Cartesian closed hull for (quasi-)metric spaces (revisited)

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Abstract. An existing description of the cartesian closed topological hull of $p\mathbf{MET}^\infty$, the category of extended pseudo-metric spaces and nonexpansive maps, is simplified, and as a result, this hull is shown to be a special instance of a "family" of cartesian closed topological subconstructs of $pqs\mathbf{MET}^\infty$, the category of extended pseudo-quasi-semi-metric spaces (also known as quasi-distance spaces) and nonexpansive maps. Furthermore, another special instance of this family yields the cartesian closed topological hull of $pq\mathbf{MET}^\infty$, the category of extended pseudo-quasi-metric spaces and nonexpansive maps (which has recently gained interest in theoretical computer science), and this hull is also shown to be a nice generalization of **Prost**, the category of preordered spaces and relation preserving maps.

Keywords: (extended) pseudo-(quasi-)metric space, (quasi-)distance space, preordered space, demi-(quasi-)metric space, cartesian closed topological, CCT hull

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1. Introduction

It is the intention of this paper to indicate a somewhat surprising analogy between topological (i.e. convergence-like) spaces and metric (i.e. quasi-distance-like) spaces in the following sense.

In [6], G. Bourdaud indicated the existence of a "family" of cartesian closed topological constructs in **CONV**, the category of convergence spaces and continuous maps, where this "family" depended on (i.e. was "indexed" by) certain choices of functors. Moreover, it turned out that the cartesian closed topological hull (CCT hull) of **TOP** (described by Ph. Antoine ([3]), A. Machado ([17]) and G. Bourdaud ([5])) is a particular instance of this family (meaning; with appropriate choices of functors). Not only that, the CCT hull of **CREG**, the category of completely regular topological spaces and continuous maps, also arises as a specific instance of this family (by again appropriate choices of functors) (shown also in [6]).

It will now first be shown that such a family of CCT constructs also arises in a quasi-distance-like setting, i.e. in $pqs\mathbf{MET}^\infty$, the category of extended pseudo-quasi-semi-metric spaces and nonexpansive maps (also known as $q\mathbf{Dist}$). Next, it turns out that the description of the CCT hull of $p\mathbf{MET}^\infty$ (which is the same

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as the CCT hull of **MET**), the category of (extended pseudo-)metric spaces and nonexpansive maps, given by J. Adámek and J. Reiterman in [2] can be simplified in such a way that this CCT hull is a particular instance of this family (indeed, as the existence of this family was inspired by this particular hull). Next, it will be shown that another specific instance of this family is the CCT hull of $pq\mathbf{MET}^\infty$, the category of extended pseudo-quasi-metric spaces (no symmetry required) and nonexpansive maps.

Since these extended pseudo-quasi-metric spaces are a generalization of both ordinary metric spaces and preordered spaces, they have recently received quite some attention of researchers working in theoretical computer science as suitable objects for domain theory (Smyth ([22], [23]), Bonsangue, van Breugel and Rutten ([4])), in spite of failing the important property of being cartesian closed, where it is also demonstrated here that the CCT hull of $pq\mathbf{MET}^\infty$ is a nice generalization not only of (obviously) $(pq)\mathbf{MET}^{(\infty)}$, but also of **Prost**, the category of preordered spaces and relation preserving maps.

2. Preliminaries

Since CCT categories (or constructs) will be considerably used, first note that a *topological* construct will stand for a concrete category over **Set** which is a *well-fibred topological c-construct* in the sense of [1], i.e.

(a) each structured source has an initial lift,
(b) every set carries only a set of structures,
(c) each constant map (or empty map) between two objects is a morphism.

Also recall that a construct **A** is *CCT (cartesian closed topological)* if **A** is a topological construct which has *canonical function spaces*, i.e. for every pair (A, B) of **A**-objects the set $\hom(A, B)$ can be supplied with the structure of an **A**-object, denoted by $[A, B]$, such that

(a) the evaluation map $\mathrm{ev} : A \times [A, B] \longrightarrow B$ is an **A**-morphism,
(b) for each **A**-object C and **A**-morphism $f : A \times C \longrightarrow B$, the map $f^* : C \longrightarrow [A, B]$ defined by $f^*(c)(a) = f(a, c)$ is an **A**-morphism (f^* is called the *transpose* of f). Observe also that given $f : A \times C \longrightarrow B$, the transpose $f^* : C \longrightarrow [A, B]$ is the map which makes the following diagram commute:

$$\begin{tikzcd} A \times [A,B] \arrow[r, "\mathrm{ev}"] & B \\ A \times C \arrow[u, "1 \times f^*"] \arrow[ur, "f"'] & \end{tikzcd}$$

In general, categorical concepts and terminology used in the sequel (and possibly not recalled here), in particular regarding categorical topology, can be found in [1] and [20]. Furthermore, a functor shall always be assumed to be concrete (unless this is clearly not the case from its definition) and subcategories to be full and isomorphism-closed.

The *CCT hull* of a construct **A** (shortly denoted by CCTH(**A**)) (if it exists) is defined as the smallest CCT construct **B** in which **A** is closed under finite products (see [10]). Also from [10], recall that given a CCT construct **C** in which **A** is finally dense (i.e. each **C**-object is a final lift of some structured (epi-)sink in **A**), the CCT hull of **A** is the full subconstruct of **C** determined by

$$\mathrm{CCTH}(\mathbf{A}) := \{C \in \mathbf{C} \mid \text{there exists an initial source } (f_i : C \longrightarrow [A_i, B_i])_{i \in I} \\ \text{where } \forall\, i \in I : A_i, B_i \in \mathbf{A}\}.$$

In short, the CCT hull of **A** is the initial hull in **C** of the power-objects of **A**-objects.

A more recent survey of such properties and hull concepts can be found in [9] and [21].

Now, let us turn to recalling some necessities regarding (extended pseudo-quasi-semi-)metric spaces, where terminology and notations will be as in [15].

Given a set X, a function $d : X \times X \longrightarrow [0, \infty]$ is called a *metric* if it fulfills the properties:

(1) $\forall\, x \in X : d(x, x) = 0$;
(2) *triangle inequality*: $\forall\, x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$;
(3) *symmetry*: $\forall\, x, y \in X : d(x, y) = d(y, x)$;
(4) *separatedness*: $\forall\, x, y \in X : d(x, y) = 0 \Rightarrow x = y$;
(5) *finiteness*: $\forall\, x, y \in X : d(x, y) < \infty$.

If d is not necessarily finite, then it is called an *extended* metric, denoted by ∞-metric for short. If it is not necessarily separated, then it is called a *pseudo-metric*, denoted by p-metric for short, and if it is not necessarily symmetric, then it is called a *quasi-metric*, denoted by q-metric for short. If it does not necessarily satisfy the triangle inequality, then it is called a *semi-metric*, denoted by s-metric for short.

Any combination of these is also possible; for instance an ∞p-metric is quite common (and also known as *écart*), as is its quasi-counterpart ∞pq-metric, in the study of (quasi-)uniform spaces (see e.g. Weil [24], Császár [7], Fletcher and Lindgren [8] and Künzi [11], [12]).

A pair (X, d) where d is an ∞pqs-metric on X is called an *extended pseudo-quasi-semi-metric space*, *∞pqs-metric space* for short. Analogous conventions regarding ∞-, p-, q- and s- hold for spaces. For instance, ∞pq-metric spaces are a common generalization of both ordinary metric spaces and preordered spaces (because a preorder relation can be viewed as a discrete quasi-distance function (see also further)), and as such they are considered suitable objects in domain theory in theoretical computer science (e.g. Lawvere ([13]), Smyth ([22], [23]) and Bonsangue, van Breugel and Rutten ([4])).

Given two ∞pqs-metric spaces, (X, d_X) and (Y, d_Y), a function $f : X \longrightarrow Y$ is said to be *nonexpansive* if it fulfills the property

$$\forall\, x, y \in X : d_Y(f(x), f(y)) \leq d_X(x, y).$$

Extended pseudo-quasi-semi-metric spaces and nonexpansive maps form the objects and morphisms of a construct which is denoted by $pqs\mathbf{MET}^\infty$. The other types of spaces which have been considered so far give rise to subconstructs of $pqs\mathbf{MET}^\infty$ and here too we follow the same notational and terminological conventions. For instance, $ps\mathbf{MET}^\infty$ is also known as **Dist**, $pqs\mathbf{MET}^\infty$ is also known as q**Dist** and **MET** consists of (ordinary) metric spaces.

Recall the following properties.

2.1 Proposition. *$pqs\mathbf{MET}^\infty$ is a cartesian closed topological construct. Moreover, given a source $(f_i : X \longrightarrow (X_i, d_i))_{i\in I}$, the initial lift d is given by*

$$d(x,y) := \sup_{i\in I} d_i(f_i(x), f_i(y)),$$

and given a sink $(f_i : (X_i, d_i) \longrightarrow X)_{i\in I}$, the final lift d is given by

$$d(x,y) := \inf\{d_i(x_i, y_i) \mid i \in I,\, x_i, y_i \in X_i,\, f_i(x_i) = x \text{ and } f_i(y_i) = y\}.$$

Given ∞pqs-metric spaces $A := (X, d_X)$ and $B := (Y, d_Y)$, the power-object $[A, B]$ is $(\hom(A, B), d)$, where

$$d(f,g) := \sup\{d_Y(f(x), g(y)) \mid x, y \in X : d_Y(f(x), g(y)) > d_X(x,y)\}.$$

The following relations hold (where r (c) : $\mathbf{A} \longrightarrow \mathbf{B}$ means that $\mathbf{A}$ is a bi(co)reflective subconstruct of $\mathbf{B}$):

$$\begin{tikzcd} ps\mathbf{MET}^\infty \arrow[r, "r", "c"'] & pqs\mathbf{MET}^\infty \\ p\mathbf{MET}^\infty \arrow[u, "r"] \arrow[r, "r", "c"'] & pq\mathbf{MET}^\infty \arrow[u, "r"'] \end{tikzcd}$$

In particular, all indicated subconstructs are topological constructs. Also, the $pq\mathbf{MET}^\infty$-bireflection of (X, d) is denoted by (X, d^*) and is given by

$$d^*(x,y) := \inf\left\{\sum_{i=0}^{n} d(u_i, u_{i+1}) \mid u_0, \dots, u_{n+1} \in X, u_0 = x, u_{n+1} = y\right\}.$$

The $ps\mathbf{MET}^\infty$-bicoreflection of (X, d) is $(X, d \vee d^{-1})$, where $d^{-1}(x,y) := d(y,x)$.

Furthermore, various preservations hold. More precisely, power-objects in $ps\mathbf{MET}^\infty$ are formed as in $pqs\mathbf{MET}^\infty$ and if (X, d) is an (extended) pseudo-semi-metric space, then (X, d^*) is an (extended) pseudo-metric space. Also note that the bicoreflection in the bottom row is the restriction of the bicoreflection in the top row. □

As mentioned earlier, one obtains some interesting subcategories of $pqs\mathbf{MET}^\infty$ by restricting one's attention to those spaces that are discrete in the following sense.

2.2 Proposition. *The topological construct* **Rere** *of reflexive relations and relation preserving maps is concretely isomorphic to the full subconstruct of* $pqs\mathbf{MET}^\infty$ *determined by* $\{(X,d) \in pqs\mathbf{MET}^\infty \mid d(X \times X) \subset \{0,\infty\}\}$ *(and these shall therefore be identified with each other whenever convenient).*

Further, using this identification, it follows that $pq\mathbf{MET}^\infty \cap \mathbf{Rere} = \mathbf{Prost}$ *(where the latter is the full subconstruct of* **Rere** *consisting of preordered spaces) and the following relations hold:*

$$\begin{array}{ccc} pq\mathbf{MET}^\infty & \xrightarrow{\ r\ } & pqs\mathbf{MET}^\infty \\ {\scriptstyle r}\uparrow{\scriptstyle c} & & {\scriptstyle r}\uparrow{\scriptstyle c} \\ \mathbf{Prost} & \xrightarrow{\ r\ } & \mathbf{Rere} \end{array}$$

In particular, all indicated constructs are topological, and moreover, **Rere** *and* **Prost** *are cartesian closed topological constructs (see e.g. [20, (4.2.4)]), where in both cases the bicoreflection* (X,d_r) *of a space* (X,d) *is obtained by letting* $d_r(x,y) := 0$ *if* $d(x,y) = 0$ *and* $d_r(x,y) := \infty$ *otherwise.* □

2.3 Proposition. **MET** *is finally dense in* $ps\mathbf{MET}^\infty$*. Moreover, letting* $D_\epsilon := (\{0,1\}, d_\epsilon)$ *where* $d_\epsilon(0,1) := \epsilon =: d_\epsilon(1,0)$*, then the class* $\{D_\epsilon \mid 0 < \epsilon < \infty\}$ *is finally dense in* $ps\mathbf{MET}^\infty$*.*

Furthermore, $q\mathbf{MET}^\infty$ *is finally dense in* $pqs\mathbf{MET}^\infty$*. Again moreover, letting* $\mathbb{S}_\epsilon := (\{0,1\}, d^q_\epsilon)$ *where* $d^q_\epsilon(0,1) := \epsilon$ *and* $d^q_\epsilon(1,0) := \infty$*, then the class* $\{\mathbb{S}_\epsilon \mid 0 < \epsilon < \infty\}$ *is finally dense in* $pqs\mathbf{MET}^\infty$*.* □

2.4 Proposition. *Let* (X,d) *be an* ∞pq*-metric space and let* $B_d(x,\epsilon) := \{y \in X \mid d(x,y) < \epsilon\}$ *and* $\mathcal{V}_d(x) := \{V \subset X \mid \exists 0 < \epsilon : B_d(x,\epsilon) \subset V\}$*, then the following hold:*

(1) $(\mathcal{V}_d(x))_{x\in X}$ *determines the neighbourhoods of a topology* $\mathcal{T}_d$*, called the topology underlying* d*;*
(2) $T : pq\mathbf{MET}^\infty \longrightarrow \mathbf{TOP} : (X,d) \mapsto (X,\mathcal{T}_d)$ *determines a concrete functor.*

□

2.5 Definition. The *right-order topology* $\mathcal{T}_r$ on $[0,\infty]$ is the topology whose open sets are $\{]a,\infty] \mid a \in [0,\infty]\} \cup \{[0,\infty]\}$.

3. Some CCT subconstructs of $pqs\mathbf{MET}^\infty$

Let us first introduce the necessary concepts in order to define the previously mentioned "family" of CCT constructs, which inspires and allows to simplify (the description of) the CCT hull of $p\mathbf{MET}^\infty$ given in [2].

Also note that $\mathrm{CCTH}(p\mathbf{MET}^\infty) = \mathrm{CCTH}(\mathbf{MET})$ (where it is actually the latter one that is considered in [2]), which follows from the well-know facts that $p\mathbf{MET}^\infty$ is the *topological hull* (that is, smallest finally dense topological extension) of **MET**, and that such related constructs have the same CCT hull (recall that a CCT hull is also the smallest finally dense CCT extension).

3.1 Definition. Let $\mathbf{A}$ be a category and denote $\mathbf{B} := \mathrm{MP}(\mathbf{A})$ for the category such that $\mathrm{Ob}(\mathbf{B}) := \mathrm{Ob}(\mathbf{A})$ and $\hom_{\mathbf{B}}(A,B) := \{(f,g) \mid f,g \in \hom_{\mathbf{A}}(A,B)\}$. Furthermore, if $(f_1,f_2) \in \hom_{\mathbf{B}}(A,B)$ and $(g_1,g_2) \in \hom_{\mathbf{B}}(B,C)$, then define $(g_1,g_2) \circ_{\mathbf{B}} (f_1,f_2) := (g_1 \circ_{\mathbf{A}} f_1, g_2 \circ_{\mathbf{A}} f_2)$. Clearly, this composition is associative and the identity elements are given by $(1^A_{\mathbf{A}}, 1^A_{\mathbf{A}})$ $(A \in \mathrm{Ob}(\mathbf{B}))$.

The resulting category $\mathrm{MP}(\mathbf{A})$ is called the *morphism pairing* (*category*) (or the *point-point-morphism category*) of $\mathbf{A}$.

3.2 Definition. Let $G : \mathrm{MP}(pqs\mathbf{MET}^\infty) \longrightarrow \mathbf{TOP}$ be a (non-concrete) functor such that $\left|G\big((f,g) : (X,d_X) \longrightarrow (Y,d_Y)\big)\right| = f \times g : X \times X \longrightarrow Y \times Y$, which will be called a *paired functor* (or a *point-point-functor*).

A space $(X,d) \in pqs\mathbf{MET}^\infty$ is called a *G-demi-metric space* if it satisfies the following condition:

$$(D): \; d : G(X,d) \longrightarrow ([0,\infty], \mathcal{T}_r) \text{ is a continuous map.}$$

The full subconstruct of $pqs\mathbf{MET}^\infty$ whose objects are G-demi-metric spaces is denoted by G-$dpqs\mathbf{MET}^\infty$.

3.3 Remark. Since the foregoing may seem (overly) complicated, let us first indicate a more usual and natural way in which such a paired functor G may arise (as they will in most of the sequel).

Let $G_i : pqs\mathbf{MET}^\infty \longrightarrow \mathbf{TOP}$ $(i = 1,2)$ be concrete functors. The paired functor $G := (G_1,G_2) : \mathrm{MP}(pqs\mathbf{MET}^\infty) \longrightarrow \mathbf{TOP}$ (called *pairing* of G_1 and G_2) is defined by

$$\begin{gathered} G\big((f,g) : (X,d_X) \longrightarrow (Y,d_Y)\big) := \\ f \times g : G_1(X,d) \times G_2(X,d) \longrightarrow G_1(Y,d_Y) \times G_2(Y,d_Y). \end{gathered}$$

3.4 Proposition. *Let $G : \mathrm{MP}(pqs\mathbf{MET}^\infty) \longrightarrow \mathbf{TOP}$ be a paired functor, then G-$dpqs\mathbf{MET}^\infty$ is cartesian closed topological. Moreover, G-$dpqs\mathbf{MET}^\infty$ is bireflective in $pqs\mathbf{MET}^\infty$ and $[(X,d_X),(Y,d_Y)]$ is a G-demi-metric space whenever (Y,d_Y) is.*

Proof: Let $(f_i : (X,d) \longrightarrow (X_i,d_i))_{i\in I}$ be an initial source in $pqs\mathbf{MET}^\infty$ such that all $(X_i,d_i) \in G$-$dpqs\mathbf{MET}^\infty$ (and assume that $I \neq \emptyset$, otherwise (X,d) is an indiscrete space which trivially satisfies (D)). To show that (X,d) satisfies (D), assume that $A \in \mathcal{T}_r$, then it follows from Proposition 2.1 that

$$d^{-1}(A) = \bigcup_{i\in I} \big(d_i \circ (f_i \times f_i)\big)^{-1}(A).$$

From the nonexpansiveness of all f_i (and therefore the continuity of all $G\big((f_i,f_i) : (X,d) \longrightarrow (X_i,d_i)\big) = f_i \times f_i : G(X,d) \longrightarrow G(X_i,d_i)$, $i \in I$) and the fact that all

(X_i, d_i), $i \in I$, satisfy (D) it follows that $d^{-1}(A)$ is open (in $G(X,d)$). Hence d is continuous and (X,d) is a G-demi-metric space.

Now let (Y, d_Y) satisfy (D) and $(X, d_X) \in pqs\mathbf{MET}^\infty$, then it needs to be shown that $(Z,d) := [(X,d_X),(Y,d_Y)]$ also satisfies (D). To this end, assume that $[0,\infty] \neq A \in \mathcal{T}_r$, then it follows from the formula of d in Proposition 2.1 that

$$d^{-1}(A) = \bigcup_{(x,y)\in X\times X} \Big(\big(d_Y \circ (\mathrm{ev}_x \times \mathrm{ev}_y)\big)^{-1} \big(A \cap]d_X(x,y),\infty]\big) \Big).$$

Hence it follows from the fact that all ev_x, $x \in X$, are contractions (and therefore the continuity of all $G\big((\mathrm{ev}_x, \mathrm{ev}_y) : (Z,d) \longrightarrow (Y,d_Y)\big) = \mathrm{ev}_x \times \mathrm{ev}_y : G(Z,d) \longrightarrow G(Y,d_Y)$, $x,y \in X$) and the fact that d_Y satisfies (D) that $d^{-1}(A)$ is open (in $G(X,d)$). Hence, (Z,d) is a G-demi-metric space. □

Using Remark 3.3 now provides us with a (natural) example.

3.5 Example. Let $R : pqs\mathbf{MET}^\infty \longrightarrow pq\mathbf{MET}^\infty : (X,d) \mapsto (X,d^*)$ be the $pq\mathbf{MET}^\infty$-bireflection and let $T : pq\mathbf{MET}^\infty \longrightarrow \mathbf{TOP}$ be as before (i.e. the underlying topology functor). Combining these properly leads to the paired functor

$$D := (T \circ R, T \circ R) : \mathrm{MP}(pqs\mathbf{MET}^\infty) \longrightarrow \mathbf{TOP}$$

which allows us to define $dps\mathbf{MET}^\infty := ps\mathbf{MET}^\infty \cap D\text{-}dpqs\mathbf{MET}^\infty$.

This example is not entirely new, if one recalls that the cartesian closed topological hull of $p\mathbf{MET}^\infty$ was described in [2] by J. Adámek and J. Reiterman as consisting of *demi-metric* spaces, where a demi-metric space is a ∞ps-metric space satisfying

(i) *positivity*: $d(x,y) = 0$ whenever $d^*(x,y) = 0$;
(ii) *lower semi-continuity*: $d : T((X,d^*) \times (X,d^*)) \longrightarrow ([0,\infty], \mathcal{T}_r)$ is continuous.

Clearly, $dps\mathbf{MET}^\infty$-objects are precisely those ∞ps-metric spaces which satisfy lower semi-continuity, a property that characterizes demi-metric spaces just by itself, as the following shows.

3.6 Proposition. *Let $(X,d) \in dps\mathbf{MET}^\infty$, then (X,d) satisfies positivity.*

Proof: Let $d^*(x,y) = 0$ for some $x,y \in X$ and assume that $d(x,y) > K > 0$. By lower semi-continuity, there exists some $\delta > 0$ such that $d^*(x,x') < \delta$ and $d^*(y,y') < \delta$ implies that $d(x',y') > K$. However, as $d^*(x,y) = 0 < \delta$ and $d^*(y,y) = 0 < \delta$, it follows that $d(y,y) > K > 0$, a contradiction. Consequently, it must be that $d(x,y) = 0$. □

In particular, the previous proposition justifies the choices in notation and terminology, as well as the following as an immediate consequence of the (main) result in [2].

3.7 Theorem. *The construct dps*$\mathbf{MET}^\infty$ *of demi-metric spaces and nonexpansive maps is the cartesian closed topological hull of* $\mathbf{MET}$, *i.e.* $\mathrm{CCTH}(\mathbf{MET}) = \mathrm{CCTH}(p\mathbf{MET}^\infty) = dps\mathbf{MET}^\infty$. □

3.8 Remark.

(1) The presence of symmetry in the previous description is quite naturally to be expected, since $ps\mathbf{MET}^\infty$ is already a topological universe containing $\mathbf{MET}$ and all natural operations such as initial lifts, power-objects, ... preserve symmetry. In particular, such a "family" of subconstructs (as presented) could also easily be considered in a $ps\mathbf{MET}^\infty$ setting.

(2) This symmetry has to be explicitly demanded, as lower semi-continuity does not imply symmetry, as shown by the following example.
Let $X := \mathbb{R}$ and define the ∞q-metric d by $d(x,y) := x-y$ if $y \leq x$ and $d(x,y) := \infty$ if $y > x$. It is then easily verified that d is a quasi-metric with $d_{\mathbb{E}}(x,y) := |x-y|$ as $p\mathbf{MET}^\infty$-bireflection. Having this, one easily sees that d is even lower-semi continuous w.r.t. $d_{\mathbb{E}}$ and d^*, but obviously not symmetric.

4. The CCT hull of $pq\mathbf{MET}^\infty$

By reasonably adapting the previous example of a particular instance to a non-symmetric (quasi) setting, one also obtains the cartesian closed topological hull of $pq\mathbf{MET}^\infty$ as a specific instance of this "family".

4.1 Definition. As before, let $R : pqs\mathbf{MET}^\infty \longrightarrow pq\mathbf{MET}^\infty : (X,d) \mapsto (X,d^*)$ be the $pq\mathbf{MET}^\infty$-bireflection and let $T : pq\mathbf{MET}^\infty \longrightarrow \mathbf{TOP}$ be the underlying topology functor. Furthermore, let $W : pq\mathbf{MET}^\infty \longrightarrow pq\mathbf{MET}^\infty$ be the concrete functor such that $W(X,d) := (X,d^{-1})$, where $d^{-1}(x,y) := d(y,x)$ $(\forall\, x,y \in X)$.

This leads to the paired functor

$$D_q := (T \circ R, T \circ W \circ R) : \mathrm{MP}(pqs\mathbf{MET}^\infty) \longrightarrow \mathbf{TOP}$$

and denote $dpqs\mathbf{MET}^\infty := D_q\text{-}dpqs\mathbf{MET}^\infty$, whose objects are called *demi-quasi-metric spaces.*

Expressing the condition (D) in this particular case, it follows that a demi-quasi-metric space must satisfy:

$$d : T((X,d^*) \times (X,(d^*)^{-1})) \longrightarrow ([0,\infty],\mathcal{T}_r) \text{ is a continuous map.}$$

The following lemma is inspired by an analogous lemma in [16].

4.2 Lemma. *Let* $d_{\mathbb{P}}$ *be the* ∞pq*-metric on* $[0,\infty]$ *defined by* $d_{\mathbb{P}}(x,y) := (x-y) \vee 0$ $(\forall\, x,y \in X)$ *and let* $(X,d) \in pqs\mathbf{MET}^\infty$, *then the following are equivalent:*

(1) d *satisfies the triangle inequality, i.e.* $(X,d) \in pq\mathbf{MET}^\infty$;

(2) $d : (X \times X, d^* \circ (\mathrm{pr}_1 \times \mathrm{pr}_1) + (d^*)^{-1} \circ (\mathrm{pr}_2 \times \mathrm{pr}_2)) \longrightarrow ([0,\infty], d_{\mathbb{P}})$ *is nonexpansive;*

(3) $\forall\, x \in X : d(x,-) : (X,(d^*)^{-1}) \longrightarrow ([0,\infty], d_{\mathbb{P}})$ *is nonexpansive;*

(4) $\forall\, y \in X : d(-,y) : (X,d^*) \longrightarrow ([0,\infty], d_{\mathbb{P}})$ *is nonexpansive.*

PROOF: $\boxed{1\Rightarrow 2}$ Let d satisfy the triangle inequality and $(x,y),(x',y')\in X\times X$, then

$$\begin{aligned}(d(x,y)-d(x',y'))\vee 0 &\le (d(x,x')+d(x',y')+d(y',y)-d(x',y'))\vee 0\\ &\le d(x,x')+d(y',y)=d^*(x,x')+(d^*)^{-1}(y,y').\end{aligned}$$

$\boxed{2\Rightarrow 3}$ and $\boxed{2\Rightarrow 4}$ This is obvious.

$\boxed{3\Rightarrow 1}$ In this case, it holds for all $x,y,z\in X$ that:

$$d(x,y)-d(x,z)\le (d(x,y)-d(x,z))\vee 0\le d^*(z,y)\le d(z,y),$$

hence $d(x,y)\le d(x,z)+d(z,y)$.

$\boxed{4\Rightarrow 1}$ In this case, it holds for all $x,y,z\in X$ that:

$$d(x,y)-d(z,y)\le (d(x,y)-d(z,y))\vee 0\le d^*(x,z)\le d(x,z),$$

hence $d(x,y)\le d(x,z)+d(z,y)$. □

4.3 Proposition. *$dpqs\mathbf{MET}^\infty$ is a cartesian closed topological construct containing $pq\mathbf{MET}^\infty$.*

PROOF: The first part immediately follows from Proposition 3.4 and the latter part follows from the previous lemma by observing that $T(X\times X, d^*\circ(\mathrm{pr}_1\times\mathrm{pr}_1)+(d^*)^{-1}\circ(\mathrm{pr}_2\times\mathrm{pr}_2))=D_q(X,d)$ (given $(X,d)\in pqs\mathbf{MET}^\infty$), hence d being a nonexpansive map (if $(X,d)\in pq\mathbf{MET}^\infty$) implies continuity of d. □

4.4 Remark. The foregoing actually gives a bit more than lower semi-continuity in the case of $pq\mathbf{MET}^\infty$-objects, as the topology on $[0,\infty]$ underlying $d_{\mathbb{P}}$ is finer than the right-order topology, in particular $\{\infty\}$ is open in $\mathcal{T}_{d_{\mathbb{P}}}$. Hence, if some distance between points is ∞, then it is also ∞ between points belonging to some properly chosen neighbourhoods.

An example given later (see Example 4.8 (4)) will however show that this property is no longer valid for a general demi-(quasi-)metric space.

4.5 Lemma. *Let (X,d) be a demi-quasi-metric space, then (X,d) satisfies positivity, i.e. $d^*(x,y)=0$ implies $d(x,y)=0$.*

PROOF: The argument is along the same lines as before.

Let $d^*(x,y)=0$ for some $x,y\in X$ and assume that $d(x,y)>K>0$. By continuity of $d: T((X,d^*)\times(X,(d^*)^{-1}))\longrightarrow([0,\infty],\mathcal{T}_r)$, there exists some $\delta>0$ such that $d^*(x,x')<\delta$ and $d^*(y',y)<\delta$ implies that $d(x',y')>K$. However, as $d^*(x,y)=0<\delta$ and $d^*(y,y)=0<\delta$, it follows that $d(y,y)>K>0$, a contradiction. Consequently, also $d(x,y)=0$. □

Now we are in a position to show that demi-quasi-metric spaces are a common generalization of both extended pseudo-quasi-metric (= generalized metric) spaces (and therefore of ordinary metric spaces) and preordered spaces.

4.6 Proposition. $dpqs\mathbf{MET}^\infty \cap \mathbf{Rere} = \mathbf{Prost}$.

Moreover, **Prost** *is bireflective and bicoreflective in* $dpqs\mathbf{MET}^\infty$ *such that the bicoreflection is the restriction of the* **Rere**-*bicoreflection in* $pqs\mathbf{MET}^\infty$.

PROOF: The inclusion $\boxed{\supset}$ of the first claim clearly holds. As for $\boxed{\subset}$, let $(X,d) \in dpqs\mathbf{MET}^\infty \cap \mathbf{Rere}$, then it follows immediately from the formula of d^* in Proposition 2.1 that also $(X,d^*) \in \mathbf{Rere}$, hence the foregoing lemma (and the fact that always $d^* \leq d$) implies that $d^* = d$ (as both d and d^* only attain the values 0 and ∞), consequently $(X,d) = (X,d^*) \in pq\mathbf{MET}^\infty \cap \mathbf{Rere} = \mathbf{Prost}$.

The claim regarding bireflectiveness follows immediately from Propositions 2.2 and 3.4 and to prove the required bicoreflectiveness, let $(X,d) \in dpqs\mathbf{MET}^\infty$, then it suffices to show that also $(X,d_r) \in dpqs\mathbf{MET}^\infty$. Since $1_X : D_q(X,d_r) \longrightarrow D_q(X,d_r)$ is continuous (as $1_X : (X,d_r) \longrightarrow (X,d)$ is a nonexpansive map) and $d : D_q(X,d) \longrightarrow ([0,\infty],\mathcal{T}_r)$ is continuous, it follows that $d : D_q(X,d_r) \longrightarrow ([0,\infty],\mathcal{T}_r)$ is continuous. In particular, let $0 \leq K < \infty$, then it holds that ${d_r}^{-1}(]K,\infty]) = d^{-1}(]0,\infty])$ is open in $D_q(X,d_r)$, hence $d_r : D_q(X,d_r) \longrightarrow ([0,\infty],\mathcal{T}_r)$ is continuous. □

In view of the following result and the fact that **Prost** is its own CCT hull (as it is cartesian closed topological), the following diagram nicely summarizes the resulting situation:

$$\begin{tikzcd}
pq\mathbf{MET}^\infty \arrow[r, "r"] & dpqs\mathbf{MET}^\infty \arrow[r, "r"] & pqs\mathbf{MET}^\infty \\
\mathbf{Prost} \arrow[u, "r", "c"'] \arrow[r, equal] & \mathrm{CCTH}(\mathbf{Prost}) \arrow[u, "r", "c"'] \arrow[r, "r"] & \mathbf{Rere} \arrow[u, "r", "c"']
\end{tikzcd}$$

4.7 Theorem. *The construct of demi-quasi-metric spaces and nonexpansive maps is the cartesian closed topological hull of* $pq\mathbf{MET}^\infty$, *i.e.* $\mathrm{CCTH}(pq\mathbf{MET}^\infty) = dpqs\mathbf{MET}^\infty$.

PROOF: For this to be the case, it is needed (as noted before) that $dpqs\mathbf{MET}^\infty$ is a cartesian closed topological construct (which is stated in Proposition 4.3) and that $pq\mathbf{MET}^\infty$ is finally dense in $dpqs\mathbf{MET}^\infty$ (which follows from Propositions 4.3 and 2.3). It also needs to be shown that the class

$$H := \{\, [(X,d_X),(Y,d_Y)] \mid (X,d_X),(Y,d_Y) \in pq\mathbf{MET}^\infty\}$$

is initially dense in $dpqs\mathbf{MET}^\infty$, and this will be done very analogously to the respective part of [2], making modifications where necessary.

Let (X,d) be a demi-quasi-metric space. First, recall that $\mathbb{S}_\epsilon = (\{0,1\}, d^q_\epsilon)$ where $d^q_\epsilon(0,1) := \epsilon$ and $d^q_\epsilon(1,0) := \infty$. Now, given a pair $(x,y) \in X \times X$ with $d(x,y) > K$ and $K > 0$, we will find some $\epsilon > 0$ and a nonexpansive map $f : (X,d) \longrightarrow [\mathbb{S}_\epsilon, ([0,\infty], d_{\mathbb{P}})]$ such that the distance of $f(x)$ and $f(y)$ in $[\mathbb{S}_\epsilon, ([0,\infty], d_{\mathbb{P}})]$ is larger or equal to K. Consequently, by Proposition 2.1, all those morphisms f form an initial source, hence $(X,d) \in \mathrm{CCTH}(pq\mathbf{MET}^\infty)$.

As (X,d) is a demi-quasi-metric space, there exists some $\delta' > 0$ such that $d^*(x,x') < \delta'$ and $d^*(y',y) < \delta'$ implies that $d(x',y') > K$. Define the pseudo-quasi-metric ρ by letting $\rho(u,v) := d^*(u,v) \wedge \frac{K}{2}$ and choose some $0 < \delta < \delta' \wedge \frac{K}{2}$. Also, define

$$f : X \longrightarrow [\mathbb{S}_{K-\delta}, ([0,\infty], d_{\mathbb{P}})] \text{ by}$$
$$f(u)(0) := K - \rho(x,u) \text{ and } f(u)(1) := \rho(u,y)$$

and denote $(Z, d_Z) := [\mathbb{S}_{K-\delta}, ([0,\infty], d_{\mathbb{P}})]$.

Now it has to be verified that

(a) $\forall\, u \in X : f(u) \in Z$,
(b) $f : (X,d) \longrightarrow (Z, d_Z)$ is a nonexpansive map,
(c) $d_Z(f(x), f(y)) \geq K$.

To this end, it will first be shown that

$$(*) \qquad (f(u)(i) - f(v)(i)) \vee 0 \leq d(u,v) \text{ for } i = 1,2$$

and that

$$(**) \quad (f(u)(0) - f(v)(1)) \vee 0 > K - \delta \text{ implies } (f(u)(0) - f(v)(1)) \vee 0 \leq d(u,v).$$

As for the first inequalities, one obtains that

$$\begin{aligned} (f(u)(0) - f(v)(0)) \vee 0 &= (K - \rho(x,u) - K + \rho(x,v)) \vee 0 \\ &= (\rho(x,v) - \rho(x,u)) \vee 0 \\ &\leq (\rho(x,u) + \rho(u,v) - \rho(x,u)) \vee 0 \\ &= \rho(u,v) \leq d^*(u,v) \leq d(u,v) \end{aligned}$$

and

$$\begin{aligned} (f(u)(1) - f(v)(1)) \vee 0 &= (\rho(u,y) - \rho(v,y)) \vee 0 \\ &\leq (\rho(u,v) + \rho(v,y) - \rho(v,y)) \vee 0 \\ &= \rho(u,v) \leq d^*(u,v) \leq d(u,v). \end{aligned}$$

As for $(**)$, assume that

$$(f(u)(0) - f(v)(1)) \vee 0 = (K - \rho(x,u) - \rho(v,y)) \vee 0 > K - \delta,$$

hence $K - \rho(x,u) - \rho(v,y) > K - \delta$, consequently $\rho(x,u) + \rho(v,y) < \delta$, in particular $\rho(x,u) < \delta$ and $\rho(v,y) < \delta$. Since $\delta < \frac{K}{2}$, the definition of ρ implies that also $d^*(x,u) < \delta < \delta'$ and $d^*(v,y) < \delta < \delta'$, hence (by choice of δ') also $d(u,v) > K$. It follows that

$$(f(u)(0) - f(v)(1)) \vee 0 = (K - \rho(x,u) - \rho(v,y)) \vee 0 \leq K < d(u,v).$$

Next, turning our attention to (a), it needs to be shown that $f(u) \in Z$ (given some $u \in X$), i.e. $(f(u)(0) - f(u)(1)) \vee 0 \leq K - \delta$. Assume the contrary, then $(**)$ (where we take $u = v$) implies that $K - \delta < (f(u)(0) - f(u)(1)) \vee 0 \leq d(u,u) = 0$, a contradiction. Also, having (a), $(*)$ and $(**)$ precisely express what is needed to show (b).

As for (c), since

$$(f(x)(0) - f(y)(1)) \vee 0 = K > K - \delta = d^q_{K-\delta}(0,1),$$

it follows that

$$d_Z(f(x), f(y)) \geq (f(x)(0) - f(y)(1)) \vee 0 = K.$$

□

Some analogous examples and remarks as in [2] can now be given.

4.8 Example.

(1) First define (X,d), where

$$X := \{(x,y) \in [0,\infty]^2 \mid (x-y) \vee 0 \leq \epsilon\}$$

and $d((x,y),(x',y'))$ is the maximum of

$$(x - x') \vee 0$$

$$(y - y') \vee 0$$

$$(x - y') \vee 0 \text{ counted only if } (x - y') \vee 0 > \epsilon.$$

One then easily sees that actually $(X,d) \cong [\mathbb{S}_\epsilon, ([0,\infty], d_{\mathbb{P}})]$, hence $\mathbb{S}^{(2)}_\epsilon := (X,d)$ is a (typical) demi-quasi-metric space.

(2) Each subspace of a product $\prod_{i\in I} \mathbb{S}^{(2)}_{\epsilon_i}$ is a demi-quasi-metric space.

(3) Conversely, each T_0 demi-quasi-metric space (i.e. such that $d(x,y) = 0 = d(y,x)$ implies that $x = y$) is a subspace of a product of $\mathbb{S}^{(2)}_\epsilon$'s. This follows from the fact that $([0,\infty], d_{\mathbb{P}})$ is initially dense in $pq\mathbf{MET}^\infty$ and $\{\mathbb{S}_\epsilon \mid \epsilon > 0\}$ is finally dense in $pq\mathbf{MET}^\infty$.

(4) Now consider an example referred to in Remark 4.4 by defining an ∞s-metric space (X,d) as follows:

$$X := \{x, y, x_n \mid n \in \mathbb{N}_0\} \quad \text{and} \quad \begin{tikzcd} x \arrow[r, dash, "\infty"] \arrow[d, dash, "\frac{1}{n}"'] & y \arrow[dl, dash, "n"] \\ x_n \end{tikzcd}$$

(where some distances are indicated and all the others are equal to ∞). It is then easily verified that (X,d) is a demi-metric space, but clearly, the stronger property mentioned in Remark 4.4 does not hold.

4.9 Remark. Although the following is (obviously) to be expected, it is nevertheless not uninteresting to be checked and noted here in light of aspects in [16] and (counter-)examples in [5].

For instance, unlike as in Lemma 4.2, one really needs combined continuity of d in the definition of demi-quasi-metric space, i.e. it does not suffice to be separately continuous in the first component and also in the second component. In fact, the s-metric space (X, d) defined by

$$X := \{x, y, x_n, y_n \mid n \in \mathbb{N}_0\} \quad \text{and} \quad \begin{tikzcd} x \arrow[r, dash, "1"] \arrow[d, dash, "\frac{1}{n}"'] & y \arrow[d, dash, "\frac{1}{n}"] \\ x_n \arrow[r, dash, "\frac{1}{2}"] & y_n \end{tikzcd}$$

(where some distances are indicated and all the others are equal to 1) has this separate continuity, but is not a demi-(quasi-)metric space.

Furthermore, one can also obtain an ∞pqs-metric d which is continuous in one component, but not in the other component (hence does not yield a demi-quasi-metric space), e.g. defined by

$$X := \{x, y, x_n \mid n \in \mathbb{N}_0\} \quad \text{and} \quad \begin{tikzcd} x \arrow[r, "1"] \arrow[d, "\frac{1}{n}"'] & y \\ x_n \arrow[ur, "\frac{1}{2}"'] & \end{tikzcd}$$

(where some distances are indicated (since there is no symmetry, the arrow $x \to y$ refers to the distance from x to y) and all the others are equal to ∞). It is then easily verified that d has the proper continuity in the second component, but not in the first component.

(Final) Remark. A similar situation to the one described here also occurs in a convergence-approach-like setting. More precisely, in [18], the author describes a family of cartesian closed topological constructs in **CAP**, the category of convergence-approach spaces and contractions (which is a unification of convergence-like and (quasi-)distance-like concepts), introduced by E. Lowen and R. Lowen in [14] as a topological quasitopos containing **AP**, the category of approach spaces and contractions (which is a unification of topological and (extended pseudo-)quasi-metric concepts) (see e.g. R. Lowen [15]).

Further similarities are the results that both the CCT hulls of **AP** and of **UAP** (a subconstruct of **AP** unifying completely regular topological spaces and (extended pseudo-)metric spaces) occur as instances of the previously mentioned family in **CAP** (see also [16], [18], [19]). In particular, a similar situation to the one illustrated by the diagram preceding Theorem 4.7 occurs in those cases (where the relation-level is replaced by a convergence-level and the quasi-distance-level by a convergence-approach-level, but where the equality in the lower level is just a bireflective embedding).

While the (quasi-)distance situation described here is one of the "components" of the combined convergence-approach-setting, the more general situation also provides some feedback, such as the (small but useful) change w.r.t. positivity. More precisely, it was brought to attention by a remark of G. Bourdaud in [5] stating that the limit set of a principal ultrafilter in an Antoine space (i.e. an object of CCTH(**TOP**)) is the same as the limit set (of that principal ultrafilter) regarded w.r.t. convergence in its **TOP**-bireflection; and: convergence is to be equated with "distance = 0".

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References

[1] Adámek J., Herrlich H., Strecker G.E., *Abstract and Concrete Categories*, Wiley, New York et al., 1990.
[2] Adámek J., Reiterman J., *Cartesian closed hull for metric spaces*, Comment. Math. Univ. Carolinae **31.1** (1990), 1–6.
[3] Antoine P., *Étude élémentaire d'ensembles structurés*, Bull. Soc. Math. Belge XVIII 2 et 4 (1966).
[4] Bonsangue M.M., van Breugel F., Rutten J.J.M.M., *Generalized metric spaces: completion, topology, and powerdomains via the Yoneda embedding*, Theoretical Computer Science **193** (1998), 1–51.
[5] Bourdaud G., *Espaces d'Antoine et semi-espaces d'Antoine*, Cahiers Topol. Géom. Diff. Cat. **16** (1975), 107–133.
[6] Bourdaud G., *Some cartesian closed topological categories of convergence spaces*, in: E. Binz and H. Herrlich (eds.), Categorical Topology (Proc. Mannheim 1975), Lecture Notes Math. **540**, Springer, Berlin et al., 1976, pp. 93–108.
[7] Császár Á., *Fondements de la topologie générale*, Pergamon Press, Oxford, 1963.
[8] Fletcher P., Lindgren W.F., *Quasi-uniform spaces*, Lecture Notes in Pure and Applied Mathematics **77**, Marcel Dekker, New York and Basel, 1982.
[9] Herrlich H., *Topological improvements of categories of structured sets*, Topology Appl. **27** (1987), 145–155.
[10] Herrlich H., Nel L.D., *Cartesian closed topological hulls*, Proc. Amer. Math. Soc. **62** (1977), 215–222.
[11] Künzi H.P.A., *Quasi-uniform spaces — eleven years later*, Topology Proc. **18** (1993), 143–171.
[12] Künzi H.P.A., *Nonsymmetric topology*, Bolyai Society in Mathematical Studies 4, Topology, Szekszárd, 1993, Hungary, (Budapest 1995), pp. 303–338.
[13] Lawvere F.W., *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. Fis. Milano **43** (1973), 135–166.
[14] Lowen E., Lowen R., *A quasitopos containing* **CONV** *and* **MET** *as full subcategories*, Internat. J. Math. & Math. Sci. **11** (1988), 417–438.
[15] Lowen R., *Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad*, Oxford Mathematical Monographs, Oxford University Press, 1997.
[16] Lowen-Colebunders E., Lowen R., Nauwelaerts M., *The cartesian closed hull of the category of approach spaces*, Cahiers Topol. Géom. Diff. Cat., to appear.
[17] Machado A., *Espaces d'Antoine et pseudo-topologies*, Cahiers Topol. Géom. Diff. Cat. **14.3** (1973), 309–327.
[18] Nauwelaerts M., *Some cartesian closed topological constructs of convergence-approach spaces*, submitted for publication.

[19] Nauwelaerts M., *The hulls of the category of uniform approach spaces*, submitted for publication.

[20] Preuß G., *Theory of Topological Structures*, Reidel (Kluwer Academic), Dordrecht, 1987.

[21] Schwarz F., Weck-Schwarz S., *Internal Description of Hulls: A Unifying Approach*, Category Theory at Work, H. Herrlich and H.-E. Porst (eds.), Heldermann Verlag, Berlin, 1991, pp. 35–45.

[22] Smyth M.B., *Quasi-uniformities: reconciling domains with metric spaces*, in Proc. of 3rd Workshop on Mathematical Foundations of Programming Language Semantics, New Orleans, 1987, ed. M. Main, A. Melton, M. Mislove, D. Schmidt, Lecture Notes in Computer Science bf 298, Springer-Verlag, Berlin, 1988, pp. 236–253.

[23] Smyth M.B., *Totally bounded spaces and compact ordered spaces as domains of computation*, in Topology and Category Theory in Computer Science, eds. G.M. Reed, A.W. Roscoe, R.F. Wachter, Clarendon Press, Oxford, 1991, pp. 207–229.

[24] Weil A., *Sur les espaces à structure uniforme et sur la topologie générale*, Hermann, Paris, 1938.

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