Homomorphism duality for rooted oriented paths

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Abstract. Let (H, r) be a fixed rooted digraph. The (H, r)-coloring problem is the problem of deciding for which rooted digraphs (G, s) there is a homomorphism $f : G \to H$ which maps the vertex s to the vertex r. Let (H, r) be a rooted oriented path. In this case we characterize the nonexistence of such a homomorphism by the existence of a rooted oriented cycle (C, q), which is homomorphic to (G, s) but not homomorphic to (H, r). Such a property of the digraph (H, r) is called *rooted cycle duality* or $*$-*cycle duality*. This extends the analogical result for unrooted oriented paths given in [6]. We also introduce the notion of *comprimed tree duality*. We show that comprimed tree duality of a rooted digraph (H, r) implies a polynomial algorithm for the (H, r)-coloring problem.

Keywords: graph homomorphism, homomorphism duality, rooted oriented path
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1. Introduction

We make use of standard concepts of graph theory. In addition we introduce in this section a few more special notions related to digraphs (see [3]). All the digraphs discussed in this paper are finite.

Let G be a digraph. We denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. We say that $M \subseteq V(G)$ is independent if for any $x, y \in M$ there is no edge joining x and y in G. *An oriented path* P is a digraph given by the sequence of its vertices $[v_1, v_2 \dots v_n]$ such that for each $i \in \{1, 2 \dots n-1\}$ either $(v_i, v_{i+1}) \in E(P)$ (a *forward* edge of P), or $(v_{i+1}, v_i) \in E(P)$ (a *backward* edge of P), and such that P has no other edges. The vertex v_1 is called *the initial vertex* $i(P)$ of P, and v_n is called *the terminal vertex* $t(P)$ of P. We denote by P^{-1} the oriented path $[v_n, v_{n-1} \dots v_1]$. If $P_1 = [v_1, v_2 \dots v_n]$ and $P_2 = [w_1, w_2 \dots w_k]$ are oriented paths with disjoint vertex sets, the *concatenation* of P_1 and P_2 is the oriented path P obtained by identifying the terminal vertex of P_1 with the initial vertex of P_2, i.e. $P = [v_1, v_2 \dots v_n = w_1, w_2 \dots w_k]$. We write $P = P_1 + P_2$, if P is the concatenation of P_1 and P_2. (Note that $P_1 + P_2$ and $P_2 + P_1$ need not be isomorphic.) *The algebraic length* $al(P)$ of an oriented path P is the number of forward edges minus the number of backward edges of P. *The net length* $nl(P)$ of P is the absolute value of $al(P)$. An oriented path P is *minimal* if it contains no proper subpath of net length $nl(P)$. An arbitrary subpath in a digraph G

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connecting two vertices u and v of G is denoted P_{uv}. *The distance* between two vertices u, v in a digraph G is the minimum length of P_{uv} which is taken without respect to the orientation of the edges. It is denoted by $d_G(u, v)$.

An oriented cycle is a digraph obtained from an oriented path by identifying its initial and terminal vertices. Algebraic and net lengths of oriented cycles are defined analogously. *A balanced cycle* is an oriented cycle of net length zero. A digraph is balanced if it contains no unbalanced cycle as its subgraph. If G is balanced then for any two vertices u, v of G, all paths P_{uv} have the same algebraic length. For a balanced digraph G we define its *height* as the maximum net length of an oriented subpath of G. We denote it $ht(G)$. *The level of a vertex* v in a balanced digraph G is $\Lambda_G(v) = \max\{al(P); P \subseteq G \text{ and } t(P) = v\}$. An oriented tree is a digraph which contains no oriented cycle as its subgraph.

Let G, H be digraphs. A mapping $f : V(G) \to V(H)$ is called *a homomorphism from G to H* if and only if f preserves edges, i.e. $(f(u), f(v)) \in E(H)$ for all edges $(u, v) \in E(G)$. If such a homomorphism exists, we say that G *is homomorphic to* H and write $G \to H$. Otherwise we write $G \not\to H$. The following is easy to see.

Observation 1.1. *Let G and H be balanced digraphs, $f : G \to H$ be a homomorphism. Then*

(a) $\Lambda_H(f(v)) = \Lambda_H(f(u)) + al(P_{uv})$ *for every two vertices u, v of G,*
(b) $ht(G) \leq ht(H)$,
(c) *if $ht(G) = ht(H)$ then f preserves the levels of the vertices, i.e. $\Lambda_H(f(v)) = \Lambda_G(v)$ for every vertex v of G.*

A rooted digraph (G, s) is a digraph G with a fixed vertex s called *the root*. We denote by $(G, s_1, s_2 \dots s_k)$ the rooted digraph with k roots. If (G, s) and (H, r) are rooted digraphs, then *a rooted homomorphism* from (G, s) to (H, r) is a homomorphism from G to H which maps s to r. If such a homomorphism exists, we say that (G, s) *is homomorphic to* (H, r) and write $(G, s) \to (H, r)$. Otherwise we write $(G, s) \not\to (H, r)$. The definition of rooted homomorphism can be extended to digraphs with more then one root as follows: $(G, s_1, s_2 \dots s_k) \to (H, r_1, r_2 \dots r_k)$ if and only if there is a homomorphism $f : G \to H$ with $f(s_i) = r_i$ for all $i \in \{1, 2 \dots k\}$.

Let G and H be digraphs. *A labeling* of G with respect to H is a mapping l of $V(G)$ to the family of subsets of $V(H)$. Such a labeling l is *consistent with an edge* $e = (x, y) \in E(G)$ if for any $p \in l(x)$ there is a $q \in l(y)$ such that $(p, q) \in E(H)$, and for any $q \in l(y)$ there is a $p \in l(x)$ such that $(p, q) \in E(H)$. We say that l is *a consistent labeling* of G with respect to H if l is consistent with all edges of G. The size of a labeling l of G with respect to H is the number $|l| = \sum_{v \in V(G)} |l(v)|$.

The following theorems about oriented paths motivated this paper.

Theorem 1.2 ([6]). *Let P be an oriented path and G be a digraph. Then $G \not\to P$ if and only if there is an oriented path W such that $W \to G$ and $W \not\to P$.*

Theorem 1.3 ([3])**.** *Let P be a fixed oriented path. The decision problem in which the instance is a digraph G and the question is whether or not $G \to P$ can be solved in polynomial time.*

Theorem 1.2 declares that oriented paths have so called path duality. It is a particular case of homomorphism duality. Theorem 1.3 is proved in [3] by means of an elegant algorithm called *consistency check*.

In this paper we consider the analogous questions for rooted homomorphisms. In particular we discuss the validity of various modifications of the homomorphism duality for rooted oriented paths. The following concepts extend the notions for unrooted digraphs (defined in [4]) to rooted digraphs. Let (H, r) be a fixed rooted digraph. *The (H, r)-coloring problem* is the following decision problem:

Instance: A rooted digraph (G, s).

Question: Is there any rooted homomorphism from (G, s) to (H, r)?

The rooted digraph (H, r) has *$*$-path duality* (*$*$-cycle* or *$*$-tree duality* respectively), if the following property holds for all rooted digraphs (G, s) : $(G, s) \not\to (H, r)$ if and only if there exists a rooted oriented path (a rooted oriented cycle or tree respectively) (P, t) such that $(P, t) \to (G, s)$ and $(P, t) \not\to (H, r)$. Thus (H, r) has not $*$-path duality ($*$-cycle or $*$-tree duality respectively) if and only if there is a rooted digraph (G, s) such that $(G, s) \not\to (H, r)$ and for any rooted oriented path (cycle or tree respectively) (P, t) the fact that $(P, t) \to (G, s)$ implies that $(P, t) \to (H, r)$. Such an oriented graph (P, t) we call *counterexample for $*$-path duality* (*$*$-cycle* or *$*$-tree duality* respectively) of a digraph (H, r).

We will show that the theorem analogous to Theorem 1.2 does not hold for rooted paths, i.e. there are rooted oriented paths without $*$-path duality. In the third section of this paper we will show that rooted oriented paths have $*$-cycle duality. We will take advantage of it to generalize the algorithm given in [3] (consistency check) to rooted homomorphisms. We also characterize the class of all rooted digraphs for which this algorithm finds a correct solution. These are exactly the rooted digraphs with comprimed tree duality.

This paper is based on the diploma thesis at Charles University [9].

2. $*$-tree duality

We will prove that the theorem analogical to Theorem 1.2 does not hold for rooted oriented paths. Moreover we prove that rooted oriented paths need not have $*$-tree duality.

Theorem 2.1. *There is a rooted oriented path (P, r) and a digraph (G, s) such that $(G, s) \not\to (P, r)$ and any rooted oriented tree (T, t) homomorphic to (G, s) is also homomorphic to (P, r).*

Proof: Consider the digraph (G, s) depicted in Figure 2.1, where the paths P_1, P_2, R_1 and R_2 have the following properties:

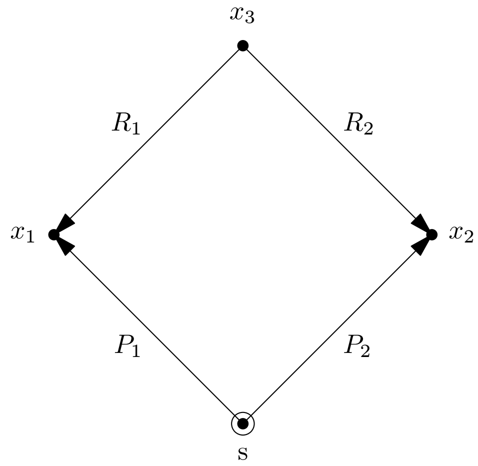

FIGURE 2.1. The digraph (G, s).

(1) P_1, P_2, R_1 and R_2 are minimal oriented paths of algebraic length n
(2) $P_i \not\to P_j$ for any i, $j \in \{1, 2\}$, $i \neq j$
(3) $P_i \to R_i$, $P_i \not\to R_j$ for any i, $j \in \{1, 2\}$, $i \neq j$
(4) $R_i \not\to P_j$ for any i, $j \in \{1, 2\}$.

We claim that (G, s) is a counterexample for $*$-tree duality of the path (P, r) depicted in Figure 2.2. To see it assume without loss of generality that $n > 0$. The denoted vertices of the digraph (G, s) and of the path (P, r) satisfy:

(i) $\Lambda_G(s) = \Lambda_G(x_3) = 0$ and $\Lambda_P(r) = \Lambda_P(z_1) = \Lambda_P(z_2) = 0$;
(ii) $\Lambda_G(x_1) = \Lambda_G(x_2) = n$ a $\Lambda_P(t_1) = \Lambda_P(t_2) = \Lambda_P(y_1) = \Lambda_P(y_2) = n$;
(iii) no other vertex of the digraph (G, s) or of the path (P, r) has level 0 or n.

Moreover $ht(G) = ht(P) = n$.

We have to show that: (a) $(G, s) \not\to (P, r)$,
(b) any rooted oriented tree (T, t) homomorphic to (G, s) is also homomorphic to (P, r).

Suppose to the contrary that (a) fails. Let $h : (G, s) \to (P, r)$ be a rooted homomorphism. The digraphs (G, s) and (P, r) are balanced. Thus by Observation 1.1. h preserves the levels of the vertices. Thus $\Lambda_P(h(x_3)) = 0$. We must have $h(x_3) = r$, $h(x_3) = z_1$ or $h(x_3) = z_2$. Let us discuss these three possibilities:

Since $R_i \not\to P_j$ for any i, $j \in \{1, 2\}$, then $h(x_3) \neq r$. If $h(x_3) = z_1$, then $h(x_2) = y_1$ because $R_2 \not\to R_1$. Necessarily $h(s) = z_1$. This contradicts the assumption that h is a rooted homomorphism from (G, s) to (P, r). The case $h(x_3) = z_2$ is similar. Thus (a) holds.

In order to prove (b) it suffices to consider rooted oriented trees whose root is a leaf. In the other case we can decompose a tree (T, t) into the branches with root

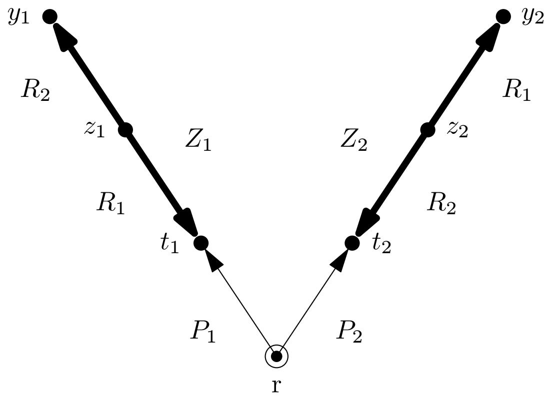

FIGURE 2.2. The path (P, r).

t as a leaf. If we find the homomorphism $f_B : (B, t) \to (P, r)$ for every branch (B, t) we can construct a new homomorphism $f : (T, t) \to (P, r)$ according to the rule $f(v) = f_B(v)$ for every $v \in V(B)$. The only vertex included in more then one branch is the root t and every f_B maps t to r. It is easy to see that f is a homomorphism from (T, t) to (P, r).

Thus let (T, t) be a rooted oriented tree with root as a leaf, let $h : (T, t) \to (G, s)$ be a rooted homomorphism. We will construct a homomorphism $f : (T, t) \to (P, r)$. Let us mention that there is only one path P_{uv} for every two vertices u, v of the tree T. First we define the sets A, A_1, A_2, B_1, B_2 of vertices of T in the following way (see Figure 2.3).

$$A = \{v \in V(T);\ \text{each vertex } w,\ w \neq v, \text{ which is in } P_{tv} \text{ satisfies } h(w) \notin \{x_1, x_2\}\},$$
$$A_i = \{v \in A; h(v) = x_i\} \text{ for } i = 1, 2,$$
$$B_i = \{v \in V(T);\ \text{there is a vertex } w \in A_i,\ w \neq v, \text{ which is in } P_{tv}\} \text{ for } i = 1, 2.$$

Denote by Z_1, Z_2 the paths marked with bold line in Figure 2.2. Let g_i be a homomorphism from G to Z_i, $i \in \{1, 2\}$. The fact that such a homomorphism exists follows from properties of paths P_1, P_2, R_1, R_2. Certainly $g_i(x_i) = t_i$, $g_i(x_j) = y_i$ and $g_i(x_3) = g_i(s) = z_i$ for i, $j \in \{1, 2\}$, $i \neq j$. Let id be an identical homomorphism which maps $P_1^{-1} + P_2$ taken as a subgraph of G onto the copy of this path in P. We have $id(x_i) = t_i$ for $i \in \{1, 2\}$ and $id(s) = r$.

Define a mapping $f : V(T) \to V(P)$ by

$$f(v) = id(h(v)) \text{ for any } v \in A,\ f(v) = g_i(h(v)) \text{ for any } v \in B_i,\ i = 1, 2.$$

We will show now that the mapping f is a homomorphism from (T, t) to (P, r). It suffices to prove:

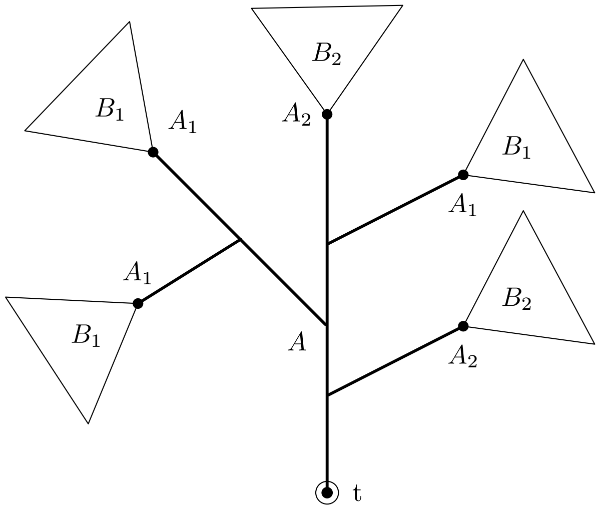

FIGURE 2.3. To the definition of the sets A, A_1, A_2, B_1, B_2.

(1) f is correctly defined, i.e. there is exactly one value of $f(v)$ for every vertex $v \in V(T)$,
(2) $(f(u), f(v)) \in E(P)$ for every edge $(u, v) \in E(T)$,
(3) $f(t) = r$.

It is clear that A_1, $A_2 \subseteq A$ and that the sets A, B_1, B_2 are mutually disjoint. Thus (1) holds. Suppose that $(u, v) \in E(T)$. It is clear that $(f(u), f(v)) \in E(P)$ if u, v are elements of the same set among A, B_1, B_2. If $u \in A$ and $v \in B_i$ for some $i \in \{1, 2\}$, then $u \in A_i$ by the definition of the sets A, A_1, A_2, B_1, B_2. Thus $h(u) = x_i$.

We have $(f(u), f(v)) = (id(h(u)), g_i(h(v))) = (id(x_i), g_i(h(v))) = (t_i, g_i(h(v))) = (g_i(x_i), g_i(h(v))) = (g_i(h(u)), g_i(h(v)))$. Since $g_i \circ h$ is a homomorphism from T to P, we obtain $(g_i(h(u)), g_i(h(v))) \in E(P)$. The case $u \in B_i$ for some $i \in \{1, 2\}$ and $v \in A$ is similar. The case $u \in B_1$, $v \in B_2$ cannot happen according to the definition of A, B_1, B_2. This proves (2).

It remains to show (3). The path P_{tt} in T contains only one vertex, namely t. Hence $t \in A$. According to the definition of f we must have $f(t) = id(h(t)) = id(s) = r$. □

Remark. $*$-tree duality is a weaker property than $*$-path duality. Hence if a path (P, r) does not have $*$-tree duality then it does not have $*$-path duality either.

3. $*$-cycle duality

We proved in Section 2 that there are rooted oriented paths without $*$-tree duality. This is as far as we can go. We will prove that every rooted oriented path has $*$-cycle duality.

Theorem 3.1. *Suppose that (G,s) is a rooted digraph and (P,r) is a rooted oriented path. Then $(G,s) \to (P,r)$ if and only if for every rooted oriented cycle (C,q) such that $(C,q) \to (G,s)$, we have $(C,q) \to (P,r)$.*

PROOF: The proof is based on the proof of Theorem 1.2 given in [6]. Let (P,r) be a rooted oriented path and (G,s) be an arbitrary rooted digraph. If $(G,s) \to (P,r)$ and (C,q) is a rooted oriented cycle such that $(C,q) \to (G,s)$, then of course $(C,q) \to (P,r)$ by composition. Thus the first implication holds. To verify the other implication is more difficult.

Let (P,r) be a rooted oriented path. Without loss of generality we may suppose that $[1,2\dots n]$ are the vertices of P (1 is the initial and n is the terminal vertex of P). We consider the root r as the respective integer. Let (G,s) be an arbitrary rooted digraph such that for every rooted oriented cycle (C,q), the existence of a homomorphism from (C,q) to (G,s) implies the existence of a homomorphism from (C,q) to (P,r). We will construct a homomorphism $f:(G,s) \to (P,r)$.

Denote by $\mathcal{W}$ the set of all rooted oriented paths (W,z) with $i(W)=z$ and such that $(W,z) \to (G,s)$. For $(W,z) \in \mathcal{W}$ we set

$$T(W,z) = \{v \in V(G); \text{there is a homomorphism } g:(W,z) \to (G,s) \text{ such that } g(t(W)) = v\} \text{ and}$$

$$\Phi(W,z) = \max\{h(t(W)); h:(W,z) \to (P,r)\}.$$

We will show that $\Phi : \mathcal{W} \to V(P)$ is well defined. Let $(W,z) \in \mathcal{W}$. Consider two copies of (W,z) and identify their initial vertices (i.e. the roots) and their terminal vertices. Denote by q the vertex arising from the two identified roots. The resulting cycle (C,q) satisfies $(C,q) \to (G,s)$. Thus $(C,q) \to (P,r)$ according to the premise. Obviously then $(W,z) \to (P,r)$. Thus $\Phi(W,z)$ is defined for every path $(W,z) \in \mathcal{W}$.

Finally, define a mapping $f: V(G) \to V(P)$ by
$f(v) = \min\{\Phi(W,z); (W,z) \in \mathcal{W} \text{ and } v \in T(W,z)\}$ for all $v \in V(G)$.

It is easy to see that the mapping f is well defined. It remains to show that f is a homomorphism from (G,s) to (P,r). We shall proceed in the following steps:

(1) If $(u,v) \in E(G)$, then $|d_G(u,s) - d_G(v,s)| = 1$.
(2) If $u \in V(G)$, then $|f(u) - r|$ and $d_G(u,s)$ have the same parity.
(3) If $(u,v) \in E(G)$, then $f(u) \neq f(v)$.
(4) If $(u,v) \in E(G)$, then $(f(u), f(v)) \in E(P)$.
(5) $f(s) = r$.

(1) If $(u,v) \in E(G)$, then $|d_G(u,s) - d_G(v,s)| \leq 1$ according to the definition of the distance $d_G(u,v)$. It suffices to prove that $|d_G(u,s) - d_G(v,s)| \neq 0$.

Assume to the contrary that $(u,v) \in E(G)$ and $d_G(u,s) = d_G(v,s)$. Let P_{su}, P_{vs} have the minimum length, i.e. their length is $d_G(u,s)$. Denote by P^* the path $P_{vs} + P_{su}$ and denote by q the terminal vertex of P_{vs} in P^*. Add the edge (u,v)

to P^*. The resulting digraph (C,q) is an odd cycle such that $(C,q) \to (G,s)$. Certainly $(C,q) \not\to (P,r)$, which is a contradiction.

(2) Suppose that $u \in V(G)$ and $(W,z) \in \mathcal{W}$ such that $\Phi(W,z) = f(u)$, $W = [z = w_1, w_2 \dots w_m]$. Necessarily $u \in T(W,z)$. Let $g : (W,z) \to (G,s)$ be a homomorphism satisfying $g(w_m) = u$. Let $i \in \{1, 2 \dots m-1\}$. Then $(g(w_i), g(w_{i+1})) \in E(G)$ or $(g(w_{i+1}), g(w_i)) \in E(G)$. We have $|d_G(g(w_i), s) - d_G(g(w_{i+1}), s)| = 1$ according to (1). Thus $d_G(g(w_i), s)$ and $d_G(g(w_{i+1}), s)$ have the opposite parity. Since $d_G(g(w_1), s) = d_G(g(z), s) = d_G(s, s) = 0$, i.e. $d_G(g(w_1), s)$ is even, we have for all $i \in \{1, 2 \dots m\}$: $d_G(g(w_i), s)$ is even if and only if i is odd. In particular, $d_G(u, s)$ is even if and only if m is odd.

Let $h : (W,z) \to (P,r)$ be a homomorphism satisfying $h(w_m) = \Phi(W,z)$. Let $i \in \{1, 2 \dots m-1\}$. Then $|h(w_i) - h(w_{i+1})| = 1$ and $|d_G(h(w_i), r) - d_G(h(w_{i+1}), r)| = 1$. It follows that $d_G(h(w_i), r)$ and $d_G(h(w_{i+1}), r)$ have the opposite parity. Since $d_G(h(w_1), r) = d_G(h(z), r) = d_G(r, r) = 0$, we have for all $i \in \{1, 2 \dots m\}$: $d_G(h(w_i), r)$ is even if and only if i is odd.

In particular, $d_G(h(w_m), r) = d_G(f(u), r) = |f(u) - r|$ is even if and only if m is odd. We know that m is odd if and only if $d_G(u, s)$ is even. Thus $d_G(u, s)$ and $|f(u) - r|$ are either both even or both odd. That is what we wanted to prove.

(3) Suppose that $(u, v) \in E(G)$. Then $|d_G(u, s) - d_G(v, s)| = 1$ according to (1). Therefore $d_G(u, s)$ and $d_G(v, s)$ have the opposite parity. Then $|f(u) - r|$ and $|f(v) - r|$ have the opposite parity according to (2). Thus $f(u) \neq f(v)$.

(4) Suppose that $(u, v) \in E(G)$. By (3) we have $f(u) \neq f(v)$. We can assume that $f(u) < f(v)$, as the other case is similar. Find $(W_u, z_u) \in \mathcal{W}$ such that $\Phi(W_u, z_u) = f(u)$. Denote $W_u = [z_u = w_1, w_2 \dots w_m]$. Transform (W_u, z_u) into the path (W_v, z_v) by adding a new vertex w_{m+1} and a new edge (w_m, w_{m+1}), with z_v as its initial vertex. Then $(W_v, z_v) \in \mathcal{W}$ and $v \in T(W_v, z_v)$. Thus $\Phi(W_v, z_v) \geq f(v)$ according to the definition of f.

Let $h : (W_v, z_v) \to (P, r)$ be a homomorphism such that $h(w_{m+1}) = \Phi(W_v, z_v)$. Since h restricted to (W_u, z_u) is a homomorphism from (W_u, z_u) to (P, r), we have $h(w_m) \leq \Phi(W_u, z_u)$. Since $(w_m, w_{m+1}) \in E(W_v)$ and h is a homomorphism, we have $(h(w_m), h(w_{m+1})) \in E(P)$. Therefore $h(w_m) - 1 \leq h(w_{m+1}) \leq h(w_m) + 1$ and $h(w_m) \neq h(w_{m+1})$. We get $h(w_m) \leq \Phi(W_u, z_u) = f(u) < f(v) \leq \Phi(W_v, z_v) = h(w_{m+1}) \leq h(w_m) + 1$. Hence $h(w_m) = f(u)$, $h(w_{m+1}) = f(v)$. Since $(h(w_m), h(w_{m+1})) \in E(P)$, it follows that $(f(u), f(v)) \in E(P)$.

(5) Any path $(W,z) \in \mathcal{W}$ such that $s \in T(W,z)$ can be transformed to a cycle (C,q) by identifying the initial and the terminal vertex of W and calling this new vertex q. Then $(C,q) \to (G,s)$. According to the premise there is a homomorphism $g : (C,q) \to (P,r)$. Then a mapping $g' : (W,z) \to (P,r)$ defined by $g'(w) = g(w)$ for any $w \in V(W)$, $w \neq z$, $w \neq t(W)$, $g'(z) = g'(t(W)) = r$, is a homomorphism. Therefore $\Phi(W,z) \geq r$ for every path $(W,z) \in \mathcal{W}$ such that $s \in T(W,z)$. We have $f(s) \geq r$.

It remains to show that $f(s) \leq r$. Let (W,z) be the path consisting of a single vertex z. Thus $(W, z) \in \mathcal{W}$, $s \in T(W, z)$ and $\Phi(W, z) = r$. We have $f(s) \leq r$ according to the definition of f. $\square$

4. An algorithm for the (H, r)-coloring problem

We describe a polynomial algorithm which is a rooted digraph modification of a procedure given in [3] (called *consistency check*). We call our algorithm $*$-*consistency check* and use an abbreviation ($*$-CC) for it. We consider only the connected rooted digraphs in this section. For the components of connectivity which do not contain the root, the procedure described for unrooted digraphs in [3] can be used. Let (H, r) be a fixed rooted digraph.

Algorithm 4.1.($*$-consistency check)

Input data: A digraph (G, s).

Question: Is there a homomorphism from (G, s) to (H, r) ?

1. Put $l(s) = \{r\}$ and $l(v) = V(H)$ for all $v \in V(G)$, $v \neq s$. Put $L = |l|$.
2. For all $e \in E(G)$ consecutively, check if labeling l is consistent with the edge $e = (u, v)$. If not, then:
   (a) remove from $l(v)$ those q for which there is no $p \in l(u)$ with $(p, q) \in E(H)$;
   (b) remove from $l(u)$ those p for which there is no $q \in l(v)$ with $(p, q) \in E(H)$.
3. If $|l| < L$, then put $L = |l|$ and return to step 2, else continue to step 4.
4. If $l(s) \neq \emptyset$ then answer YES, else answer NO.

Remarks. If the size of l does not decrease in step 2, then l is consistent with all edges. In the end of the algorithm l is a maximal (with respect to coordinate-wise inclusion) consistent labeling of G with respect to H which satisfies $l(s) \subseteq \{r\}$. It is clear that $l(s) \neq \emptyset$ if and only if $l(v) \neq \emptyset$ for all $v \in V(G)$, because the digraph G is connected.

Let $n = |V(G)|$, $k = |E(G)|$. It is easy to see that the running time of ($*$-CC) is $O(n \cdot k)$ in the worst case. However ($*$-CC) need not solve the (H, r)-coloring problem for all digraphs (H, r). We will give a condition for a digraph (H, r) which is necessary and sufficient for correctness of ($*$-CC).

Definition 4.2. A rooted digraph (G, m) is called a comprimed tree if there exist an oriented tree S and an independent subset M of $V(S)$ such that $V(G) = (V(S) - M) \cup \{m\}$, $m \notin V(S)$, $E(G) = \{(x, y); (x, y) \in E(S)$ and $x, y \notin M\} \cup \{(m, y)$; there is $x \in M$ such that $(x, y) \in E(S)\} \cup \{(x, m)$; there is $y \in M$ such that $(x, y) \in E(S)\}$.

Remark. In other words, a comprimed tree is a homomorphism image of an oriented tree in the case that the homomorphism maps a subset of vertices to one vertex, which becomes the root, and is $1-1$ on the remaining vertices.

Definition 4.3. We say that a rooted digraph (H,r) has *comprimed tree duality* if the following property holds for all digraphs (G,s): $(G,s) \to (H,r)$ if and only if for every comprimed tree (T,t) which satisfies $(T,t) \to (G,s)$, we have $(T,t) \to (H,r)$.

We will first show that comprimed tree duality of a rooted digraph (H,r) is sufficient for correctness of the algorithm ($*$-CC) for solving (H,r)-coloring problem.

Theorem 4.4. *Suppose that a rooted digraph (H,r) has comprimed tree duality. Then the following statements are equivalent:*

(1) $(G,s) \to (H,r)$,
(2) *the algorithm ($*$-CC) for (G,s) with respect to (H,r) will answer YES,*
(3) *for every comprimed tree (T,t) which satisfies $(T,t) \to (G,s)$, we have $(T,t) \to (H,r)$.*

Proof: Obviously (3) $\Rightarrow$ (1). We will first prove that (1) $\Rightarrow$ (2). Let $h : (G,s) \to (H,r)$ be a homomorphism. Let l be the labeling of (G,s) with respect to (H,r) arising in the course of the algorithm ($*$-CC). After the first step of ($*$-CC) we have $h(v) \in l(v)$ for every $v \in V(G)$. Namely $h(s) = r \in l(s)$. For every edge $(u,v) \in E(G)$ we have $(h(u),h(v)) \in E(P)$. That is why a vertex $h(v)$ cannot be removed from $l(v)$ in the course of the algorithm ($*$-CC) for every vertex $v \in V(G)$. Thus $l(v) \neq \emptyset$ for every vertex $v \in V(G)$ in the end of the algorithm and ($*$-CC) will answer YES.

It remains to prove that (2) $\Rightarrow$ (3). Let l be the labeling of (G,s) with respect to (H,r) obtained in the end of the algorithm ($*$-CC). We have $l(v) \neq \emptyset$ for every vertex $v \in V(G)$ because ($*$-CC) answers YES. Let (T,t) be a comprimed tree such that $(T,t) \to (G,s)$. We shall call this homomorphism h.

We will construct a homomorphism $f : (T,t) \to (H,r)$. The case $V(T) = \{t\}$ is trivial. We will suppose that $V(T) - \{t\} \neq \emptyset$. Let us choose $v \in V(T) - \{t\}$ and $p \in l(h(v))$. Put $f(v) = p$. Choose a neighbor u of v in T, $u \neq t$. The mapping h is a homomorphism, then we have $(h(u),h(v)) \in E(G)$, $(h(v),h(u)) \in E(G)$ respectively. The labeling l is consistent, thus there is an element $q \in l(h(u))$ such that $(p,q) \in E(H)$, $(q,p) \in E(H)$ respectively. Put $f(u) = q$. We can proceed in such a way to the neighbors of the vertices discussed which are different of t and define the mapping f on these neighbors. Since the subgraph T' of T induced by the vertex set $V(T') = V(T) - \{t\}$ is a tree, the mapping f is a homomorphism from T' to H. Put $f(t) = r$. For any vertex $w \in V(T)$ such that $(t,w) \in E(T)$ and any $q \in l(h(w))$, we have $(r,q) \in E(H)$. It is guaranteed by the fact that $l(s) = \{r\}$ and that l is consistent. Thus $(f(t),f(w)) \in E(H)$. Similarly for any vertex $u \in V(T)$ such that $(u,t) \in E(T)$ and any $q \in l(h(u))$, we have $(q,r) \in E(H)$. It follows that $(f(u), f(t)) \in E(H)$ in this case. Now it is clear that f is a homomorphism from (T,t) to (H,r). □

It remains to show that the comprimed tree duality of (H,r) is necessary for the algorithm ($*$-CC) to work correctly. We will start by proving the following lemma.

Lemma 4.5. *Let (G,s), (H,r) be rooted digraphs. Let l be the labeling of (G,s) with respect to (H,r) obtained in any step of the algorithm ($*$-CC). Let $l(s) \neq \emptyset$ in this step. Let $v \in V(G)$, $w \in V(H)$ such that $v \neq s$ and $w \notin l(v)$. Then there is a comprimed tree (T,p) and a vertex $t \in V(T)$ such that $(T,p,t) \to (G,s,v)$ and $(T,p,t) \not\to (H,r,w)$.*

Proof: Suppose to the contrary that there are $v \in V(G)$ and $w \in V(H)$ with $v \neq s$, $w \notin l(v)$, such that for any comprimed tree (T,p) and any vertex $t \in V(T)$ satisfying $(T,p,t) \to (G,s,v)$, we have $(T,p,t) \to (H,r,w)$. The vertex w was removed from $l(v)$ in the j-th step of the algorithm ($*$-CC). We choose the vertices v a w so that the assumptions are satisfied and j is minimal. Suppose that in the j-th step an edge $(u,v) \in E(G)$ was checked. The other case (in the j-th step an edge $(v,u) \in E(G)$ was checked) is similar. We have to consider two different cases.

Let first $u = s$. Then $(s,v) \in E(G)$. Neccesarily $(r,w) \notin E(H)$, because w was removed from $l(v)$ in this step. Let (T,p,t) be a tree with only two vertices, namely p, t, and with the only edge (p,t). Then $(T,p,t) \to (G,s,v)$ and $(T,p,t) \not\to (H,r,w)$. This is a contradiction with the choice of v, w.

Now we will consider the case $u \neq s$. If w was removed from $l(v)$, then there is not a vertex $x \in l(u)$ with $(x,w) \in E(H)$. According to the assumption for every vertex y, $y \notin l(u)$, there is a comprimed tree (T_y, p_y) and a vertex $t_y \in V(T)$ such that $(T_y, p_y, t_y) \to (G,s,u)$ and $(T_y, p_y, t_y) \not\to (H,r,y)$. Otherwise we would have chosen u, y instead of v, w. In addition if $u \neq s$, then $p_y \neq t_y$ for every $y \notin l(u)$.

Construct a rooted digraph (T,p) by identifying all vertices t_y for $y \notin l(u)$ and calling the new vertex t and by identifying all vertices p_y for $y \notin l(u)$ and calling the new vertex p. Construct a rooted digraph (T^*,p) by adding a new vertex t^* and a new edge (t,t^*) to the digraph (T,p). The set $\{p_y; y \in V(H)$ and $y \notin l(u)\}$ is independent in T and T^*. It follows that (T,p) and (T^*,p) are comprimed trees. We have $(T^*,p,t^*) \to (G,s,v)$. According to the assumption there is a homomorphism $h : (T^*,p,t^*) \to (H,r,w)$. Let $x = h(t)$. If $x \notin l(u)$, then there is a comprimed tree (T_x, p_x) and a vertex $t_x \in V(T_x)$ with (T_x, p_x, t_x) which is a subgraph of (T,p,t) and which is not homomorphic to (H,r,x). Hence $(T^*,p,t) \not\to (H,r,x)$. This contradicts the existence of the homomorphism h. Thus $x \in l(u)$ and $(x,w) = (h(t), h(t^*)) \in E(H)$. This is a contradiction with the assumption that w was removed from $l(v)$ in this situation in the course of ($*$-CC). □

Theorem 4.6. *Let (G, s), (H, r) be rooted digraphs. Let l be the labeling of (G, s) with respect to (H, r) obtained in the end of the algorithm ($*$-CC). Let $l(s) = \emptyset$. Then there is a comprimed tree (T, p) with $(T, p) \to (G, s)$ and $(T, p) \not\to (H, r)$.*

Proof: Suppose that the vertex r was removed from $l(s)$ by checking an edge (s, v). The other case (the vertex r was removed from $l(s)$ by checking an edge (s, v)) is similar. Let l^* be the labeling in the respective step. Then $(r, w) \notin E(H)$ for all $w \in l^*(v)$. According to Lemma 4.5 there is a comprimed tree (T_w, p_w) and a vertex $t_w \in V(T_w)$ such that $(T_w, p_w, t_w) \to (G, s, v)$ and $(T_w, p_w, t_w) \not\to (H, r, w)$, for all vertices $w \notin l^*(v)$. We know that $t_w \neq p_w$ for all $w \notin l^*(v)$ because $s \neq v$.

Construct the rooted digraph (T, p, t) as follows: first, identify all vertices p_w for $w \in V(H)$, $w \notin l^*(v)$, and call the new vertex p; next, identify all vertices t_w for $w \in V(H)$, $w \notin l^*(v)$, and call the new vertex t; in the end add the edge (p, t) if it is not in $E(T)$ yet. The digraph (T, p, t) clearly satisfies $(T, p, t) \to (G, s, v)$, $(T, p, t) \not\to (H, r, w)$ for all $w \in V(H)$. It is easy to see that (T, p) is a comprimed tree with $(T, p) \to (G, s)$ and $(T, p) \not\to (H, r)$. □

Corollary 4.7. *Let (H, r) be a rooted digraph. Then the following statements are equivalent:*

(1) *(H, r) has comprimed tree duality,*
(2) *for any rooted digraph (G, s) is true that ($*$-CC) for (G, s) with respect to (H, r) answers YES if and only if $(G, s) \to (H, r)$.*

Proof: The implication (1) $\Rightarrow$ (2) was proved as a part of Theorem 4.4. It remains to show (2) $\Rightarrow$ (1). Let (H, r) be a digraph without comprimed tree duality, for contradiction. Then there is a counterexample which guarantees this property, i.e. there is a digraph (G, s) such that $(G, s) \not\to (H, r)$ and for any comprimed tree (T, p) with $(T, p) \to (G, s)$, we have $(T, p) \to (H, r)$. Suppose that the algorithm ($*$-CC) for (G, s) with respect to (H, r) answers NO. Let l be the labeling of (G, s) with respect to (H, r) obtained by the algorithm ($*$-CC). Then $l(s) = \emptyset$.

Then by Theorem 4.6 there is a comprimed tree (T, p) such that $(T, p) \to (G, s)$ and $(T, p) \not\to (H, r)$. This contradicts the assumption that (G, s) is the counterexample for comprimed tree duality of (H, r). □

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