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BGG sequences on spheres

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Abstract. BGG sequences on flat homogeneous spaces are analyzed from the point of view of decomposition of appropriate representation spaces on irreducible parts with respect to a maximal compact subgroup, the so called K-types. In particular, the kernels and images of all standard invariant differential operators (including the higher spin analogs of the basic twistor operator), i.e. operators appearing in BGG sequences, are described.

Keywords: BGG sequences, invariant differential operators, branching rules, K-types, complexes, homogeneous spaces

Classification: 35P15, 43A85, 22E46

1. Introduction

Properties of conformally invariant operators on manifolds with a given conformal structure are studied in many papers ([21], [14], [12], [17], [18], [26], [1], [2], [24], [13], [5], [6]). A special subclass of them — so called standard invariant operators, are coming together in sequences called (generalized) Bernstein-Gelfand-Gelfand sequences (BGG sequences for a short). A general construction of such sequences from differential geometry point of view was given recently in [10].

In this paper BGG sequences on the homogeneous model (i.e. on the sphere) are studied using elementary tools from representation theory. The aim of the paper is twofold. Firstly, it is shown that any sequence of conformally invariant operators acting among global sections of the same natural bundles on the sphere as the BGG sequence does necessarily form a complex. It is a consequence of representational theoretical properties of the spaces of global sections and no specific information concerning a form of invariant operators is needed.

Secondly, it is shown that exactness of the sequence on the sphere (up to the last place) is equivalent to certain spectral properties of corresponding invariant operators. It opens a possibility that there can be a direct verification of the mentioned spectral properties of corresponding invariant operators which would give a simple and elementary proof of the exactness of the BGG resolution on the sphere. Moreover — in fact this was the original aim of the study — the equivalence can be used in other direction for explicit computation of a form of kernels and images of all standard operators on the sphere.

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In Section 2 we summarize the representation theoretical tools used in later computations. Then using exactness property of BGG sequences, we determine in Section 3 the kernels and images of higher spin twistor operators on sphere. The language of complexes in Section 4 allow us to formulate conjecture relating spectral properties of standard invariant operators and exactness of BGG sequences. This conjecture is then proved for the even case in subsection 5.1, and for the odd case in subsection 5.2.

2. Branching rules and Frobenius reciprocity

Let V_λ be an irreducible H-module with highest weight λ and let V_α be an irreducible G-module with a highest weight α. The nonnegative integer number

$$dim\, \mathsf{Hom}_H(V_\lambda, V_\alpha\,|_H) \in \mathbb{N}_0 \tag{1}$$

is the dimension of H-equivariant homomorphisms of H-modules. The rules describing possible targets and computing their dimension are called the **branching rules**. In the paper [6], branching rules are described in cases which will be needed in our work. Let $G = Spin(n+1)$ and $H = Spin(n)$. Then

$$dim\, \mathsf{Hom}_{Spin(n)}(V_\lambda, V_\alpha\,|_{Spin(n)}) \in \{0,1\}, \tag{2}$$

i.e. either a given irreducible $Spin(n)$-module V_λ is present in the decomposition with multiplicity 1 or it is not present at all.

The case $dim\, \mathsf{Hom}_{Spin(n)}(V_\lambda, V_\alpha\,|_{Spin(n)}) = 1$ happens iff

- n is odd,n=2l+1

$$\alpha_1 - \lambda_1 \in \mathbb{Z} \ \wedge\ \alpha_1 \geq \lambda_1 \geq \alpha_2 \geq \cdots \geq \lambda_l \geq |\alpha_{l+1}|; \tag{3}$$

- n is even,n=2l

$$\alpha_1 - \lambda_1 \in \mathbb{Z} \ \wedge\ \alpha_1 \geq \lambda_1 \geq \alpha_2 \geq \cdots \geq \lambda_{l-1} \geq \alpha_l \geq |\lambda_l|. \tag{4}$$

We shall use the notation $\alpha \nearrow \lambda$ or $\lambda \searrow \alpha$ if the highest weights λ and α are related through (3) or (4). These rules are also the basic ingredient for the decomposition of induced representations.

Theorem 2.1 (Frobenius reciprocity theorem)**.** *Let λ, α be the highest weight of $Spin(n)$, $Spin(n+1)$, respectively. Then there is a bijection between the set of homomorphisms of $Spin$-modules,*

$$\mathsf{Hom}_{Spin(n+1)}(V_\alpha, Ind_{Spin(n)}^{Spin(n+1)} V_\lambda) \simeq \mathsf{Hom}_{Spin(n)}(V_\alpha\,|_{Spin(n)}, V_\lambda). \tag{5}$$

The symbol $Ind_{Spin(n)}^{Spin(n+1)} V_\lambda$ denotes the representation **induced** by V_λ. In terms of associated vector bundles, the induced representation is the space of $Spin(n+1)$-finite sections of associated $Spin(n)$ vector bundle $\mathcal{V}_\lambda$ on the n-dimensional sphere $S^n \simeq Spin(n+1)/Spin(n)$. Using the bijection (5), the space $\oplus_{\alpha \in Spin(n+1)} V_{\alpha \searrow \lambda}$ decomposes to $Spin(n+1)$-modules V_α via branching rules (3) or (4) (depending on the parity of the dimension considered).

3. Higher twistor operators on spheres

In this section we shall investigate **higher spin** analogs of the basic twistor operator. We shall divide the discussion into two cases, distinguished by the parity of dimension. For the representation theoretical notation of parabolic subalgebras and corresponding Dynkin diagrams, see [21], [9], [19], [24].

Notation 3.1. *In our case, the homogeneous space will be $S^n \simeq G/P \simeq G/MAN$, where $G = Spin(n+1,1,\mathbb{R})$ and P is the maximal parabolic subgroup with Langlands decomposition corresponding to $M = Spin(n,\mathbb{R})$. The maximal compact subgroup of G is $K = Spin(n+1,\mathbb{R})$.*

The property of exactness of BGG sequence (in much more wider context of curved analogs of flat homogeneous models) is discussed in [10]. The pictures of BGG sequences look as follows:

- $n = 2l$

$$0 \longrightarrow C_{-1} \xrightarrow{i} C_0 \xrightarrow{D_0} \dots \qquad C_{l-1} \xrightarrow{D_{l-1}} C_l^+ \oplus C_l^- \xrightarrow{D_l} C_{l+1} \xrightarrow{D_{l+1}} \dots$$

$$C_{2l} \xrightarrow{\pi} C_{2l+1} \longrightarrow 0$$

- $n = 2l+1$

$$0 \longrightarrow C_{-1} \xrightarrow{i} C_0 \xrightarrow{D_0} \dots \qquad C_{l-1} \xrightarrow{D_{l-1}} C_l \xrightarrow{D_l} C_{l+1} \xrightarrow{D_{l+1}} \dots$$

$$C_{2l+1} \xrightarrow{\pi} C_{2l+2} \longrightarrow 0$$

The meaning of particular symbols on these pictures is summarized in the following exposition:

$$
\begin{aligned}
& n = 2l \ : \\
& C_v \, , \, 0 \leq v \leq 2l \ \ , \quad D_v \, , \, 0 \leq v \leq 2l-1 \, , \\
& C_{-1} \equiv \mathbb{V}_{(\lambda_1+k-1,\lambda_1,\dots,\lambda_l)_G} \, , \\
& C_0 \equiv \Gamma(\mathcal{V}_{(\lambda_1,\dots,\lambda_l)_M}) \, , \\
& C_1 \equiv \Gamma(\mathcal{V}_{((\lambda_1+k)_1,\lambda_2,\dots,\lambda_l)_M}) \, , \\
& D_0 \equiv D_{(\lambda_1,\dots,\lambda_l)_M,e_1,k} \, , \\
& C_{v-1} \equiv \Gamma(\mathcal{V}_{(\lambda_1+k,\lambda_1+1,\lambda_2+1,\dots,(\lambda_{v-2}+1)_{v-1},\lambda_v,\dots,\lambda_l)_M}) \, , \\
& D_{v-1} \equiv D_{(\lambda_1+k,\lambda_1+1,\dots,(\lambda_{v-2}+1)_{v-1},\lambda_v,\dots,\lambda_l)_M,e_v,(\lambda_{v-1}-\lambda_v+1)} \, , \\
& C_l^+ \equiv \Gamma(\mathcal{V}_{(\lambda_1+k,\lambda_1+1,\lambda_2+1,\dots,(\lambda_{l-2}+1)_{l-1},(\lambda_{l-1}+1)_l)_M}) \, , \\
& C_l^- \equiv \Gamma(\mathcal{V}_{(\lambda_1+k,\lambda_1+1,\lambda_2+1,\dots,(\lambda_{l-2}+1)_{l-1},-(\lambda_{l-1}+1)_l)_M}) \, , \\
& C_l \equiv C_l^+ \oplus C_l^- \, ; \\
& n = 2l+1 \ : \\
& C_v \, , \, 0 \leq v \leq 2l+1 \ \ , \quad D_v \, , \, 0 \leq v \leq 2l \, , \\
& C_l \simeq C_{l+1} \, . \\
& D_l \equiv D_{(\lambda_1+k,\lambda_1+1,\dots,\lambda_l+1)_M,0,2\lambda_l+1} \, .
\end{aligned}
\tag{6}
$$

The previous notation comes from [9]. The algebra $\mathcal{G}$ is $|1|$-graded, $\mathcal{G} \simeq \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$. The commutative subalgebra $\mathcal{G}_1$ is a $\mathcal{G}_0^s$-module, where $\mathcal{G}_0^s$ is the semisimple part of reductive subgroup $\mathcal{G}_0$, and the corresponding extremal weights are denoted by e_i. The positive natural number k is the order of invariant differential operator.

We add a few remarks useful in later computations.

Remark 3.2. *The BGG-sequence in the even case $n = 2l$ and $G = Spin(2l+1,1,\mathbb{R})$-module $\lambda = (\frac{2k-1}{2}{}_1, \frac{1}{2}{}_2, \dots, \frac{1}{2}{}_{l+1})_G$ is fully characterized by the values of $\lambda+\delta$ inscribed over the nodes of the first crossed Dynkin diagram* (*compare to* [21])*:*

$$b = k \, , \, d_1 = 1 \, , \, \dots \, , \, d_{l-1} = 1 \, , \, a = 2 \, , \, c = 1.$$

Remark 3.3. *The BGG-sequence in the odd case $n = 2l+1$ and $G = Spin(2l+2,1,\mathbb{R})$-module $\lambda = (\frac{2k-1}{2}{}_1, \frac{1}{2}{}_2, \dots, \frac{1}{2}{}_{l+1})_G$ is fully characterized by the values of $\lambda+\delta$ inscribed over the nodes of the first crossed Dynkin diagram* (*compare* [21])*:*

$$b = k \, , \, d_1 = 1 \, , \, \dots \, , \, d_{l-1} = 1 \, , \, a = 2.$$

Remark 3.4. *We use in all computations the spaces of K-finite sections $\Gamma_K(\mathcal{V}_\lambda)$. The spaces of all sections $\Gamma(\mathcal{V}_\lambda)$ are then completions of $\Gamma_K(\mathcal{V}_\lambda)$ with respect to any scalar product, so $\Gamma_K(\mathcal{V}_\lambda)$ are dense subspaces of $\Gamma(\mathcal{V}_\lambda)$, see* [20]. *From the representational theoretic point of view, these spaces have the same content and so we use the unified notation $\Gamma(\mathcal{V}_\lambda)$.*

In this section, we shall exclusively specialize to the case $\lambda_M \equiv (\lambda_1, \dots, \lambda_l)_M = (\frac{1}{2}, \dots, \frac{1}{2})_M$. Note that in the even case $n = 2l$, the discussion and the results concerning the highest weight $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})_M$ of the second half $Spin(n)$-module are the same as in the case of $(\frac{1}{2}, \dots, \frac{1}{2})_M$ and we shall not repeat it.

Lemma 3.5. *Let $n = 2l$ and let $\lambda_M = (\frac{1}{2}, \dots, \frac{1}{2})_M$. The kernel of k-th order invariant differential operator D_0 (corresponding to k-th spin twistor operator),*

$$D_0 : C_0 \to C_1, \tag{7}$$

is given by a sum of irreducible $K = Spin(2l+1)$-modules:

$$\mathsf{Ker} D_0 \simeq \oplus_{m=1}^{k} (\frac{2m-1}{2}{}_1, \frac{1}{2}{}_2, \dots, \frac{1}{2}{}_l)_K. \tag{8}$$

The image of this higher twistor operator is

$$\mathsf{Im} D_0 \simeq \oplus_{m=k}^{\infty} (\frac{2m+1}{2}{}_1, \frac{1}{2}{}_2, \dots, \frac{1}{2}{}_l)_K. \tag{9}$$

Proof: First of all, we decompose the source and target spaces of $M = Spin(n)$-valued sections over $S^n \simeq K/M$ on $K = Spin(n+1)$-types. The use of Frobenius reciprocity and branching rules for $n = 2l$ (4) gives the possible target $Spin(n+1)$-irreducible modules:

$$C_0 \simeq \oplus_{m=0}^{\infty} (\frac{2m+1}{2}{}_1, \frac{1}{2}{}_2, \dots, \frac{1}{2}{}_l)_K\,, \tag{10}$$

$$C_1 \simeq \oplus_{p=0}^{k} \oplus_{m=k}^{\infty} \{(\frac{2m+1}{2}{}_1, \frac{2p+1}{2}{}_2, \frac{1}{2}{}_3, \dots, \frac{1}{2}{}_l)_K\}. \tag{11}$$

Rewriting the definition of D_0 in terms of homomorphism of irreducible $K = Spin(n+1)$-modules,

$$\oplus_{m=0}^{\infty} (\frac{2m+1}{2}{}_1, \frac{1}{2}{}_2, \dots, \frac{1}{2}{}_l)_K \xrightarrow{D_0} \oplus_{p=0}^{k} \oplus_{m=k}^{\infty} \{(\frac{2m+1}{2}{}_1, \frac{2p+1}{2}{}_2, \frac{1}{2}{}_3, \dots, \frac{1}{2}{}_l)_K\}, \tag{12}$$

we see that the $Spin(n+1)$-modules $(\frac{2m+1}{2}{}_1, \frac{1}{2}{}_2, \dots, \frac{1}{2}{}_l)_K,\ m \in \{0, \dots, k-1\}$, are present in the source space of K-types, but they are missing in the target

space of D_0. These irreducible modules are hence necessarily in the kernel of D_0. The operator D_0 acts by Schur Lemma as a multiple of the identity map between the irreducible K-modules which appear on both sides of (12). However, we do not know whether a multiplication constant on a given irreducible K-module appearing on both sides of (12) is zero or not. This information can be extracted from the BGG-sequence for a suitable $G = Spin(n+1, 1, \mathbb{R})$-module (for information on BGG sequence, see [21]).
The first three terms of the BGG-resolution for the representation of $G = Spin(n+1, 1, \mathbb{R})$ with the dominant weight $(\frac{2k-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_{l+1})_G$ in even dimensions $n = 2l$ are:

$$\mathbb{V}_{(\frac{2k-1}{2}{}_1, \frac{1}{2}{}_2 \ldots, \frac{1}{2}{}_{l+1})_G} \xrightarrow{i} \underset{C_0}{\bullet} \xrightarrow{D_0} \underset{C_1}{\bullet} \xrightarrow{D_1} \ldots$$

The operator i denotes an embedding of the G-module $(\frac{2k-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_{l+1})_G$ into the space of $P \equiv \{(\frac{1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_l)_M, k - \frac{1}{2}\}_P$-valued sections over the homogeneous space G/P. From the exactness of the BGG sequence it follows that

$$\mathsf{Ker} D_0 \simeq \mathsf{Im}\, i \tag{13}$$

is an irreducible G-module with highest weight $(\frac{2k-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_{l+1})_G$. Applying repeatedly the odd case of branching rules (3) for the even case, $n = 2l$, to the couple $(G, K) = (Spin(2l+2, \mathbb{R}), Spin(2l+1, \mathbb{R}))$ and the dominant weight $(\frac{2k-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_{l+1})_G$, we get the inequality for possible highest weights λ_K of K,

$$(\frac{2k-1}{2} - \lambda_1) \in \mathbb{Z} \wedge \frac{2k-1}{2} \geq \lambda_1 \geq \frac{1}{2} \cdots \geq \lambda_l \geq \frac{1}{2}\,. \tag{14}$$

This implies the appearance of k irreducible K-types $\lambda_m = (\frac{2m-1}{2}{}_1, \ldots, \frac{1}{2}{}_l)_K$, $m \in \{1, \ldots, k\}$, in the decomposition of $(\frac{2k-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_{l+1})_G$ on K-types,

$$(\frac{2k-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_{l+1})_G \xrightarrow{G \searrow K} \oplus_{m=1}^{k} (\frac{2m-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_l)_K, \tag{15}$$

and this finally proves the assertion of the lemma. □

Lemma 3.6. *Let $n = 2l + 1$ and let $\lambda_M = (\frac{1}{2}, \dots, \frac{1}{2})_M$. The kernel of the k-th order invariant differential operator D_0 (corresponding to the k-th spin twistor operator),*

$$D_0 : C_0 \to C_1, \tag{16}$$

is given by the collection of irreducible $K = Spin(2l+2)$-modules:

$$\mathsf{Ker} D_0 \simeq \oplus_{m=1}^{k} \{ (\frac{2m-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_{l+1})_K \oplus (\frac{2m-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_l, -\frac{1}{2}_{l+1})_K \}. \tag{17}$$

Similarly, the image of D_0 is

$$\mathsf{Im} D_0 \simeq \oplus_{j \in \{-1,1\}} \oplus_{m=k}^{\infty} \{ (\frac{2m+1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_l, \frac{j}{2}_{l+1})_K \}. \tag{18}$$

Proof: The structure of the proof is the same as in the even case $n = 2l$. We have

$$\begin{aligned} C_0 &\simeq \oplus_{j \in \{-1,1\}} \oplus_{m=0}^{\infty} \{ (\frac{2m+1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_l, \frac{j}{2}_{l+1})_K \}, \\ C_1 &\simeq \oplus_{j \in \{-1,1\}} \oplus_{p=0}^{k} \oplus_{m=k}^{\infty} \{ (\frac{2m+1}{2}_1, \frac{2p+1}{2}_2, \frac{1}{2}_3, \dots, \frac{1}{2}_{l-1}, \frac{j}{2}_l)_K \}. \end{aligned} \tag{19}$$

The $Spin(n+1)$-modules $(\frac{2m+1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_{l-1}, \frac{j}{2}_l)_K$, $m \in \{0, \dots, k-1\}, j \in \{-1, 1\}$, are the only K-modules present in the source space of K-types but missing in the target space of K-types. The first three terms of the BGG sequence for fundamental spinor representation of $Spin(2l+1)$ can be represented by similar picture to the one in the previous even case. In the odd case, $n = 2l+1$, we use the even case of branching rules (4) on the couple $(G, K) = (Spin(2l+3, \mathbb{R}), Spin(2l+2, \mathbb{R}))$ and irreducible G-module with highest weight $(\frac{2k-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_{l+1})_G$, and we get

$$\begin{aligned} (\frac{2k-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_{l+1})_G &\xrightarrow{G \searrow K} \\ &\oplus_{m=1}^{k} \{ (\frac{2m-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_{l+1})_K \\ &\qquad \oplus (\frac{2m-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_l, -\frac{1}{2}_{l+1})_K \}. \end{aligned} \tag{20}$$

□

The results of presented computations can be summarized in the following assertion.

Theorem 3.7. *Let D (without any subscript) be a higher spin twistor operator. Then it holds (we shall refer to this property as the $\star$-property):*
($\star$) If a given K-type appears both in the source space and in the target space of higher twistor operator D, then D is injective on this K-type.

The property proved in Theorem 3.7 makes it possible to find Ker and Im of an invariant differential operator D. Moreover, the only crucial property of a BGG sequence, used to derive it, is its exactness. It is also easy to check this property is, in fact, equivalent to exactness of the BGG resolution in the first place. It is then natural to make the following conjecture, saying, that the relation between exactness of the BGG sequence and the property ($\star$) proved in Theorem 3.7, can be true for all standard invariant differential operators appearing in BGG sequence.

Conjecture 3.8. *The statements* (1) *and* (2) *are* **equivalent***:*

(1) *BGG resolution is exact sequence;*
(2) *Theorem* 3.7 *holds true for all standard invariant differential operators.*

This implies that the information concerning the behavior of homomorphisms of P-modules inside a BGG sequence of a given G-module — expressed for example in terms of properties of invariant differential operators on K-types — is an alternative way how to prove exactness of a resolvent of P-modules with standard invariant differential operators acting among them.

4. BGG resolutions and complexes

We shall consider here sequences of linear maps D_i acting between vector spaces C_i,

$$\dots \xrightarrow{D_{i-2}} C_{i-1} \xrightarrow{D_{i-1}} C_i \xrightarrow{D_i} C_{i+1} \xrightarrow{D_{i+1}} \dots . \tag{21}$$

They will usually be complexes, i.e. the composition of any two successive homomorphisms is zero,

$$D_{i+1} \circ D_i = 0 \,,\ \forall\, i. \tag{22}$$

In our case, the vector spaces C_i will be direct sums of irreducible K-modules (K-finite sections of associated homogeneous vector bundles over homogeneous space G/P), and the operators acting between any two neighboring vector spaces will be invariant differential operators. In particular, let us consider a BGG sequence (see [21], [10]). Let $i = 0, \dots, 2l-1$, with exception in the even case $n = 2l,\ i \neq l$.

Definition 4.1. *We denote*

$$C^i_{i,\,i+1} := \oplus_{\lambda \in A} V_\lambda \,,\ V_\lambda \in C_i \,, \tag{23}$$

$$C^{i+1}_{i,\,i+1} := \oplus_{\lambda\in A} V_\lambda\,,\; V_\lambda \in C_{i+1}, \tag{24}$$

where A is the set of all K-types appearing both in the spaces C_i and C_{i+1} and lying in the space C_i, C_{i+1}, respectively.

The structure of the middle part of a BGG sequence is different from the remaining part. Let us first consider the even case $n = 2l$.

Definition 4.2. *The set of K-types $C^l_{l-1,\,l}$ respectively $C^l_{l,\,l+1}$ is defined by the following two conditions:*

1. $\mathsf{Im} D_{l-1} \subset C^l_{l-1,\,l} \subset \mathsf{Ker} D_l$;
2. $C^l_{l-1,\,l} \simeq \oplus_K \alpha_K$, *such that every K-type appearing in this decomposition has multiplicity* 1.

The space $C^l_{l,\,l+1}$ is defined as the complement of the set of K-types $C^l_{l-1,\,l}$ in C_l, i.e. $C_l \simeq C^l_{l-1,\,l} \oplus C^l_{l,\,l+1}$. In particular, every K-type appearing in its decomposition has also multiplicity one.

The middle part of the odd case $n = 2l + 1$ also requires a more careful treatment.

Definition 4.3. *The sets of K-types $C^l_{l-1,\,l}$ respectively $C^{l+1}_{l,\,l+1}$ are defined by:*

1. $C^l_{l-1,\,l} \subset \mathsf{Ker}\ \mathsf{D}_l$;
2. $\mathsf{Im}\ \mathsf{D}_l \subset C^{l+1}_{l,\,l+1}$.

In the following theorems we shall discuss the distribution of K-types inside particular terms of a BGG sequence.

Theorem 4.4. *It holds true*

$$\begin{aligned} C_i \setminus C^i_{i,i+1} &\subset \mathsf{Ker} D_i, \\ \mathsf{Im} D_i &\subset C^{i+1}_{i,i+1}. \end{aligned} \tag{25}$$

Proof: Any K-type, which is present in the source space C_i but is not present in the target space C_{i+1}, must lie in the kernel of the operator D_i acting between them. On the other hand, the image of D_i must be included in the union of K-types belonging both to C_i and C_{i+1}. □

The assertion of the previous theorem is rather trivial. But it is much more less obvious, whether the previous inclusions are proper or not. A key property of the sequences studied is stated in the following theorem.

Theorem 4.5. *It holds true for all i,*

$$C_i = C^i_{i-1,i} \oplus C^i_{i,i+1}. \tag{26}$$

The proof of this theorem is the heart of this section, it will be given in next sections. The assertion of this theorem is illustrated by the figure:

$$\begin{array}{ccccc}
\boxed{\mathsf{Ker} D_i} & & & & \\
\boxed{\mathsf{Im} D_i} & \xrightarrow{D_i} & \boxed{\mathsf{Ker} D_{i+1}} & & \\
C_i & & \boxed{\mathsf{Im} D_{i+1}} & \xrightarrow{D_{i+1}} & \boxed{\mathsf{Ker} D_{i+2}} \\
 & & C_{i+1} & & \boxed{\mathsf{Im} D_{i+2}} \\
 & & & & C_{i+2}
\end{array}$$

This theorem has several direct consequences.

Corollary 4.6. *The following three conditions hold true.*

1.
$$\mathsf{Im} D_i \subset C^{i+1}_{i,i+1} \subset \mathsf{Ker} D_{i+1},\ \forall\, i.$$
2. *The BGG sequence is a complex.*
3. *There is a set of equivalent conditions:*
$\mathsf{Ker} D_i = C^i_{i-1,i} \Longleftrightarrow \mathsf{Im} D_i = C^{i+1}_{i,i+1} \Longleftrightarrow (\star)$ *of* 3.7 *holds true for* i.

Proof: 1. From Theorem 4.5 we know that $C^{i+1}_{i,i+1} = C_{i+1} \setminus C^{i+1}_{i+1,i+2}$, and from Theorem 4.4 it follows $C_{i+1} \setminus C^{i+1}_{i+1,i+2} \subset \mathsf{Ker} D_{i+1}$. Putting this together,

$$C^{i+1}_{i,i+1} \subset \mathsf{Ker} D_{i+1}. \tag{28}$$

The second inclusion is the content of Theorem 4.4.

2. It is an immediate consequence of the first condition, because from $\mathsf{Im} D_i \subset \mathsf{Ker} D_{i+1}$ (for all i) it follows that the composition of two consecutive operators is an identically zero operator.

3. Using the first part, i.e. Theorems 4.4 and 4.5, one can see that $\mathsf{Ker} D_i = C^i_{i-1,i}$ holds true if $\mathsf{Ker} D_i \cap C^i_{i,i+1} = 0$ and this is equivalent to condition $(\star)$ of 3.7, and moreover this is also equivalent to $\mathsf{Im} D_i = C^{i+1}_{i,i+1}$. □

Theorem 4.7. *The property* $(\star)$ *of Theorem* 3.7 *is true for all operators* $D_i \iff \forall\, i$, $\mathsf{Ker} D_i = C^i_{i-1,i} \iff \forall\, i$, $\mathsf{Im} D_i = C^{i+1}_{i,i+1}$.

PROOF: The proof follows from the third assertion of Corollary 4.6. □

Theorem 4.8. *A BGG sequence is an exact sequence iff the property* $(\star)$ *holds true for all* i.

PROOF: The proof follows from the first and the third assertion of Corollary 4.6. □

5. BGG sequences on spheres S^n

The assertion of Theorem 4.5 will be proved now. We shall divide the discussion into two separate parts — the even case $n = 2l$ and the odd case $n = 2l+1$. In both cases, we shall use the BGG sequence slightly different from the one presented in [21], in the sense that we add at the beginning of the resolvent the kernel of the first invariant differential operator, and we add at the end the cokernel of the last invariant differential operator.

5.1 BGG sequences on even dimensional spheres.

The situation in the even case $n = 2l$ is described in the following picture:

$$0 \longrightarrow C_{-1} \xrightarrow{i} C_0 \xrightarrow{D_0} \dots \qquad C_{l-1} \xrightarrow{D_{l-1}} C_l^+ \oplus C_l^- \xrightarrow{D_l} C_{l+1} \xrightarrow{D_{l+1}} \dots$$

$$C_{2l} \xrightarrow{\pi} C_{2l+1} \longrightarrow 0$$

Theorem 5.1. *Let* $n = 2l$ *and* $v \in \mathbb{Z}$ *be an integer fulfilling* $0 \le v \le 2l$. *Then there holds true the direct sum decomposition of the* C_v*-th term of the BGG sequence:*

$$C_v \simeq C^v_{v-1,\,v} \oplus C^v_{v,\,v+1}. \tag{29}$$

The definition of the middle, respectively terminal parts, i.e. C_l, *respectively* C_{-1} *and* C_{2l+1}, *has been given in the previous section. Note that the mapping* i *in* *the last picture denotes the embedding of the G-module C_{-1} into C_0; similarly the map π denotes the projection with kernel* $\mathsf{Im}D_{2l-1}$.

Proof: Let us consider the C_v-th term of the BGG sequence, $0 \leq v \leq (l-1)$ or $l+1 \leq v \leq 2l$. The space C_v is the space of sections of the vector bundle associated to the M-module with highest weight $(\lambda_1+k, \lambda_1+1, \lambda_2+1, \ldots, (\lambda_{v-1}+1)_v, \lambda_{v+1}, \ldots, \lambda_l)_M$. In order to decompose C_v into $K = Spin(2l+1)$-types, we use the branching rules for even case $n = 2l$ (4),

$$\alpha_1 - \lambda_1 \in \mathbb{Z} \ \wedge \ \alpha_1 \geq \lambda_1 + k \geq \alpha_2 \geq \lambda_1 + 1 \geq \alpha_3 \geq \lambda_2 + 1 \cdots \geq \alpha_v \geq \lambda_{v-1} + 1 \\ \geq \alpha_{v+1} \geq \lambda_{v+1} \geq \alpha_{v+2} \cdots \geq \lambda_{l-1} \geq \alpha_l \geq |\lambda_l|,$$

with the result:

$$\begin{aligned} C_v \simeq & \oplus_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}}^{\infty} \oplus_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}}^{\lambda_1+k} \\ & \oplus_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}}^{\lambda_1+1} \cdots \oplus_{\alpha_v=\lambda_{v-1}+1\,,\,\alpha_v-\lambda_v\in\mathbb{Z}}^{\lambda_{v-2}+1} \oplus_{\alpha_{v+1}=\lambda_{v+1}\,,\,\alpha_{v+1}-\lambda_{v+1}\in\mathbb{Z}}^{\lambda_{v-1}+1} \\ & \oplus_{\alpha_{v+2}=\lambda_{v+2}\,,\,\alpha_{v+2}-\lambda_{v+2}\in\mathbb{Z}}^{\lambda_{v+1}} \cdots \oplus_{\alpha_{l-1}=\lambda_{l-1}\,,\,\alpha_{l-1}-\lambda_{l-1}\in\mathbb{Z}}^{\lambda_{l-2}} \\ & \oplus_{\alpha_l=sgn(\lambda_l)\lambda_l\,,\,\alpha_l-\lambda_l\in\mathbb{Z}}^{\lambda_{l-1}} (\alpha_1, \ldots, \alpha_l)_K. \end{aligned} \tag{30}$$

Similar decomposition can be done immediately for C_{v-1} respectively for C_{v+1}. The sets of K-types common with C_v are

(31)

$$\begin{aligned} C_{v-1,v}^v \simeq & \oplus_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}}^{\infty} \oplus_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}}^{\lambda_1+k} \oplus_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}}^{\lambda_1+1} \cdots \\ & \oplus_{\alpha_{v-1}=\lambda_{v-2}+1\,,\,\alpha_{v-1}-\lambda_{v-1}\in\mathbb{Z}}^{\lambda_{v-3}+1} \oplus_{\alpha_v=\lambda_{v-1}+1\,,\,\alpha_v-\lambda_v\in\mathbb{Z}}^{\lambda_{v-2}+1} \oplus_{\alpha_{v+1}=\lambda_{v+1}\,,\,\alpha_{v+1}-\lambda_{v+1}\in\mathbb{Z}}^{\lambda_v} \\ & \cdots \oplus_{\alpha_{l-1}=\lambda_{l-1}\,,\,\alpha_{l-1}-\lambda_{l-1}\in\mathbb{Z}}^{\lambda_{l-2}} \oplus_{\alpha_l=sgn(\lambda_l)\lambda_l\,,\,\alpha_l-\lambda_l\in\mathbb{Z}}^{\lambda_{l-1}} (\alpha_1, \ldots, \alpha_l)_K, \end{aligned}$$

and

(32)

$$\begin{aligned} C_{v,v+1}^v \simeq & \oplus_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}}^{\infty} \oplus_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}}^{\lambda_1+k} \oplus_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}}^{\lambda_1+1} \cdots \\ & \oplus_{\alpha_v=\lambda_{v-1}+1\,,\,\alpha_v-\lambda_v\in\mathbb{Z}}^{\lambda_{v-2}+1} \oplus_{\alpha_{v+1}=\lambda_v+1\,,\,\alpha_{v+1}-\lambda_{v+1}\in\mathbb{Z}}^{\lambda_{v-1}+1} \oplus_{\alpha_{v+2}=\lambda_{v+2}\,,\,\alpha_{v+2}-\lambda_{v+2}\in\mathbb{Z}}^{\lambda_{v+1}} \\ & \cdots \oplus_{\alpha_{l-1}=\lambda_{l-1}\,,\,\alpha_{l-1}-\lambda_{l-1}\in\mathbb{Z}}^{\lambda_{l-2}} \oplus_{\alpha_l=sgn(\lambda_l)\lambda_l\,,\,\alpha_l-\lambda_l\in\mathbb{Z}}^{\lambda_{l-1}} (\alpha_1, \ldots, \alpha_l)_K. \end{aligned}$$

The comparison with (30) then implies

$$C_v \simeq C_{v-1,v}^v \oplus C_{v,v+1}^v, \tag{33}$$

which proves the assertion of the theorem for the specified region of index v.

Let us discuss now the middle part of the BGG sequence. The space of sections of the vector bundle associated to the reducible M-module with highest weight

$(\lambda_1 + k, \lambda_1 + 1, \dots, (\lambda_{l-2} + 1)_{l-1}, (\lambda_{l-1} + 1)_l)_M \oplus (\lambda_1 + k, \lambda_1 + 1, \dots, (\lambda_{l-2} + 1)_{l-1}, -(\lambda_{l-1} + 1)_l)_M$, which appears in the middle term $C_l \simeq C_l^+ \oplus C_l^-$, can be also decomposed into K-types

$$C_l \simeq C_l^+ \oplus C_l^- \simeq 2\{\oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}} \cdots \tag{34}$$
$$\cdots \oplus^{\lambda_{l-3}+1}_{\alpha_{l-1}=\lambda_{l-2}+1\,,\,\alpha_{l-1}-\lambda_{l-1}\in\mathbb{Z}} \oplus^{\lambda_{l-2}+1}_{\alpha_l=sgn(\lambda_{l-1}+1)(\lambda_{l-1}+1)\,,\,\alpha_l-\lambda_l\in\mathbb{Z}} (\alpha_1, \dots, \alpha_l)_K\},$$

with the result that every K-type appearing in the decomposition (34) has multiplicity 2. Note that due to 4.2, it follows that the spaces $C^l_{l-1,\,l}$ and $C^l_{l,\,l+1} \simeq C_l/C^l_{l-1,\,l}$ are isomorphic as K-modules,

$$\begin{aligned} C^l_{l,\,l+1} \simeq C^l_{l-1,\,l} \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}} \\ \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-4}+1}_{\alpha_{l-2}=\lambda_{l-3}+1\,,\,\alpha_{l-2}-\lambda_{l-2}\in\mathbb{Z}} \\ \oplus^{\lambda_{l-3}+1}_{\alpha_{l-1}=\lambda_{l-2}+1\,,\,\alpha_{l-1}-\lambda_{l-1}\in\mathbb{Z}} \oplus^{\lambda_{l-2}+1}_{\alpha_l=\lambda_{l-1}+1\,,\,\alpha_l-\lambda_l\in\mathbb{Z}} (\alpha_1, \dots, \alpha_l)_K, \end{aligned} \tag{35}$$

so that

$$C_l \simeq C^l_{l-1,\,l} \oplus C^l_{l,\,l+1}. \tag{36}$$

Note that in comparison with the BGG sequence introduced in [21], we have included in it the kernel of the first homomorphism D_0 corresponding to the injection of K-modules coming from the decomposition of G-module $(\lambda_1 + k - 1, \lambda_1, \dots, \lambda_l)_G$ on K-types, and we also added the last projection corresponding to the cokernel of the homomorphism D_{2l-1} (equal to the quotient space of $C_{2l}/\mathsf{Im}D_{2l-1}$). The corresponding injection respectively projection were denoted i respectively π in the introductory picture of this subsection. This completes the proof. □

Remark 5.2. *If we work with the BGG sequence as formulated in* [21], *then the finite dimensional set of K-types $C_{2l} \setminus C^{2l}_{2l-1,\,2l}$ is an obstruction for the BGG sequence to be an exact sequence on $2l$-th term. The presence of this set of K-types in C_{2l} which are not in the image of D_{2l-1} demonstrates the well-known fact, that the only one* (*co*)*homologically nontrivial dimension of the base space S^n* (*except for degree zero*) *is n.*

Corollary 5.3. *As an immediate consequence of the previous decomposition Theorem* 5.1 *we get a description of the kernel and the image of homomorphism D_{v-1},*

$$D_{v-1} : C_{v-1} \to C_v\,,\; v = 1, \dots, l, \tag{37}$$

(38)

$$\mathsf{Ker} D_{v-1} \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{v-3}+1}_{\alpha_{v-1}=\lambda_{v-2}+1\,,\,\alpha_{v-1}-\lambda_{v-1}\in\mathbb{Z}} \oplus^{\lambda_{v-1}}_{\alpha_v=\lambda_v\,,\,\alpha_v-\lambda_v\in\mathbb{Z}} \oplus^{\lambda_v}_{\alpha_{v+1}=\lambda_{v+1}\,,\,\alpha_{v+1}-\lambda_{v+1}\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-2}}_{\alpha_{l-1}=\lambda_{l-1}\,,\,\alpha_{l-1}-\lambda_{l-1}\in\mathbb{Z}} \oplus^{\lambda_{l-1}}_{\alpha_l=sgn(\lambda_l)\lambda_l\,,\,\alpha_l-\lambda_l\in\mathbb{Z}} (\alpha_1,\ldots,\alpha_l)_K,$$

(39)

$$\mathsf{Im} D_{v-1} \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{v-3}+1}_{\alpha_{v-1}=\lambda_{v-2}+1\,,\,\alpha_{v-1}-\lambda_{v-1}\in\mathbb{Z}} \oplus^{\lambda_{v-2}+1}_{\alpha_v=\lambda_{v-1}+1\,,\,\alpha_v-\lambda_v\in\mathbb{Z}} \oplus^{\lambda_v}_{\alpha_{v+1}=\lambda_{v+1}\,,\,\alpha_{v+1}-\lambda_{v+1}\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-2}}_{\alpha_{l-1}=\lambda_{l-1}\,,\,\alpha_{l-1}-\lambda_{l-1}\in\mathbb{Z}} \oplus^{\lambda_{l-1}}_{\alpha_l=sgn(\lambda_l)\lambda_l\,,\,\alpha_l-\lambda_l\in\mathbb{Z}} (\alpha_1,\ldots,\alpha_l)_K,$$

i.e. there is an identification

$$\mathsf{Ker} D_{v-1} \simeq C^{v-1}_{v-2,\,v-1}, \qquad \mathsf{Im} D_{v-1} \simeq C^{v-1}_{v-1,\,v}. \tag{40}$$

For $0 \le v \le (l-1)$*, there is an isomorphism (substitute* $D_{-1} := i$ *and* $D_{2l} := \pi$*):*

$$\mathsf{Ker} D_v \simeq \mathsf{Im} D_{v-1} \simeq \mathsf{Ker} D_{2l-v+1} \simeq \mathsf{Im} D_{2l-v}. \tag{41}$$

Concerning the middle part of the sequence, the kernel and image of D_l *have the form*

$$\mathsf{Im} D_l \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-4}+1}_{\alpha_{l-2}=\lambda_{l-3}+1\,,\,\alpha_{l-2}-\lambda_{l-2}\in\mathbb{Z}} \oplus^{\lambda_{l-3}+1}_{\alpha_{l-1}=\lambda_{l-2}+1\,,\,\alpha_{l-1}-\lambda_{l-1}\in\mathbb{Z}} \oplus^{\lambda_{l-2}+1}_{\alpha_l=\lambda_{l-1}+1\,,\,\alpha_l-\lambda_l\in\mathbb{Z}} (\alpha_1,\ldots,\alpha_l)_K, \tag{42}$$

and

$$\mathsf{Ker} D_l \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-4}+1}_{\alpha_{l-2}=\lambda_{l-3}+1\,,\,\alpha_{l-2}-\lambda_{l-2}\in\mathbb{Z}} \oplus^{\lambda_{l-3}+1}_{\alpha_{l-1}=\lambda_{l-2}+1\,,\,\alpha_{l-1}-\lambda_{l-1}\in\mathbb{Z}} \oplus^{\lambda_{l-2}+1}_{\alpha_l=\lambda_{l-1}+1\,,\,\alpha_l-\lambda_l\in\mathbb{Z}} (\alpha_1,\ldots,\alpha_l)_K. \tag{43}$$

In this case, one has the isomorphism of K*-modules:*

$$C^l_{l,\,l+1} \simeq \mathsf{Ker} D_l \simeq \mathsf{Im} D_{l-1} \simeq C^l_{l-1,\,l}. \tag{44}$$

PROOF: It is a consequence of Theorems 5.1 and 4.8. □

5.2 BGG sequence on odd dimensional sphere.

The odd dimensional BGG sequence ($n = 2l + 1$) has the following simple form:

$$0 \longrightarrow C_{-1} \xrightarrow{i} C_0 \xrightarrow{D_0} \cdots \quad C_{l-1} \xrightarrow{D_{l-1}} C_l \xrightarrow{D_l} C_{l+1} \xrightarrow{D_{l+1}} \cdots$$

$$C_{2l+1} \xrightarrow{\pi} C_{2l+2} \longrightarrow 0$$

Theorem 5.4. *Let $n = 2l+1$ and $v \in \mathbb{Z}$ be an integer fulfilling $0 \leq v \leq (2l+1)$. Then the term of the BGG sequence corresponding to C_v decomposes as follows:*

$$C_v \simeq C^v_{v-1,\, v} \oplus C^v_{v,\, v+1}. \tag{45}$$

The definition of the middle, respectively terminal parts, i.e. C_l, respectively C_{-1} and C_{2l+2}, has been explained in the previous section. The meaning of the mappings i and π is the same as in the even dimensional case (*see* 5.1).

Proof: The underlying M-module structure of C_v corresponds to highest weight $(\lambda_1 + k, \lambda_1 + 1, \lambda_2 + 1, \ldots, (\lambda_{v-1} + 1)_v, \lambda_{v+1}, \ldots, \lambda_l)_M$. In order to decompose C_v on $K = Spin(2l+2)$-types, we use the branching rules for the odd case $n = 2l+1$ (3),

$$\begin{aligned} \alpha_1 - \lambda_1 \in \mathbb{Z} \ \wedge \ \alpha_1 \geq \lambda_1 + k \geq \alpha_2 \geq \lambda_1 + 1 \geq \alpha_3 \geq \lambda_2 + 1 \cdots \geq \alpha_v \geq \lambda_{v-1} + 1 \\ \geq \alpha_{v+1} \geq \lambda_{v+1} \geq \alpha_{v+2} \geq \cdots \geq \lambda_{l-1} \geq \alpha_l \geq \lambda_l \geq |\alpha_{l+1}|, \end{aligned}$$

and we get:

$$\begin{aligned} C_v \simeq {} & \oplus^{\infty}_{\alpha_1 = \lambda_1 + k\,,\, \alpha_1 - \lambda_1 \in \mathbb{Z}} \oplus^{\lambda_1 + k}_{\alpha_2 = \lambda_1 + 1\,,\, \alpha_2 - \lambda_2 \in \mathbb{Z}} \\ & \oplus^{\lambda_1 + 1}_{\alpha_3 = \lambda_2 + 1\,,\, \alpha_3 - \lambda_3 \in \mathbb{Z}} \cdots \oplus^{\lambda_{v-2}+1}_{\alpha_v = \lambda_{v-1}+1\,,\, \alpha_v - \lambda_v \in \mathbb{Z}} \oplus^{\lambda_{v-1}+1}_{\alpha_{v+1} = \lambda_{v+1}\,,\, \alpha_{v+1} - \lambda_{v+1} \in \mathbb{Z}} \\ & \oplus^{\lambda_{v+1}}_{\alpha_{v+2} = \lambda_{v+2}\,,\, \alpha_{v+2} - \lambda_{v+2} \in \mathbb{Z}} \cdots \oplus^{\lambda_{l-2}}_{\alpha_{l-1} = \lambda_{l-1}\,,\, \alpha_{l-1} - \lambda_{l-1} \in \mathbb{Z}} \oplus^{\lambda_{l-1}}_{\alpha_l = \lambda_l\,,\, \alpha_l - \lambda_l \in \mathbb{Z}} \\ & \oplus^{\lambda_l}_{\alpha_{l+1} = -\lambda_l\,,\, \alpha_l - \lambda_l \in \mathbb{Z}} (\alpha_1, \ldots, \alpha_{l+1})_K. \end{aligned} \tag{46}$$

In the same way, one can decompose also the terms C_{v-1} and C_{v+1}. The set of

K-types common with C_v is

$$C^v_{v,v+1} \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{v-2}+1}_{\alpha_v=\lambda_{v-1}+1\,,\,\alpha_v-\lambda_v\in\mathbb{Z}} \oplus^{\lambda_{v-1}+1}_{\alpha_{v+1}=\lambda_v+1\,,\,\alpha_{v+1}-\lambda_{v+1}\in\mathbb{Z}} \oplus^{\lambda_{v+1}}_{\alpha_{v+2}=\lambda_{v+2}\,,\,\alpha_{v+2}-\lambda_{v+2}\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-1}}_{\alpha_l=\lambda_l\,,\,\alpha_l-\lambda_l\in\mathbb{Z}} \oplus^{\lambda_l}_{\alpha_{l+1}=-\lambda_l\,,\,\alpha_{l+1}-\lambda_l\in\mathbb{Z}} (\alpha_1,\dots,\alpha_{l+1})_K, \tag{47}$$

and

$$C^v_{v-1,v} \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{v-3}+1}_{\alpha_{v-1}=\lambda_{v-2}+1\,,\,\alpha_{v-1}-\lambda_{v-1}\in\mathbb{Z}} \oplus^{\lambda_{v-2}+1}_{\alpha_v=\lambda_{v-1}+1\,,\,\alpha_v-\lambda_v\in\mathbb{Z}} \oplus^{\lambda_v}_{\alpha_{v+1}=\lambda_{v+1}\,,\,\alpha_{v+1}-\lambda_{v+1}\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-1}}_{\alpha_l=\lambda_l\,,\,\alpha_l-\lambda_l\in\mathbb{Z}} \oplus^{\lambda_l}_{\alpha_{l+1}=-\lambda_l\,,\,\alpha_{l+1}-\lambda_l\in\mathbb{Z}} (\alpha_1,\dots,\alpha_{l+1})_K. \tag{48}$$

It is then easy to verify the direct sum decomposition

$$C_v \simeq C^v_{v-1,\,v} \oplus C^v_{v,\,v+1}, \tag{49}$$

which finally proves the assertion of the theorem. We shall work out in details the case of the middle operator D_l. The decomposition of $C_l \simeq C_{l+1}$ gives

$$C_l \simeq C_{l+1} \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{v-2}+1}_{\alpha_v=\lambda_{v-1}+1\,,\,\alpha_v-\lambda_v\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-3}+1}_{\alpha_{l-1}=\lambda_{l-2}+1\,,\,\alpha_{l-1}-\lambda_{l-1}\in\mathbb{Z}} \oplus^{\lambda_{l-2}+1}_{\alpha_l=\lambda_{l-1}+1\,,\,\alpha_l-\lambda_l\in\mathbb{Z}} \oplus^{\lambda_{l-1}+1}_{\alpha_{l+1}=-(\lambda_{l-1}+1)\,,\,\alpha_{l+1}-\lambda_l\in\mathbb{Z}} (\alpha_1,\dots,\alpha_{l+1})_K. \tag{50}$$

Definition 4.3 allows us to conclude

$$C^{l+1}_{l+1,\,l+2} \simeq C^l_{l-1,\,l} \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-3}+1}_{\alpha_{l-1}=\lambda_{l-2}+1\,,\,\alpha_{l-1}-\lambda_{l-2}\in\mathbb{Z}} \oplus^{\lambda_{l-2}+1}_{\alpha_l=\lambda_{l-1}+1\,,\,\alpha_l-\lambda_l\in\mathbb{Z}} \oplus^{\lambda_l}_{\alpha_{l+1}=-\lambda_l\,,\,\alpha_{l+1}-\lambda_l\in\mathbb{Z}} (\alpha_1,\dots,\alpha_{l+1})_K, \tag{51}$$

and

$$C^l_{l,\,l+1} \simeq C^{l+1}_{l,\,l+1} \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\,\alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\,\alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\,\alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-3}+1}_{\alpha_{l-1}=\lambda_{l-2}+1\,,\,\alpha_{l-1}-\lambda_{l-2}\in\mathbb{Z}} \oplus^{\lambda_{l-2}+1}_{\alpha_l=\lambda_{l-1}+1\,,\,\alpha_l-\lambda_l\in\mathbb{Z}} \{\oplus^{\lambda_{l-1}+1}_{\alpha_{l+1}=(\lambda_l+1)\,,\,\alpha_{l+1}-\lambda_l\in\mathbb{Z}} \oplus^{-(\lambda_l+1)}_{\alpha_{l+1}=-(\lambda_{l-1}+1)\,,\,\alpha_{l+1}-\lambda_l\in\mathbb{Z}}\}(\alpha_1,\dots,\alpha_{l+1})_K, \tag{52}$$

so that

$$C_l \simeq C^l_{l-1,\,l} \oplus C^l_{l,\,l+1}\,, \qquad C_{l+1} \simeq C^{l+1}_{l,\,l+1} \oplus C^{l+1}_{l+1,\,l+2}. \tag{53}$$

It is remarkable to note that we have an isomorphism of K-modules

$$C^{l+1}_{l+1,\,l+2} \simeq C^l_{l-1,\,l}, \tag{54}$$

and so the homomorphism D_l acts by 'flipping' of K-modules isomorphic to $C^l_{l-1,\,l}$ and $C^l_{l,\,l+1}$ with respect to the decomposition of homomorphism D_{l+1} on $C^{l+1}_{l+1,\,l+2}$ and $C^{l+1}_{l,\,l+1}$ (see the picture below):

$$\begin{tikzcd}
 & C^l_{l-1,\,l} & C^{l+1}_{l,\,l+1} & {} \\
{} \arrow[ur, "\mathrm{Im}\ \mathrm{D}_{l-1}"] & C^{l+1}_{l,\,l+1} \arrow[ur, "\mathrm{Im}\ \mathrm{D}_{l}"] & C^{l+1}_{l+1,\,l+2} \arrow[ur, "\mathrm{Im}\ \mathrm{D}_{l+1}"] & \\
C_{l-1} & C_l & C_{l+1} & C_{l+2}
\end{tikzcd}$$

The last term of the BGG sequence C_{2l+2} has the content of K-types

$$\begin{aligned} C_{2l+2} \simeq C_{2l+1} \setminus C^{2l+1}_{2l,\,2l+1} \simeq \oplus^{\lambda_1+k-1}_{\omega_1=\lambda_1\,,\,\omega_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1}_{\omega_2=\lambda_2\,,\,\omega_2-\lambda_2\in\mathbb{Z}} \cdots \\ \cdots \oplus^{\lambda_{i-1}}_{\omega_i=\lambda_i\,,\,\omega_i-\lambda_i\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-1}}_{\omega_l=\lambda_l\,,\,\omega_l-\lambda_l\in\mathbb{Z}} \oplus^{\lambda_l}_{\omega_{l+1}=-\lambda_l\,,\,\omega_{l+1}-\lambda_l\in\mathbb{Z}} (\omega_1,\ldots,\omega_{l+1})_K. \end{aligned} \tag{55}$$

□

Remark 5.5. *Note that the same remark as* 5.2 *in the even case holds true for* $n = 2l + 1$.

Corollary 5.6. *Let* $0 \le v \le (l-1)$ *or* $(l+1) \le v \le (2l+1)$*. Then*

$$\mathsf{Ker} D_v \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k,\, \alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1,\, \alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1,\, \alpha_3-\lambda_3\in\mathbb{Z}} \cdots \cdots \oplus^{\lambda_{v-2}+1}_{\alpha_v=\lambda_{v-1}+1,\, \alpha_v-\lambda_v\in\mathbb{Z}} \oplus^{\lambda_v}_{\alpha_{v+1}=\lambda_{v+1},\, \alpha_{v+1}-\lambda_{v+1}\in\mathbb{Z}} \oplus^{\lambda_{v+1}}_{\alpha_{v+2}=\lambda_{v+2},\, \alpha_{v+2}-\lambda_{v+2}\in\mathbb{Z}} \cdots \cdots \oplus^{\lambda_{l-1}}_{\alpha_l=\lambda_l,\, \alpha_l-\lambda_l\in\mathbb{Z}} \oplus^{\lambda_l}_{\alpha_{l+1}=-\lambda_l,\, \alpha_{l+1}-\lambda_l\in\mathbb{Z}} (\alpha_1,\dots,\alpha_{l+1})_K, \tag{56}$$

$$\mathsf{Im} D_v \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k,\, \alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1,\, \alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1,\, \alpha_3-\lambda_3\in\mathbb{Z}} \cdots \cdots \oplus^{\lambda_{v-2}+1}_{\alpha_v=\lambda_{v-1}+1,\, \alpha_v-\lambda_v\in\mathbb{Z}} \oplus^{\lambda_{v-1}+1}_{\alpha_{v+1}=\lambda_v+1,\, \alpha_{v+1}-\lambda_{v+1}\in\mathbb{Z}} \oplus^{\lambda_{v+1}}_{\alpha_{v+2}=\lambda_{v+2},\, \alpha_{v+2}-\lambda_{v+2}\in\mathbb{Z}} \cdots \cdots \oplus^{\lambda_{l-1}}_{\alpha_l=\lambda_l,\, \alpha_l-\lambda_l\in\mathbb{Z}} \oplus^{\lambda_l}_{\alpha_{l+1}=-\lambda_l,\, \alpha_{l+1}-\lambda_l\in\mathbb{Z}} (\alpha_1,\dots,\alpha_{l+1})_K, \tag{57}$$

and we have the isomorphisms

$$\begin{aligned} \mathsf{Ker} D_v &\simeq C^v_{v-1,\,v}\,, \\ \mathsf{Im} D_v &\simeq C^v_{v,\,v+1}. \end{aligned} \tag{58}$$

There is also the set of isomorphisms, allowing the identification of kernels and images of homomorphisms before and beyond the middle part of the BGG sequence:

$$\mathsf{Ker} D_{l+v+1} \simeq \mathsf{Im} D_{l+v} \simeq \mathsf{Im} D_{l-v} \simeq \mathsf{Ker} D_{l-v+1}\,,\ 1 \le v \le l. \tag{59}$$

The only exception from this scheme occurs in the middle part of the sequence for homomorphism $D_l : C_l \to C_{l+1}$*, where we work with abelian group* $C_l \simeq C_{l+1}$*:*

(60)

$$\mathsf{Ker} D_l \simeq \mathsf{Im} D_{l-1} \simeq C^l_{l-1,\,l} \simeq C^{l-1}_{l-1,\,l} \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\, \alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\, \alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\, \alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-3}+1}_{\alpha_{l-1}=\lambda_{l-2}+1\,,\, \alpha_{l-1}-\lambda_{l-2}\in\mathbb{Z}} \oplus^{\lambda_{l-2}+1}_{\alpha_l=\lambda_{l-1}+1\,,\, \alpha_l-\lambda_l\in\mathbb{Z}} \oplus^{\lambda_l}_{\alpha_{l+1}=-\lambda_l\,,\, \alpha_{l+1}-\lambda_l\in\mathbb{Z}} (\alpha_1,\dots,\alpha_{l+1})_K,$$

and the image of D_l *is equal to*

$$C^{l+1}_{l,\,l+1} \simeq C^l_{l,\,l+1} \simeq \mathsf{Im} D_l \simeq \oplus^{\infty}_{\alpha_1=\lambda_1+k\,,\, \alpha_1-\lambda_1\in\mathbb{Z}} \oplus^{\lambda_1+k}_{\alpha_2=\lambda_1+1\,,\, \alpha_2-\lambda_2\in\mathbb{Z}} \oplus^{\lambda_1+1}_{\alpha_3=\lambda_2+1\,,\, \alpha_3-\lambda_3\in\mathbb{Z}} \cdots \oplus^{\lambda_{l-3}+1}_{\alpha_{l-1}=\lambda_{l-2}+1\,,\, \alpha_{l-1}-\lambda_{l-2}\in\mathbb{Z}} \oplus^{\lambda_{l-2}+1}_{\alpha_l=\lambda_{l-1}+1\,,\, \alpha_l-\lambda_l\in\mathbb{Z}} \{\oplus^{\lambda_{l-1}+1}_{\alpha_{l+1}=(\lambda_l+1)\,,\, \alpha_{l+1}-\lambda_l\in\mathbb{Z}} \oplus^{-(\lambda_l+1)}_{\alpha_{l+1}=-(\lambda_{l-1}+1)\,,\, \alpha_{l+1}-\lambda_l\in\mathbb{Z}}\}(\alpha_1,\dots,\alpha_{l+1})_K. \tag{61}$$

The identifications in the middle part can be summarized as

$$C^{l+1}_{l+1,\,l+2} \simeq \mathsf{Ker} D_{l+2} \simeq \mathsf{Im} D_{l+1} \simeq \mathsf{Im} D_{l-1} \simeq \mathsf{Ker} D_l \simeq C^l_{l-1,\,l}, \tag{62}$$

and

$$C_l \simeq C^l_{l-1,\,l} \oplus C^l_{l,\,l+1}\,, C_{l+1} \simeq C^{l+1}_{l,\,l+1} \oplus C^{l+1}_{l+1,\,l+2}, \tag{63}$$

where $C^l_{l-1,\,l} \simeq C^{l+1}_{l+1,\,l+2}$*.*

Proof: It is a consequence of Theorems 5.4 and 4.8. □

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