hand, it is easily seen that all remained posets do not have atoms and the $^*_1$-posets do not have coatoms. This shows that we must only prove that the $^*_1$-posets are non-isomorphic. This will follow from the following lemmas.

Lemma 3.1. *Let $X \in (\omega)^\omega$. Then the posets $((X)_\leq, \leq)$ and $((\omega), \leq)$ are isomorphic. The corresponding isomorphism f can be chosen so that for any $Y, Z \in (\omega)$ we have $Y =^*_1 Z$ iff $f(Y) =^*_1 f(Z)$. In particular, the posets $(((\omega)/ =^*_1), \leq^*_1)$ and $(((X)_\leq / =^*_1), \leq^*_1)$ are isomorphic.*

Proof: Let $X = \{x_i : i \in \omega\}$. For $Y \in (\omega)$ and $y \in Y$ put $A_y = \bigcup\{x_i : i \in y\}$. Then the function $f(Y) = \{A_y : y \in Y\}$ is a required isomorphism.

It suffices to show that for any $Y \in (\omega)$ and $d \in [\omega]^{<\omega}$ we have $f(Y) =^*_1 f(Y^d)$ and $f^{-1}(f(Y)^d) =^*_1 Y$. Let $b = \{minA_y : (y \in Y) \wedge (y \cap d \neq \emptyset)\}$ and $c = \{i : x_i \cap d \neq \emptyset\}$. It is easy to see that b, c are finite subsets of ω and $f(Y)^b = f(Y^d)$ and $f^{-1}(f(Y)^d) = Y^c$. □

For $X \in (\omega)$ let $([X]) = \{[Y] \in ((\omega)/ =^*_1) : [Y] \leq^*_1 [X]\}$. It is clear, that $(((X)_\leq / =^*_1), \leq^*_1)$ and $(([X]), \leq^*_1)$ are isomorphic. We have arrived at the following

Corollary 3.2. *Let $X \in (\omega)^\omega$. Then $(([X]), \leq^*_1)$ and $(((\omega)/ =^*_1), \leq^*_1)$ are isomorphic.*

Lemma 3.3. *Let $X \in (\omega)^c_1$ have at most two-element pieces. Then*

$(((X)_\geq / =^*_1), \preceq^*_1)$ is isomorphic to $(P(\omega)/Fin, \subseteq^*)$.

In particular, $(P(\omega)/Fin, \subseteq^*)$ is isomorphic to $(([X])_\geq, \preceq^*_1)$, where

$([X])_\geq = \{[Y] \in ((\omega)/ =^*_1) : [Y] \preceq^*_1 [X]\}$.

Proof: Let $X' = \{x_i : i \in \omega\}$ be an enumeration of all two-element pieces of X and $X = X' \cup Y$ where $Y = \{x \in X : card(x) = 1\}$. For $A \in [\omega]^\omega$ put

$$X_A = \{x_i : i \in A\} \cup \{\{n\} : n \in x_i, i \in \omega \setminus A\} \cup Y.$$

Then the function $f([A]) = [X_A]$ is a required isomorphism. $\square$

Theorem 3.4. *The posets* $(((\omega)/=_1^*), \leq_1^*)$ *and* $(((\omega)/=_1^*), \preceq_1^*)$ *are not isomorphic.*

Proof: Assume that f is an isomorphism from the first poset onto the second one. Let X be an infinite non-trivial partition whose pieces are at most two-element. Then $[X] = f([Y])$ for some infinite Y and the posets $(([X])_{\geq}, \preceq_1^*)$ and $(([Y]), \leq_1^*)$ are isomorphic. This yields by Lemmas 3.1 and 3.3 the existence of an isomorphism between $(P(\omega)/Fin, \subseteq^*)$ and $(((\omega)/=_1^*), \leq_1^*)$, which is impossible because the former is a Boolean algebra. $\square$

The above arguments show that all these posets are pairwise elementary non-equivalent.

4. Diagrams

The general idea of van Douwen's diagram can be described as follows.

Let L be a lattice with 0 and 1, and let $\leq_a$ be a relation of almost containedness. As we study a very particular case, we do not axiomatize almost containedness. In the case of the lattice (or the converse lattice) of partitions $\leq_a$ is one of the relations: $\leq_1^*$, $\preceq_1^*$, $\leq_2^*$, $\preceq_2^*$. We write $a =_a b$ if $a \leq_a b$ and $b \leq_a a$. Let $I = \{a \in L : a \leq_a 0\}$. For any $a \in L$ we put $a_I = \{b : b =_a a\}$. We say that $a, b \in L \setminus I$ are *orthogonal* if $a \wedge b \leq_a \mathbf{0}$.

In general, to characterize the lattice L under these relations we need some further notions. We say that a *splits* b if there are $c, d \leq b$ not in I such that $c \leq a$ and d, a are orthogonal. A family $\Gamma \subset L \setminus I$ is a *splitting family* if for every $b \in L \setminus I$ there exists $a \in \Gamma$ that splits b. We say that Γ is a *reaping family* if for each $a \in L \setminus I$ there is some $b \in \Gamma$ such that $b \leq_a a$ or a, b are orthogonal. We also define a family $\Gamma \subset L \setminus I$ to be $\leq$-*centered* if any finite intersection of its elements is not in I.

We can now associate with L the following cardinals. Define $\mathbf{a}_I$ to be the least cardinality of a maximal family of pairwise orthogonal elements from $L \setminus 1_I$. Let $\mathbf{p}_I$ be the least cardinality of a $\leq$-centered family without lower bound under $\leq_a$ (from $L \setminus I$) such that the family extended by it (if it is not in the family) is still $\leq$-centered. Similarly, define $\mathbf{t}_I$ as the least cardinality of a $\leq_a$-decreasing $\leq$-centered chain without lower $\leq_a$-bound consistent (in the sense of $\leq$-centeredness) with the family. The cardinals $\mathbf{s}_I, \mathbf{r}_I$ are the corresponding cardinals for splitting families and reaping families respectively. It is worth noting that $\mathbf{p}_I$ and $\mathbf{t}_I$ can be undefined (for example if for any $a \in L$ the set $\{b : b \leq a\}$ is finite). Also, $(L, \leq_a)$ does not necessarily have a splitting family (for example if L is an atomic boolean algebra and I is trivial). So, $\mathbf{s}_I$ can be undefined too. On the other hand, it is clear that $\mathbf{p}_I \le \mathbf{t}_I$ if they are defined.

The last cardinal $\mathbf{h}_I$ is defined as follows. A family Σ of maximal families of pairwise orthogonal elements in $L \setminus 1_I$ is *shattering* if for every $a \in L \setminus I$ there are $\Gamma \in \Sigma$ and distinct $b, c \in \Gamma$ which are not orthogonal to a. Let $\mathbf{h}_I$ be the least cardinality of a shattering family in L.

Lemma 4.1. *If $\mathbf{s}_I$ is defined then $\mathbf{h}_I \le \mathbf{s}_I$.*

Proof: Take a splitting family $\Gamma = \{c_\nu : \nu < \mathbf{s}_I\}$. For each $\nu < \mathbf{s}_I$ choose Ψ_ν a maximal family of pairwise orthogonal elements such that $c_\nu \in \Psi_\nu$. Let us check that the set of these families is shattering. Let $c \in L \setminus I$. Since Γ is a splitting family there is ν and $a, b \le c$ such that $a \le c_\nu$ and b is orthogonal to c_ν. By our construction there is $d \in \Psi_\nu$ not orthogonal to b. So, Ψ_ν shatters c by c_ν and d. $\square$

By $\mathbf{a}_i, \mathbf{p}_i, \dots$ we denote the cardinals defined in the case of $(L, \le_i)$, $i = 1, 2$. By $\mathbf{a}_i^c, \mathbf{p}_i^c, \dots$ we denote the cardinals defined in the converse case: $(L, \preceq_i)$, $i = 1, 2$.

Remark. In the case of the lattice $(P(\omega), \cup, \cap)$ and the natural relation of almost containedness the set I is the ideal $[\omega]^{<\omega}$ of all finite subsets of ω. The cardinals introduced are exactly the cardinals of van Douwen's diagram. Indeed, our definitions of $\mathbf{a}_I$, $\mathbf{r}_I$, $\mathbf{s}_I$, $\mathbf{h}_I$ are formulated as the classical ones in [4]. The classical $\mathbf{t}$ is the least cardinality of a $\le_a$-decreasing chain in $P(\omega)$ without $\le_a$-bound. The classical $\mathbf{p}$ is defined as follows. We say that a family $\Gamma \subseteq [\omega]^\omega$ is $\le_a$-centered if every its finite subset Γ' has an infinite pseudointersection: a set $X \in [\omega]^\omega$ almost contained in each element of Γ'. Then the classical $\mathbf{p}$ is the least cardinality of a $\le_a$-centered family from $P(\omega)$ without lower $\le_a$-bound. So, there is no assumption on $\subseteq$-centeredness as in the definitions of $\mathbf{p}_I$ and $\mathbf{t}_I$. On the other hand we do not need such assumptions now because any $\le_a$-centered family from $P(\omega)$ is centered. So, $\mathbf{p} = \mathbf{p}_I$ and $\mathbf{t} = \mathbf{t}_I$. $\square$

4.1 The order $\le_2^*$.

It has been already mentioned that the classes of partitions into two infinite pieces are atoms in $(((\omega)/=_2^*), \le_2^*)$. Therefore the following holds.

Proposition 4.2. $\mathbf{p}_2 = \mathbf{t}_2 = \omega$, $\mathbf{r}_2 = 1$, $\mathbf{a}_2 = 2^\omega$, *$\mathbf{h}_2$ and $\mathbf{s}_2$ are undefined.*

Proof: The cardinals $\mathbf{h}_2$ and $\mathbf{s}_2$ are undefined since an atom can be neither shattered nor splitted.

$\mathbf{p}_2 = \mathbf{t}_2 = \omega$. Let $X = \{N_i : i \in \omega\}$ be any partition into infinitely many infinite pieces. For $k \in \omega$, put $X_k = \{\bigcup\{N_i : i \le k\}\} \cup \{N_i : i > k\}$. Obviously, the family $\{X_k : k \in \omega\}$ is centered and decreasing under $\le_2^*$. It is also easy to see that there is no atom bounding the family.

$\mathbf{r}_2 = 1$. The family consisting of any atom in $(((\omega)), \le_2^*)$ is a $\le_2^*$-reaping family. Indeed, let X be an arbitrary atom and $Y \in (\omega)_2^c$. Suppose that X and Y are not orthogonal. Then $X \wedge Y \ne_2^* \mathbf{0}$. Hence, $X \le_2^* Y$.

$\mathbf{a}_2 = 2^\omega$. The idea of the following proof comes from the Krawczyk's proof of $\mathbf{a}_1 = 2^\omega$ (see [3]). Let $\{X_\alpha : \alpha < \kappa\}$, $\kappa > 1$, be a maximal family of pairwise orthogonal partitions ($\alpha \neq \beta \to X_\alpha \wedge X_\beta \leq_2^* \mathbf{0}$). Observe that the maximality of the family $\{X_\alpha : \alpha < \kappa\}$ is equivalent to the property that for every atom A (a partition into two infinite pieces) there is $\alpha < \kappa$ such that $A \leq X_\alpha$. It is worth noting that there are 2^ω different atoms and they are obviously pairwise orthogonal.

If each X_α is finite, where $\alpha < \kappa$, then $\kappa = 2^\omega$ because there are only finitely many atoms below any finite partition.

So we shall only consider the case when there is $\alpha_0 < \kappa$ such that X_{α_0} is infinite and obviously different from $\mathbf{1}$. We shall use the following

Claim. Let X be an arbitrary infinite partition not equal to $\mathbf{1}$ and let $F \subseteq [\omega]^\omega$ be an almost disjoint family of power 2^ω. Then there is a family $\{Y_A : A \in F\}$ of atoms orthogonal to X such that for any distinct $A, B \in F$, $X \wedge (Y_A \vee Y_B) \neq_2^* \mathbf{0}$.

Proof: Let $X = \{x_i : i \in \omega\} \cup \{a\}$ where a is a piece of X having at least two elements and let $a = b \cup c$ be any partition of a into two nonempty sets. For $A \in F$ put $b_A = \bigcup\{x_i : i \in A\} \cup b$, $c_A = \bigcup\{x_i : i \in \omega \setminus A\} \cup c$ and $Y_A = \{b_A, c_A\}$.

Obviously the family $\{Y_A : A \in F\}$ consists of pairwise distinct atoms orthogonal to X and for any distinct $A, B \in F$ we have $X \wedge (Y_A \vee Y_B) = \{d_0, d_1, d_2\}$, a partition into three infinite pieces, where

$$d_0 = \bigcup\{x_i : i \in (A \cap B \cup \omega \setminus A \cup B)\} \cup a;$$
$$d_1 = \bigcup\{x_i : i \in A \setminus B\};$$
$$d_2 = \bigcup\{x_i : i \in B \setminus A\}. \qquad \square$$

Now to finish the proof of the lemma let $\{Y_\zeta : \zeta < 2^\omega\}$ be a family of pairwise distinct atoms satisfying the claim for $X = X_{\alpha_0}$. Since the family $\{X_\alpha : \alpha < \kappa\}$ is maximal, for every $\zeta < 2^\omega$ there is exactly one $\alpha < \kappa$, say α_ζ, such that $Y_\zeta \leq X_{\alpha_\zeta}$. We claim that for distinct $\zeta, \xi < 2^\omega$ we have $\alpha_\zeta \neq \alpha_\xi$. Indeed, if $\alpha_\zeta = \alpha_\xi$, then obviously $\alpha_\zeta \neq \alpha_0$ and $X \wedge (Y_\xi \vee Y_\zeta) \leq X \wedge X_{\alpha_\zeta}$. This contradicts orthogonality of X_{α_0} and X_{α_ζ}. As a result we have $\kappa = 2^\omega$. $\square$

Remark. Theorem 4.1 of [11] states that the cardinality of a maximal non-trivial family of partitions where any two members have meet $\mathbf{0}$ is 2^ω. Notice that such a family Γ is a maximal orthogonal family with respect to $\leq_2^*$. Indeed, Γ consists of pairwise orthogonal partitions and for any partition A into two infinite pieces there is $X \in \Gamma$ such that $X \wedge A \neq \mathbf{0}$ and then $A \leq X$. As above we see that Γ is maximal.

This shows that $\mathbf{a}_2$ is not greater than the cardinality of a maximal non-trivial family of partitions where any two members have meet $\mathbf{0}$. So the corresponding part of the proof of the theorem above provides a new (and easier) proof of Theorem 4.1 of [11].

4.2 The order $\preceq_2^*$.

Recall that $(\omega)_2^c = (\omega) \setminus ([\mathbf{1}]_2 \cup [\mathbf{0}]_2)$. As we mentioned earlier the ordering $\leq_1^*$ is an extension of $\leq_2^*$ and they are equal on the set of partitions having only finite pieces. Therefore $[\mathbf{1}]_1 = [\mathbf{1}]_2$ and the relations of orthogonality $\top_1$ and $\top_2$ are the same, where $X \top_2 Y$ iff $\mathbf{1} \leq_2^* X \vee Y$. This implies that many properties of $\preceq_1^*$ hold for $\preceq_2^*$. For example,

Proposition 4.3. *For non-trivial X, Y if $X \not\preceq_2^* Y$ then there is a non-trivial $X' \preceq X$ such that $X' \top_2 Y$.* □

Also, it is a routine to check that the arguments of Lemmas 2.3–2.5 work for $\preceq_2^*$. So we have

Proposition 4.4. *The following inequalities hold:* $\mathbf{p}_2^c \leq \mathbf{p}$, $\mathbf{t}_2^c \leq \mathbf{t} \leq \mathbf{h}_2^c \leq \mathbf{h}$, $\mathbf{s} \leq \mathbf{s}_2^c$, $\mathbf{r}_2^c \leq \mathbf{r}$, $\mathbf{a} \leq \mathbf{a}_2^c$. □

However there is a difference between $\preceq_1^*$ and $\preceq_2^*$. Note that by the remark from Section 3, $\preceq_1^*$-centeredness does not imply c-centeredness in the sense of Definition 2.1. On the other hand:

Lemma 4.5. *Any $\preceq_2^*$-centered family is c-centered.*

Proof: Let $\{X_1, \ldots, X_n\}$ be a family $\preceq_2^*$-centered by $X \in (\omega)_2^c$. Let $X_i \leq X_{d_i}, 1 \leq i \leq n$, where d_i are finite. Let $d = \bigcup d_i$. Then $X_i \leq X_d, 1 \leq i \leq n$. □

We use this fact in the proposition below. Note that some of the inequalities from Proposition 4.4 also follow from this proposition.

Proposition 4.6. $\mathbf{t}_2^c$ *is a regular cardinal and the following relations hold:* $\mathbf{t}_2^c \leq \mathbf{t}_1^c$, $\mathbf{a}_1^c = \mathbf{a}_2^c$, $\mathbf{h}_1^c = \mathbf{h}_2^c$, $\mathbf{r}_1^c \leq \mathbf{r}_2^c$, *and* $\mathbf{s}_2^c \leq \mathbf{s}_1^c$.

Proof: To prove regularity of $\mathbf{t}_2^c$ suppose that for some κ with $cof(\kappa) < \mathbf{t}_2^c$ the family $\mathcal{F} = \{X_\alpha : \alpha < \kappa\}$ is c-centered and well-ordered by $\leq_2^*$. Let $(\alpha_\xi)_{\xi < cof(\kappa)}$ be any sequence of ordinals cofinal in κ. Then $\mathcal{F}' = \{X_{\alpha_\xi} : \xi < cof(\kappa)\}$ is a c-centered and well-ordered by $\leq_2^*$ family of power less than $\mathbf{t}_2^c$. So, there is $X \in (\omega)_2^c$ such that for each $\xi < cof(\kappa)$, $X \preceq_2^* X_{\alpha_\xi}$ and $\mathcal{F}' \cup \{X\}$ is c-centered. By Lemma 4.5, X is a c-bound of $\{X_\alpha : \alpha < \kappa\}$ too.

$\mathbf{a}_2^c = \mathbf{a}_1^c$ and $\mathbf{h}_2^c = \mathbf{h}_1^c$ are obvious.

$\mathbf{r}_1^c \leq \mathbf{r}_2^c$. Let $\{X_\alpha : \alpha < \kappa\} \subset (\omega)_2^c$ be a c-reaping family for $\preceq_2^*$. Notice that if for each $\alpha < \kappa$ we have $X_\alpha \leq X'_\alpha$ then the family $\{X'_\alpha : \alpha < \kappa\}$ is also c-reaping. So we may assume that $\{X_\alpha : \alpha < \kappa\} \subseteq (\omega)_1^c$. If X is non-trivial then there exists $\alpha < \kappa$ such that $X_\alpha \top_2 X$ or $X_\alpha \preceq_2^* X$. Then for this α we have $X_\alpha \top_1 X$ or $X_\alpha \preceq_1^* X$.

$\mathbf{s}_2^c \leq \mathbf{s}_1^c$. Let $\{X_\alpha : \alpha < \kappa\}$ be a family c-splitting with respect to $\preceq_1^*$. Let X be non-trivial. Then there exists $\alpha < \kappa$ such that none of the relations $X \top_1 X_\alpha, X \preceq_1^* X_\alpha$ holds. Then $\neg(X \top_2 X_\alpha \vee X \preceq_2^* X_\alpha)$.

To prove $\mathbf{t}_2^c \leq \mathbf{t}_1^c$ we need the following lemma.

Lemma 4.7. *Let κ be an uncountable regular cardinal and $\{X_\alpha : \alpha < \kappa\} \subset (\omega)^{<\omega}$ be a family well-ordered by $\leq_2^*$. Then there is a non-trivial X such that $X \preceq_2^* X_\alpha$ for each $\alpha < \kappa$.*

PROOF: By the definition of $\preceq_2^*$ we may assume that for each $\alpha < \kappa$, X_α has only infinite pieces. Let n_α be the corresponding number of pieces. For any $\alpha < \beta < \kappa$ we have:

$$\text{(i)} \qquad y \in X_\beta \to (\exists\, x \in X_\alpha)(y \subseteq^* x),$$

$$\text{(ii)} \qquad n_\alpha \leq n_\beta.$$

Since κ is regular, there is a natural number n such that $|\{\alpha < \kappa : n_\alpha = n\}| = \kappa$. Thus, by (i) and (ii) we may assume that for each $\alpha < \kappa, n_\alpha = n$. This guarantees that for each $\alpha < \beta < \kappa$, $X_\alpha =_2^* X_\beta$. □

To finish the proposition let $\{X_\alpha : \alpha < \kappa\}$ be a c-tower family for $\preceq_2^*$, where κ is regular. Then, there is no non-trivial X such that $X \preceq_2^* X_\alpha$, for each $\alpha < \kappa$. The family is c-centered and well-ordered by $\leq_1^*$ either. We claim that there is no non-trivial X such that $X \preceq_1^* X_\alpha$ for all $\alpha < \kappa$.

Suppose the contrary. Then we may assume that X consists of at most two-element pieces. Put $X' = \{x \in X : |x| = 2\}$, Since X is not a $\preceq_2^*$-bound of the family, there is $\alpha_0 < \kappa$ and an infinite $X'' \subseteq X'$ such that each element of X'' is not contained in a piece of X_{α_0}. Let $N = \bigcup X''$. We may assume that $\alpha_0 = 0$. Then for each $\alpha < \kappa$, each, except finitely many elements of X'', is not contained in any piece of X_α. On the other hand the condition $(\forall\, \alpha < \kappa)(X \preceq_1^* X_\alpha)$ implies that for each $\alpha < \kappa$ the set $\{x \in X_\alpha : x \cap N \neq \emptyset\}$ is finite. Thus, for each $\alpha < \kappa$, the set $\{x \cap N : x \in X_\alpha\} \setminus \{\emptyset\}$ is a finite partition of N. By Lemma 4.7 we obtain a contradiction. □

In general the van Douwen's diagram for $\preceq_2^*$ looks as in the case of $\preceq_1^*$ (the same proofs work). So, it remains to find possible values of these cardinals in models of **ZFC**. As we noted above the orderings $\preceq_1^*$ and $\preceq_2^*$ are the same on the set of partitions having only finite pieces and the sets $\{X \in (\omega) : X \preceq_1^* \mathbf{1}\}$ and $\{X \in (\omega) : X \preceq_2^* \mathbf{1}\}$ are the same. On the other hand each notion of forcing defined in Section 2.2 for $((\omega), \preceq_1^*)$ adds a partition having only finite pieces. Now it is easy to verify that all those notions of forcing work for $\preceq_2^*$. As a result we have

Theorem 4.8. (a) *Under Martin's Axiom,* $\mathbf{a}_2^c = \mathbf{r}_2^c = \mathbf{p}_2^c = 2^\omega$.

(b) $Con(\mathbf{ZFC} + \neg\mathbf{CH} + \mathbf{a}_2^c = \mathbf{r}_2^c = \mathbf{s}_2^c = \omega_1)$.

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