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On topological and algebraic structure of extremally disconnected semitopological groups

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Abstract. Starting with a very simple proof of Frolík's theorem on homeomorphisms of extremally disconnected spaces, we show how this theorem implies a well known result of Malychin: that every extremally disconnected topological group contains an open and closed subgroup, consisting of elements of order 2. We also apply Frolík's theorem to obtain some further theorems on the structure of extremally disconnected topological groups and of semitopological groups with continuous inverse. In particular, every Lindelöf extremally disconnected semitopological group with continuous inverse and with square roots is countable, and every extremally disconnected topological field is discrete.

Keywords: extremally disconnected, semitopological group, order 2, Souslin number, Lindelöf space

Classification: Primary 54H11; Secondary 54A25, 54C05, 54G15

All spaces considered in this note are assumed to be Hausdorff. A semitopological group is a group with a topology such that all left and right translations are continuous. The neutral element of a group is always denoted by e.

A topological space X is called *extremally disconnected*, if the closure of any open subset of X is open in X. It is still an open question, formulated for the first time in [1], in 1967, whether there exists in ZFC a non-discrete extremally disconnected topological group.

The first consistent example of a non-discrete extremally disconnected topological group was constructed by S. Sirota in [10]. Further work in this direction was done by V.I. Malychin [8]. In particular, he was the first to notice a remarkable fact: that every extremally disconnected topological group G contains an open (and closed) Abelian subgroup H such that $a^2 = e$, for each $a \in H$.

Below we provide a very simple and transparent proof of Malychin's theorem, based on a general theorem on homeomorphisms of extremally disconnected spaces belonging to Z. Frolík [4], [5], [6] (Theorem 1) (see also [9] and [7]). For the sake of completeness, we present here a direct proof of Theorem 1, all the more so since it is amazingly simple and short.

Theorem 1 is just what we need to get Malychin's result and to obtain further theorems on the structure of extremally disconnected topological groups, and to partially extend these to semitopological groups.

Theorem 1. *Let X be an extremally disconnected space, and h a homeomorphism of X onto itself. Then the set $M = \{x \in X : h(x) = x\}$ of all fixed points under h is an open and closed subset of X.*

Proof: A subset A of X will be called *h-simple*, if $h(A) \cap A = \emptyset$. Let (using Zorn's Lemma) C be a maximal chain of h-simple open subsets of X. Put $U = \cup C$. Then, by an obvious standard argument, U is h-simple. Thus, the sets U and $h(U)$ are disjoint. Therefore, since h is a homeomorphism, the sets $h^{-1}(U)$ and $h(U)$ are also h-simple open sets. Since X is extremally disconnected, it follows that the closures of U and $h(U)$ are disjoint open sets as well. Thus, $\bar{U}$ is h-simple. Notice, that maximality of the chain C and the definition of U imply that U is a maximal h-simple open subset of X. Therefore, $\bar{U}$ coincides with U, that is, U is closed. It follows that the sets $h(U)$ and $h^{-1}(U)$ are also closed. Hence, the set $F = U \cup h(U) \cup h^{-1}(U)$ is closed.

Now, it is obvious that the intersection of M with any h-simple set is empty. Since F is the union of three h-simple sets, it follows that $M \cap F = \emptyset$. Therefore, $X \setminus F$ is an open set containing M. Let us show that $M = X \setminus F$ (which will obviously make the proof of Theorem 1 complete).

Assume the contrary. Then there exists $a \in X \setminus F$ such that $h(a) \neq a$. Since X is Hausdorff and h is continuous, there exists an open neighbourhood W of a such that $h(W) \cap W = \emptyset$ and $W \cap F = \emptyset$. Then W is h-simple, and $W \cap U = \emptyset$, $W \cap h(U) = \emptyset$, $W \cap h^{-1}(U) = \emptyset$, from which it follows that $U \cup W$ is an h-simple open set that properly contains U. On the other hand, by maximality of U this is impossible, a contradiction. □

Remark 1. The sets U and $h(U)$ are disjoint, as well as the sets U and $h^{-1}(U)$, while the sets $h(U)$ and $h^{-1}(U)$ may have a non-empty intersection. If we wish to have a disjoint covering of the complement to M by open and closed h-simple subsets of X, we only have to replace $h^{-1}(U)$ by the set $h^{-1}(U) \setminus h(U)$. Compare the result obtained with Problem 183 in [2, Chapter 6, Section 5].

Theorem 2 (V.I. Malychin [8]). *Let G be an extremally disconnected topological group. Then there exists an open and closed Abelian subgroup H of G such that $a^2 = e$, for each $a \in H$.*

Proof: By Theorem 1, the set $U = \{a \in G : a^2 = e\}$ is an open neighbourhood of the neutral element e. Since G is a topological group, there exists an open neighbourhood V of e such that $V^2 \subset U$. Every two elements a and b of V commute. Indeed, $abab = e$, since $ab \in U$. Now from $a^2 = e$ and $b^2 = e$ it follows that $ab = ba$. Therefore, the subgroup H generated by V in G is Abelian. Since V is open, the subgroup H is also open, and therefore closed in X. Finally, since H is Abelian and is generated by V, and all elements of V are of order 2, it follows that $a^2 = e$, for every $a \in H$. □

The proof above heavily depends on the assumption that G is a topological group, in particular, on the joint continuity of multiplication in G. If we replace this assumption with a weaker one that the multiplication is separately continuous, we cannot derive a conclusion as strong as in Theorem 2, but we still can obtain some interesting information on the topologo-algebraic structure of G. Recall that *a semitopological group* is a group with a topology such that left and right multiplications are separately continuous.

Theorem 3. *Let G be an extremally disconnected semitopological group with continuous inverse. Then the set W of all elements a of G such that $a^2 = e$ is an open (and closed) neighbourhood of the neutral element e of G.*

Proof: Indeed, the inverse mapping of G onto itself is a homeomorphism, and e is a fixed point of this mapping. It remains to apply Theorem 1. $\square$

If G is a group and a is an element of G such that a^2 is the neutral element e of G, we say that a is an element *of order* 2. It is well known that elements of order 2 need not constitute a subgroup. This happens because they do not have to commute. In this light, the next result is of some interest.

Proposition 4. *Let G be an extremally disconnected semitopological group with continuous inverse. Then, for every element a of G of order 2, there exists an open neighbourhood V of the neutral element e such that a commutes with every element of $V \cup aV$.*

Proof: By Theorem 3, the set U of all elements of G of order 2 is open in G. Since G is semitopological, there exists an open neighbourhood V of the neutral element e such that $aV \subset U$. Let $b \in V$. Then $ab \in U$; therefore, $abab = e$. Since $a^2 = e$ and $b^2 = e$, it follows that $ab = ba$. Thus, a commutes with every element of V.

Now, let $c \in aV$. Then $c = ab$, for some $b \in V$, and $ac = aab$, $ca = aba = aab$. Therefore, $ac = ca$. $\square$

In the proof of Theorem 6 below we apply Proposition 4. However, it seems worth noting the next stronger statement that is proved by a slightly more elaborate argument. If G is a group and $a \in G$, we denote by C_a the set of all $b \in G$ which commute with a (that is, satisfy the condition $ab = ba$).

Theorem 5. *Let G be an extremally disconnected semitopological group with continuous inverse. Then, for any $a \in G$, the set C_a of all $b \in G$ that commute with a is an open and closed subgroup of G (containing a).*

Proof: Let ϕ be the mapping of G into G given by the rule: $\phi(x) = a^{-1}xa$, for each $x \in G$. Clearly, ϕ is a homeomorphism of the space G onto itself. Therefore, by Theorem 1, the set F of all fixed points under ϕ is open and closed. Since it is well known that C_a is always a subgroup of G, it remains to check that $C_a = F$. We have: $\phi(x) = x$ if and only if $a^{-1}xa = x$ if and only if $ax = xa$ if and only if $x \in C_a$. $\square$

Remark 2. Theorem 5 allows to strengthen Theorem 2 in the following way. *Let G be an extremally disconnected topological group. Then, for any $a \in G$, there exists an open (and closed) Abelian subgroup A of G such that, for every element b of A, $ab = ba$ and $b^2 = e$.*

Theorem 6. *Let G be an extremally disconnected semitopological group with continuous inverse, such that G is generated by every open neighbourhood of the neutral element e. Then G is Abelian, and $a^2 = e$, for each $a \in G$.*

PROOF: Let $U = \{a \in G : a^2 = e\}$. Take any $a \in U$ and any $b \in G$. By Proposition 4, there exists an open neighbourhood V of e such that a commutes with every element of V. Then, obviously, a commutes with every element of the subgroup H algebraically generated by V. However, H coincides with G by the assumption. Hence, a commutes with every element of G. It follows, in particular, that any two elements of U commute. By Theorem 3, U is an open neighbourhood of e. Therefore, by the assumption, U generates G. On the other hand, if $a \in U$ and $b \in U$, then $a^{-1} = a \in U$, and $ab \in U$, since $abab = abba = aea = a^2 = e$. Therefore, U is a subgroup of G. It follows that $G = U$. □

Corollary 7 ([9])**.** *Let h be a homeomorphism of the Stone-Čech compactification βN of an infinite discrete space N onto βN such that there are no fixed points of h in N. Then no point of βN is fixed under h.*

PROOF: Since the space βN is extremally disconnected, this follows from Theorem 1. □

Several other results in the paper [9] of R. Raimi can be similarly deduced from Theorem 1; such proofs seem to be more elementary and transparent, than the original ones.

Theorem 8. *Let G be a separable extremally disconnected semitopological group with continuous inverse. Then there exists an Abelian subgroup H of G such that H is a closed G_δ-subset of G. Moreover, H can be chosen so that every element of H commutes with every element of G.*

PROOF: Fix a countable dense subset A of G. By Theorem 5, for each $a \in A$ there exists an open and closed subgroup H_a of G such that every $x \in H_a$ commutes with a. Put $H = \cap\{H_a : a \in A\}$. Then H is a closed subgroup of G and a G_δ-subset of G; it is also clear that every $x \in H$ commutes with every element of A. Since A is dense in G, and left and right translations are continuous, it follows that every $x \in H$ commutes with each element of G. In particular, H is Abelian. □

Theorem 9. *Let G be an extremally disconnected semitopological group with continuous inverse, and b any element of G. Then the set $M_b = \{x \in G : x^2 = b\}$ is open and closed in G.*

Proof: Let h_b be the mapping of G into itself given by the rule: $h_b(x) = x^{-1}b$, for each $x \in G$. Obviously, h_b is a homeomorphism of the space G onto itself. Therefore, the set F of all fixed points under h_b is an open and closed subset of G. Now, F coincides with M_b. Indeed, for $a \in G$, $h_b(a) = a$ if and only if $a = a^{-1}b$ if and only if $a^2 = b$. □

Corollary 10. *Let G be an extremally disconnected semitopological group with continuous inverse, and let $S_2 = S_2(G) = \{M_b : b \in G\}$, where the sets M_b are defined as in Theorem 9. Then S_2 is a disjoint open covering of the space G.*

Everywhere below S_2 and M_b have the same meaning as in Theorem 9 and Corollary 10. We say that *the discrete Souslin number* of a space X is countable if every discrete in X family of non-empty open subsets of X is countable.

Proposition 11. *Let G be an extremally disconnected semitopological group with continuous inverse such that the discrete Souslin number of the space G is countable. Then the set of all $b \in G$, for which there exists $a \in G$ such that $a^2 = b$, is countable.*

Proof: This follows from Corollary 10 which guarantees that, under the restrictions on G in Proposition 11, for only countably many b in G the set M_b is non-empty. □

We will call a group G *a group with square roots*, if for each $b \in G$ there exists $a \in G$ such that $a^2 = b$. From Proposition 11 we immediately get the next result:

Theorem 12. *Let G be an extremally disconnected semitopological group with continuous inverse and with square roots such that the discrete Souslin number of the space G is countable. Then G is countable.*

Corollary 13. *Let G be a pseudocompact extremally disconnected semitopological group with continuous inverse and with square roots. Then G is finite.*

Proof: By Theorem 12, G is countable. Therefore it is Lindelöf, and G being pseudocompact, it has to be compact. Since every countable compact Hausdorff space has an isolated point, G must be discrete. Therefore, G is finite. □

Corollary 14. *Let G be a Lindelöf extremally disconnected semitopological group with continuous inverse and with square roots. Then G is countable.*

Corollary 15. *Let G be an extremally disconnected semitopological group with continuous inverse and with square roots such that the Souslin number of G is countable. Then G is countable.*

Theorem 16. *Let G be an extremally disconnected semitopological group with continuous inverse and with square roots such that the discrete Souslin number of the space G is countable and the space G has the Baire property. Then G is countable and discrete.*

Proof: This assertion follows from Theorem 12, since every countable T_1 -space with the Baire property has an isolated point. Indeed, then the space G, being homogeneous, must be discrete. □

Remark 3. Notice, that if G is an extremally disconnected group, then the set $L = \{x \in G : x^3 = e\}$ need not be open in G. Indeed, if L is open, then L is a neighbourhood of e; therefore, $L \cap M_e$ is also an open neighbourhood of the neutral element e in G. On the other hand, it is clear that $M_e \cap L = \{e\}$; therefore, e is isolated in G, which implies that G is discrete.

Observe that the next old question, put forward in [1], remains open:

Problem 1. Is there in ZFC an example of a non-discrete extremally disconnected topological group?

Even the answer to the following, much more general, question seems to be unknown:

Problem 2. Is there in ZFC an example of a non-discrete extremally disconnected semitopological group with continuous inverse?

It is well known that if we do not require the continuity of inverse in Problem 2, then the answer is "yes" ([3]).

In connection with Theorem 2, it is natural to ask the following question:

Problem 3. Let G be an extremally disconnected semitopological group with continuous inverse. Is then true that there exists an open and closed Abelian subgroup of G?

Notice that Theorem 8 is a partial result just in this direction.

In connection with Problems 2 and 3, we should mention that Theorem 3 is strong enough to derive the following statement which shows that many (algebraic) groups do not admit a topology of the kind we are looking for.

An element a of a group G we call *Boolean* if $a \neq e$ and $a^2 = e$. Then Theorem 1 immediately implies the following statements:

Corollary 17. *Let G be a group such that the set of all Boolean elements of G is finite, and $\mathcal{T}$ is a topology on G such that the inverse operation is continuous and the one-point set e does not belong to $\mathcal{T}$. Then $\mathcal{T}$ is not extremally disconnected.*

Corollary 18. *Let G be a group such that the set of all Boolean elements of G is finite, and $\mathcal{T}$ is a topology on G such that the inverse operation is continuous and the space $(G, \mathcal{T})$ is homogeneous. Then the space $(G, \mathcal{T})$ is extremally disconnected if and only if it is discrete.*

Example 19. The assumption that the space $(G, \mathcal{T})$ is homogeneous cannot be dropped in Corollary 18. Indeed, let us fix an extremally disconnected topology $\mathcal{T}$ on the set P of all positive real numbers such that P is dense in itself. For $V \subset P$, put $-V = \{-x : x \in V\}$. Then the family $\mathcal{B} = \mathcal{T} \cup \{-V : V \in \mathcal{T}\} \cup \{\{0\}\}$ is a base of a non-discrete extremally disconnected topology $\mathcal{T}^*$ on the set R of all real numbers such that the inverse mapping is continuous.

In conclusion, we present the following theorem:

Theorem 20. *If a topological skew field F is extremally disconnected, then it is discrete.*

Proof: A topological skew field F is a topological ring in which multiplication is not necessarily commutative and the set G of all non-zero elements is a topological group under the multiplication. Let 0 and 1 denote the zero element and the unit element of F. Notice that $G = F \setminus \{0\}$ is dense in F and, therefore, the space G is also extremally disconnected.

Since F is an extremally disconnected topological group with respect to addition, there exists an open neighbourhood V of 0 such that $a + a = 0$, for each $a \in V$.

Since G is an extremally disconnected topological group with respect to multiplication, there exists an open neighbourhood W of 1 such that $b^2 = 1$, for each $b \in W$. Clearly, W is open in F since G is open in F.

Since F is a semitopological group with respect to addition, there exits an open neighbourhood U of 0 such that

$$1 + U \subset W.$$

Then for any $a \in U$ we have: $(1+a)(1+a) = 1+(a+a)+a^2 = 1+0+a^2 = 1+a^2$, since $a \in V$. On the other hand, $(1 + a)^2 = 1$, since $1 + a \in W$. Therefore, $1 = 1 + a^2$ which implies that $a^2 = 0$. Since all non-zero elements of F are invertible, it follows that $a = 0$. Therefore, $U = \{0\}$, and, hence, F is discrete. □

Remark 4. Theorem 20, as it is clear from its proof, remains true if we only assume that F is an extremally disconnected semitopological skew field, that is, both F and G are semitopological groups with continuous inverse.

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