On the semilinear multi-valued flow under constraints and the periodic problem*

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Abstract. ** In the paper we will be concerned with the topological structure of the set of solutions of the initial value problem of a semilinear multi-valued system on a closed and convex set. Assuming that the linear part of the system generates a C_0-semigroup we show the R_δ-structure of this set under certain natural boundary conditions. Using this result we obtain several criteria for the existence of periodic solutions for the semilinear system. As an application the problem of controlled heat transfer in an isotropic rigid body is considered.

Keywords: multi-valued maps, C_0-semigroup, initial value problem under constraints, R_δ-sets, periodic solutions, equilibria, control problem

Classification: Primary 49K24, 34C30, 34C25; Secondary 47D06

Introduction

Let E be a Banach space, D a closed, convex subset of E, A the infinitesimal generator of a C_0-semigroup $\{U(t)\}_{t\geq 0}$ and $F : [0,T] \times D \to 2^E$ an upper semi-continuous multi-valued map. We consider the semilinear differential inclusion

$$x'(t) \in Ax(t) + F(t, x(t)). \tag{1}$$

First, assuming as boundary conditions, that the semigroup $\{U(t)\}_{t\geq 0}$ leaves D invariant and that F satisfies a condition expressed in terms of the Bouligand contingent cone, we will study the topological structure of the set of mild solutions to (1) and we show that this set carries an R_δ-structure, i.e. it can be represented as the intersection of a decreasing sequence of compact absolute retracts (see [13]).

Problem (1) with single- or multi-valued perturbation F was studied by many authors (see [15], [16], [18], [6]). Also the structure of the solution set of differential inclusions in infinite dimensional spaces was considered in various works (see [7] and references given there). In particular, we mention [6], where it was shown that the set of strict solutions of (1) is an R_δ-set in the case $A = 0$ (in [12] it was observed that this set is compact and connected) and [7], where the same result

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was obtained for (1) but F defined on E, i.e. problem (1) was considered without constraints.

In the second part, we let $F : [0,\infty) \times D \to 2^E$ be T-periodic and consider the problem of T-periodic, mild solutions to (1). Applying our results on the integral funnel it is possible to apply certain fixed point principles of multi-valued mappings (see [3]) on the translation operator along trajectories (see [11] for this attitude in the finite dimensional setting and [4], [14] in the present case but without constraints). Using this approach we obtain various assumptions on D, F and A assuring the existence of periodic solutions. Thus we obtain multi-valued generalizations of the existence principles formulated in [17] for single-valued nonlinearities. Also some refinements of results given in [4] and [14] are considered. In case F is autonomous, we can derive sufficient conditions for the existence of equilibria from the results on periodic solutions. Finally we will be concerned with a control problem arising in the study of controlled heat transfer in an isotropic, homogeneous body. We give sufficient conditions in terms of the heating sources, which can be regulated, in order to obtain a periodic evolution of the system.

1. Preliminaries

Given a metric space (X,d) let $B_X(x,r) := \{z \in X : d(x,z) < r\}$ be the open ball with center x and radius r (the subscript X is omitted unless it leads to ambiguity) and $d(x,D) := \inf_{z\in D} d(x,z)$ the distance from x to a set $D \subset X$. The interior, closure and boundary of D will be denoted by $\operatorname{int} D$, $\overline{D}$ and ∂D, respectively. In the sequel, E will always be a Banach space over $\mathbb{C}$ or $\mathbb{R}$ with the norm $|\cdot|$ and E^* is the normed dual of E. If U is a linear operator in E the resolvent set of U is denoted by $\varrho(U)$ and the Banach space of all bounded linear operators is $\mathcal{L}(E)$ with the operator norm $|\cdot|_0$. Given reals $a < b$ we let $C([a,b],X)$ be the Banach space of continuous $y : [a,b] \to X$ equipped with the maximum norm and by $L^1([a,b],E)$ we mean the Banach space of all Bochner integrable maps $f : [a,b] \to E$ with the norm $|f|_1 := \int_a^b |f(s)|\,ds$.

For nonempty, bounded subsets Ω of E the Hausdorff measure of noncompactness (MNC) χ is given by

$$\chi(\Omega) := \inf\Big\{ r > 0 : \text{there are finitely many points } x_1, \dots, x_k \in E \text{ with } \Omega \subset \bigcup_{i=1}^{k} B(x_i, r) \Big\}.$$

Let us also provide some terminology concerning multi-valued maps. Given metric spaces X and Y a multi-valued map $F : X \to 2^Y \setminus \emptyset$ is called upper semi-continuous (usc), if $F^{-1}(V) := \{x \in X : F(x) \subset V\}$ is open in X whenever V is open in Y and it is lower semi-continuous (lsc) if $F^{-1}(A)$ is closed in X whenever A is closed in Y. Given another multi-valued map $G : Y \to 2^Z$ the composition of F and G is given by $G \circ F : X \to 2^Z$, $G \circ F(x) := \bigcup_{y \in F(x)} G(y)$. A multi-valued map $F : D \to 2^E \setminus \emptyset$, where D is a subset of the Banach space E, is called a k-set-contraction, if F is usc and for every bounded $\Omega \subset D$ the set $F(\Omega) := \bigcup_{x \in \Omega} F(x)$ is bounded and $\chi(F(\Omega)) \leq k\chi(\Omega)$. In case $k = 0$ we say that F is a compact map.

2. Existence of solutions for the semilinear multi-valued flow under constraints

Let A be a closed linear (in general unbounded) operator on E, which is the infinitesimal generator of a C_0-semigroup $\{U(t)\}_{t \geq 0}$ (see [15]). Let D be a closed subset of E and let $F : [0, T] \times D \to 2^E \setminus \emptyset$ be a multi-valued mapping. Given $x_0 \in D$ we consider the initial value problem

$$\begin{cases} x'(t) \in Ax(t) + F(t, x(t)), \\ x(0) = x_0. \end{cases} \tag{2}$$

A continuous mapping $x : [0, T] \to D$ is called a mild solution of (2) if x satisfies the integral equation

$$x(t) = U(t)x_0 + \int_0^t U(t-s)f(s)\,ds \quad \text{for every } t \in [0, T], \tag{3}$$

where

$$f \in N_F(x) := \{g \in L^1([0, T], E) : g(t) \in F(t, x(t)) \text{ a.e. on } [0, T]\}.$$

The set of all mild solutions of (2) will be denoted by $S(x_0)$. Recall that strong solutions of (2), i.e. continuous $x : [0, T] \to D$ such that $x(0) = x_0$, x is almost everywhere (a.e.) differentiable with $x' \in L^1([0, T], E)$ and x satisfies (2) a.e. on $[0, T]$, must not exist in general. Clearly, if we have $A = 0$, a mild solution to (2) is also a strong solution.

In order to state the boundary condition on F sufficient for the existence of solutions we consider the Bouligand contingent cone, i.e.

$$T_D(x) = \{y \in E : \liminf_{h \to 0, h > 0} \frac{d(x + hy, D)}{h} = 0\}.$$

Recall that in case D is convex the equality

$$T_D(x) = \overline{\bigcup_{t > 0} \frac{1}{t}(D - \{x\})}$$

holds and the following characterization in terms of hyperplanes is available (see [9, p. 32]): $y \in T_D(x)$ iff

$$x^* \in E^*, \ |x^*| = 1, \ \operatorname{Re} x^*(x) = \sup_D \operatorname{Re} x^*(z) \text{ implies } \operatorname{Re} x^*(y) \leq 0.$$

We also will use the condition that the semigroup $\{U(t)\}_{t \geq 0}$ leaves the set D invariant, i.e. for each $t \geq 0$ we have $U(t)D \subset D$. Recall that this condition can be characterized solely in terms of the generator A:

Lemma 1 (see [15, p. 304]). *Let A be the generator of a C_0-semigroup $\{U(t)\}_{t\geq 0}$ on E and let D be a closed, convex subset of E. Then $U(t)D \subset D$ for each $t \geq 0$ iff $(I - \lambda A)^{-1} D \subset D$ for each $\lambda > 0$ sufficiently small.*

We have the following existence result.

Theorem 2. *Let E be a Banach space, D be a closed subset of E and let A be the generator of a C_0-semigroup $\{U(t)\}_{t\geq 0}$ such that $U(t)D \subset D$ for each $t \geq 0$. Let $F : [0,T] \times D \to 2^E \setminus \emptyset$ be usc with compact convex values such that*

$$\|F(t,x)\| := \sup\{|z| : z \in F(t,x)\} \leq c(t)(1 + |x|) \quad \text{for every } t \in [0,T], x \in D,$$

where $c \in L^1([0,T], \mathbb{R})$ and

$$F(t,x) \cap T_D(x) \neq \emptyset \text{ for every } t \in [0,T] \text{ and } x \in D. \tag{4}$$

Then for each $x_0 \in D$ the initial value problem (2) has a mild solution provided that one of the following conditions holds:

(i) *for bounded $\Omega \subset D$*

$$\lim_{h\to 0, h>0} \chi(F(B_{[0,a]}(t,h) \times \Omega)) \leq k(t)\chi(\Omega) \text{ for each } t \in [0,T], \tag{5}$$

where $k \in L^1([0,T], \mathbb{R})$

(ii) *the semigroup $\{U(t)\}_{t\geq 0}$ is compact.*

Proof: Theorem 2 was shown in [6] under analogous assumptions, except that instead of our separated boundary conditions, i.e. $U(t)D \subset D$ for each $t \geq 0$ and (4), it was assumed that

$$F(t,x) \cap T_D^U(x) \neq \emptyset \text{ for every } t \in [0,a] \text{ and } x \in D, \tag{6}$$

where

$$T_D^U(x) := \left\{ y \in E : \liminf_{h\to 0, h>0} \frac{d(U(h)x + hy, D)}{h} = 0 \right\}. \tag{7}$$

Now the theorem follows since under the above assumptions $y \in T_D(x)$ implies $y \in T_D^U(x)$ ([1]). To see this, recall that $y \in T_D(x)$ iff $x + h_n y_n \in D$ for some $h_n \to 0$, $h_n > 0$ and some $y_n \to y$. Then

$$\begin{aligned} d(U(h_n)x + h_n y, D) &\leq d(U(h_n)(x + h_n y_n), D) + h_n|U(h_n)y_n - y| \\ &= h_n|U(h_n)y_n - y| \end{aligned}$$

since $U(t)D \subset D$ for every $t \geq 0$. Thus, using the strong continuity of $\{U(t)\}_{t\geq 0}$, it follows that

$$\frac{1}{h_n} d(U(h_n)x + h_n y, D) \to 0 \text{ as } n \to \infty,$$

i.e. $y \in T_D^U(x)$. □

We will also use the following continuity property of the solution operator.

[1]The argument used in the subsequent proof of this implication was communicated by D. Bothe.

Corollary 3. *Let the assumptions of Theorem 2 be fulfilled. Then the multi-valued map $S : D \to 2^{C([0,T],D)} \setminus \emptyset$ is usc with compact values ($C([0,T],D)$ denotes $\{x : [0,T] \to D : x$ is continuous$\}$).*

Proof: Follows easily by using arguments from the existence theory to (2) (see [6], [9]). $\square$

Remark. The "cone" defined in (7) does not have the useful property to define a lsc set-valued map $D \ni x \mapsto T_D^U(x) \subset E$ in the case where D is convex (as it happens for $T_D(\cdot)$). To see this let $E := \{x : [0,\infty) \to \mathbb{R} : x$ is bounded uniformly continuous$\}$ with the supremum norm and consider $(Af)(t) := f'(t)$ for $f \in D(A) := \{f \in E : f' \in E\}$. The operator A generates a nonexpansive C_0-semigroup. Now let $D := \{f \in E : f(0) = 0\}$ closed and convex. One easily sees $T_D(x) = D$ for every $x \in D$. We consider

$$f_n(t) := \frac{1}{n} \sin nt.$$

Then $f_n \to 0$ and $(Af_n)(t) = \cos nt$. Thus we see that $f_n \in D(A) \cap D$ and since for such elements g we have $T_D^U(g) = T_D(g) - Ag$ we get that $T_D^U(f_n) = D - \cos n\cdot$ and $T_D^U(0) = D$. Therefore, given an open $V := B(0,1)$, we have $T_D^U(0) \cap V \neq \emptyset$ but $T_D^U(f_n) \cap V = \emptyset$ for all $n = 1, 2, \ldots$, and we see that $T_D^U(\cdot)$ cannot be lsc.

3. Topological characterization of the solution set

In the section we will prove that if we assume in addition that $\operatorname{int} D \neq \emptyset$, the set of mild solutions to problem (2) is actually an R_δ-set, i.e. the intersection of a decreasing sequence of compact absolute retracts (see [13]). To prove this result we need the following two lemmata.

First we have the following generalization of the Michael selection theorem.

Lemma 5 (see [5]). *Let X be a metric space and E a Banach space. Let $F : X \to 2^E \setminus \emptyset$ be usc with closed, convex values and $G : X \to 2^E \setminus \emptyset$ be lsc with closed, convex values such that $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$. Then, given $\epsilon > 0$, there exists a continuous map $f : X \to E$ such that $f(x) \in F(B(x,\epsilon)) + B(0,\epsilon)$ for each $x \in X$ and f is a selection of G (i.e. $f(x) \in G(x)$ for each $x \in X$).*

Similarly as in [6], we will also apply the following simple consequence of the characterizations of R_δ-sets given in [13].

Lemma 6. *Let E be a Banach space and let $A_n \subset E$ be nonempty, closed and contractible such that $A_n \supset A_{n+1}$ for $n \geq 1$ and $\chi(A_n) \to 0$. Then the set*

$$A := \bigcap_{n \geq 1} A_n$$

is an R_δ-set.

Theorem 7. *Let E be a Banach space, $D \subset E$ closed, convex and bounded with $\operatorname{int} D \neq \emptyset$ and let A be the generator of a C_0-semigroup $\{U(t)\}_{t\geq 0}$ such that $U(t)D \subset D$ for each $t \geq 0$. Let $F : [0,T] \times D \to 2^E \setminus \emptyset$ be usc with compact convex values such that $\|F(t,x)\| \leq c$ on $[0,T] \times D$ and (4) holds. Then for each $x_0 \in D$ the set of mild solutions $S(x_0)$ is an R_δ-set provided that one of the conditions* (i) *or* (ii) *of Theorem* 2 *holds.*

Proof: 1. Let M be given such that $|U(t)|_0 \leq M$ for every $t \in [0,T]$.

Since D is a closed, convex set the map $T_D : D \to E$ is lsc with closed, convex values (see [9, p. 32]). Thus, by Lemma 5, for each $n \geq 1$ there exists a continuous map $g_n : [0,T] \times D \to X$ such that $g_n(t,x) \in T_D(x)$ and $g_n(t,x) \in F(B_{[0,T]}(t, \frac{1}{n}) \times B_D(x, \frac{1}{n})) + B_E(0, \frac{1}{n})$ for each $t \in [0,T]$ and $x \in D$.

Now let $x_1 \in \operatorname{int} D$ and $D_1 := D - \{x_1\}$. Then $0 \in \operatorname{int} D_1$ and we find $\mu > 0$ such that $B_E(0,\mu) \subset D_1$. In view of the Lasota-Yorke theorem choose a locally Lipschitz map $\tilde{f}_n : [0,T] \times D \to E$ such that

$$\sup_{[0,T]\times D} |\tilde{f}_n(t,x) - g_n(t,x)| < \frac{\mu}{n}$$

and we define

$$f_n : [0,T] \times D \to E, \;\; f_n(t,x) := \tilde{f}_n(t,x) - \frac{1}{n}(x - x_1).$$

Clearly, the map f_n is also locally Lipschitz and we claim that in addition $f_n(t,x) \in T_D(x)$ on $[0,T] \times D$ holds. To this end let $t \in [0,T]$, $x \in D$ be arbitrary and let $x^* \in E^*$ be such that $|x^*| = 1$, $\operatorname{Re} x^*(x) = \sup_D \operatorname{Re} x^*(z)$. Then

$$\begin{aligned}\operatorname{Re} x^*(f_n(t,x)) &\leq \operatorname{Re} x^*(\tilde{f}_n(t,x) - g_n(t,x)) - \frac{1}{n}\operatorname{Re} x^*(x - x_1)\\ &\leq |\tilde{f}_n(t,x) - g_n(t,x)| - \frac{\mu}{n} \leq 0\end{aligned}$$

since $x^*(x - x_1) \geq \mu$.

Finally let $d := 2 + \|D_1\|$. Then it is clear that for each $n \geq 1$ the map f_n is a selection of a multi-valued map $F_n : [0,T] \times D \to 2^E \setminus \emptyset$, given by

$$F_n(t,x) := F\left(B_{[0,T]}(t, \frac{1}{n}) \times B_D(x, \frac{1}{n})\right) + B_E(0, \frac{d}{n}).$$

Observe also that

$$(8) \qquad F(t,x) \subset F_{n+1}(t,x) \subset F_n(t,x) \;\text{ on }\; [0,T] \times D \;\text{ for every } n \geq 1$$

and all mappings F_n are bounded by $c + d$. Denoting by S_n the set of solutions of (2) where F is replaced by F_n, we have also from (8) that $S_n(x_0) \supset S_{n+1}(x_0)$ and $S(x_0) \subset \bigcap_{n\geq 1} S_n(x_0)$.

2. Recall that from $u_n \in S_n(x_0)$ for all $n \geq 1$ it follows that $u_{n_k} \to u \in S(x_0)$ for some subsequence $\{u_{n_k} : k \geq 1\}$ of $\{u_n : n \geq 1\}$. This implication follows from standard methods occurring in the existence theory for semilinear systems.

Now observe that the above mentioned fact shows that

$$S(x_0) = \bigcap_{n \geq 1} \overline{S}_n(x_0),$$

but, moreover, we also obtain $\chi_0(S_n(x_0)) \to 0$, where χ_0 denotes the Hausdorff MNC on $C([0,T],E)$; notice that we get

$$\varrho_n := \sup_{S_n(x_0)} d(v, S(x_0)) \to 0,$$

and thus $S_n(x_0) \subset S(x_0) + \overline{B}(0,\varrho_n)$, which implies $\chi_0(S_n(x_0)) \leq \varrho_n \to 0$, since $S(x_0)$ is compact by Corollary 3.

3. We show that for each $n \geq 1$ the set $\overline{S}_n(x_0)$ is contractible. Observe that in this case the proof is finished in view of Lemma 6. Let $f := f_n$ be the locally Lipschitz selection of F_n which exists by step 1. Then, for each $t_0 \in [0,T]$ and $x_0 \in D$ let us denote by $v(t;t_0,x_0)$ the unique solution of the integral equation

$$y(t) = U(t-t_0)x_0 + \int_{t_0}^{t} U(t-s)f(s,y(s))\,ds. \tag{9}$$

Define a homotopy $h : [0,1] \times \overline{S}_n \to \overline{S}_n$ by the formula

$$h(\lambda,u)(s) := \begin{cases} u(s), & \text{if } s \in [0,\lambda T], \\ v(s;\lambda,u(\lambda)), & \text{if } s \in [\lambda T, T]. \end{cases}$$

Since the semigroup $U(t)$ is strongly continuous and f_n is locally Lipschitz it is easy to see that the solution of equation (9) depends continuously on t_0 and x_0 and therefore h is continuous. We also have $h([0,1] \times S_n) \subset S_n$ and thus, from the continuity of h we infer, $h([0,1] \times \overline{S}_n) \subset \overline{S}_n$. Finally observe that $h(0,u) = v(\cdot;0,x_0)$ and $h(1,u) = u$ for every $u \in \overline{S}_n$, i.e. $\overline{S}_n$ is contractible. $\square$

4. Periodic solutions of the semilinear system

Let $F : [0,\infty) \times D \to 2^E \setminus \emptyset$ be T-periodic, which means that $F(t,x) \subset F(t+T,x)$ for every $t \in [0,\infty)$ and every $x \in D$. We will be concerned with the existence of T-periodic, mild solutions to

$$x'(t) \in Ax(t) + F(t,x(t)), \tag{10}$$

i.e. continuous, T-periodic maps $x : [0,\infty) \to D$ such that

$$x(t) = U(t)x(0) + \int_0^t U(t-s)f(s)\,ds \text{ for each } t \in [0,\infty) \tag{11}$$

with $f \in L^1_{loc}([0,\infty),E)$ such that $f(s) \in F(s,x(s))$ a.e. on $[0,\infty)$ (by $L^1_{loc}([0,\infty),E)$ we denote the set of all locally Bochner integrable maps $[0,\infty) \to E$).

Recall that a semigroup $\{U(t)\}_{t\geq 0}$ is said to be of type (C,ω) for some constants $C \geq 0, \omega \in \mathbb{R}$, if $|U(t)|_0 \leq C \exp(\omega t)$ for each $t \geq 0$. In the sequel, we will only consider semigroups of type $(1,\omega)$. However, this does not seem to be a serious restriction since there is an equivalent norm on E such that $C = 1$ holds. Our assumptions will be such that they remain valid under equivalent renorming.

In order to apply the results of the previous section to the periodic problem (10), we shall take advantage of the following fixed point result.

Proposition 8 (see [3]). *Let $D \subset E$ be closed, convex and bounded and let $D' \subset E'$ be a closed, convex subset of a Banach space E'. Let $\varphi : D \to 2^{D'} \setminus \emptyset$ be an usc map such that $\varphi(x)$ is an R_δ-set for each $x \in D$ and let $f : D' \to D$ be a continuous map. Finally let $\Phi := f \circ \varphi$ be a k-set-contraction for $k < 1$. Then the mapping Φ has a fixed point, i.e. there exists $x_0 \in D$ such that $x_0 \in \Phi(x_0)$.*

We start with the following result.

Theorem 9. *Let E be a Banach space and $D \subset E$ closed, convex and bounded with $\operatorname{int} D \neq \emptyset$. Let A be the generator of a C_0-semigroup of type $(1,\omega)$ such that $U(t)D \subset D$ for each $t \geq 0$. Assume that $F : [0,\infty) \times D \to 2^E \setminus \emptyset$ is usc, T-periodic with compact, convex values such that $\|F(t,x)\| \leq c$ on $[0,T] \times D$ and (4) hold. Then the periodic problem (10) has a mild solution in each of the following cases:*

- (i) *condition (i) of Theorem 2 holds, $t \mapsto U(t)$ is continuous with respect to the norm in $\mathcal{L}(E)$ for $t > 0$ and $\omega T + 4|k|_1 < 0$;*
- (ii) *condition (i) of Theorem 2 holds, E is separable and $\omega T + |k|_1 < 0$;*
- (iii) *$\{U(t)\}_{t\geq 0}$ is a compact semigroup.*

Proof: From Corollary 3 and Theorem 7 we know that the solution operator $S : D \to 2^{C([0,T],D)} \setminus \emptyset$ defines an usc map with R_δ-sets as its values. Next, let $e_T : C([0,T],D) \to D$ be the evaluation map, i.e. $e_T(u) := u(T)$ and consider the operator of translation along trajectories given by $P := e_T \circ S$. In view of the results of [4] it follows that in all three cases P is a k-set contraction with $k < 1$. Hence, by Proposition 8 we see that P has a fixed point, i.e. there is a continuous map $x : [0,T] \to D$ satisfying (3) such that $x(0) = x(T)$. Clearly, x can be extended to a continuous T-periodic map $\tilde{x} : [0,\infty) \to D$ and, by T-periodicity of F, we see that $\tilde{x}$ satisfies also (11) with a map f chosen in the obvious way. $\square$

Observe that the assumptions in the cases (i) and (ii) above imply $\omega < 0$. In the next result we consider also the case $\omega = 0$.

Theorem 10. *Let the suppositions in front of Theorem 9 be fulfilled. Then the periodic problem (10) has a solution provided*

$$0 \in D,\ \omega \leq 0,\ F \text{ is compact and } 1 \in \varrho(U(T)).$$

Proof: For $\epsilon > 0$ we consider $y' \in Ay - \epsilon y + F(t,y)$. Then the C_0-semigroup $U_\epsilon(t) := \mathrm{e}^{-\epsilon t}U(t)$ generated by $A - \epsilon I$ satisfies $U_\epsilon(t)D \subset D$ since $0 \in D$. By Theorem 9 we thus get the existence of a periodic solution x_ϵ to the perturbed equation. Since the resolvent set $\varrho(U(T))$ is open we have invertibility of $I - U_\epsilon(T)$ for $\epsilon > 0$ sufficiently small. Thus the following representation holds:

$$x_\epsilon(t) = U_\epsilon(t)(I - U_\epsilon(T))^{-1}\int_0^T U_\epsilon(T-s)f_\epsilon(s)\,ds + \int_0^t U_\epsilon(t-s)f_\epsilon(s)\,ds \tag{12}$$

for each $t \in [0,T]$ with some $f_\epsilon \in N_F(x_\epsilon)$.

Let $\epsilon_n \to 0$. Since F is compact, we get from (12) that $\{x_{\epsilon_n}(t) : n \geq 1\}, t \in [0,T]$ is relatively compact and, since equicontinuity is then also clear, it follows w.l.o.g. $x_{\epsilon_n} \to x \in C([0,T],E)$. Finally, applying arguments as in the second step of the proof of Theorem 7, we see that x is actually a solution of (10). □

In [8] there is an example showing that without the assumption "$1 \in \varrho(U(T))$" the theorem may be false (in this example there is actually $A = 0$).

We now consider the case where F is defined on the whole Banach space E. The above given results allow us to relax the growth condition on the nonlinearity F from results given in [4] and [14].

Corollary 11. *Let E be a Banach space, A the generator of a C_0-semigroup of type $(1,\omega)$. Assume that $F : [0,\infty) \times E \to 2^E \setminus \emptyset$ is compact, T-periodic with compact convex values. Assume also that there is $c \geq 0$ such that*

$$\frac{1}{|x|}\inf\{|z| : z \in F(t,x)\} \to c \quad \textit{uniformly in } t \in [0,T] \textit{ as } |x| \to \infty. \tag{13}$$

Then the periodic problem (10) *has a solution if $c < -\omega$.*

Proof: It is clear that the translation operator along trajectories P associated with (10) is an $e^{\omega T}$-set contraction ($\omega < 0$ by our assumptions). Now let $B := A - \omega I$ and $G(t,x) := F(t,x) + \omega x$. Since a solution to $x' \in Bx + G(t,x)$ is also a solution to (10), it follows that P_0 — the translation operator associated with the latter system — is also an $e^{\omega T}$-set contraction. In view of (13) we now may choose $R > 0$ such that for $|x| > R$ and $t \in [0,T]$ there exists $z \in F(t,x)$ such that $\frac{|z|}{|x|} < -\omega$. Now let $D := \overline{B}(0,R+1)$. Then our choice of R assures that $G(t,x) \cap T_D(x) \neq \emptyset$ for every $t \in [0,T]$, $x \in D$ and we have $V(t)D \subset D$ where V is the semigroup generated by B. Thus by arguments as in Theorem 9 we see the existence of a periodic solution. □

Recall that already scalar examples show that our assumption "$c < -\omega$" cannot be replaced by "$c \leq -\omega$" (e.g. let $Ax := \omega x$ and $f(x) := |x| + 1$; then $c = 1$ and for $\omega = -1$ there is no periodic solution).

Let us again consider the case $\omega = 0$. Recall that $\|A\| = \sup\{|x| : x \in A\}$ for a set $A \subset E$.

Corollary 12. *Let E be a Banach space, A the generator of a C_0-semigroup of type $(1,\omega)$. Assume that $F : [0,\infty) \times E \to 2^E \setminus \emptyset$ is compact, T-periodic with compact convex values and*

$$\frac{1}{|x|}\|F(t,x)\| \to 0 \quad \text{uniformly in } t \in [0,T] \text{ as } |x| \to \infty. \tag{14}$$

Then the periodic problem (10) *has a solution if $\omega \le 0$ and $1 \in \varrho(U(T))$.*

Proof: Let $\epsilon > 0$. Then Corollary 11 shows the existence of a periodic solution x_ϵ to the system $y' \in Ay - \epsilon y + F(t,y)$. From the representation (12) it follows that there exists $a > 0$ such that $|x_\epsilon(0)| \le a \int_0^T |f_\epsilon(s)|\, ds$ where $f_\epsilon \in N_F(x_\epsilon)$. Then by formula (11), growth condition (14) and an application of the Gronwall inequality we see that there are $b, d > 0$ such that $|x_\epsilon(t)| \le b|x_\epsilon(0)| + d$ for each $t \in [0,T]$.

Choose $\delta > 0$ such that $abT\delta < 1$. Then, again by (14), there exists $\beta > 0$ such that

$$\|F(t,x)\| \le \beta + \delta|x| \text{ for each } t \in [0,T], x \in E.$$

Thus it follows for a.e. $t \in [0,T]$

$$|f_\epsilon(t)| \le \beta + \delta|x_\epsilon(t)| \le \beta + \delta(b|x_\epsilon(0)| + d) = \beta + \delta d + \delta b|x_\epsilon(0)|$$

and hence

$$|x_\epsilon(0)| \le aT(\beta + \delta d + \delta b|x_\epsilon(0)|).$$

From the latter we infer

$$|x_\epsilon(0)| \le \frac{a(T\beta + T\delta d)}{1 - aT\delta b}$$

which shows that $\{x_\epsilon(0) : \epsilon > 0\}$ and therefore also $\{x_\epsilon(t) : \epsilon > 0\}$ is bounded. Now, using arguments as in the proof of Theorem 10 we see that x_ϵ converges to a solution x of (10) as $\epsilon \to 0$. □

In the remaining results we would like to dispense with the assumption $\operatorname{int} D \ne \emptyset$. Explicitly, we can replace this assumption by the condition that the metric retraction on D exists, i.e. there is a continuous map $r : E \to D$ such that $|r(x) - x| = d(x, D)$.

Theorem 13. *Let E be a Banach space and $D \subset E$ closed, convex, bounded and let the metric retraction r exist. Let A be the generator of a C_0-semigroup of type $(1,0)$ such that $U(t)D \subset D$ for each $t \ge 0$. Assume that $F : [0,\infty) \times D \to 2^E \setminus \emptyset$ is usc, T-periodic with compact, convex values such that $\|F(t,x)\| \le c$ on $[0,T] \times D$ and* (4) *holds. Then the periodic problem* (10) *has a solution in each of the following cases:*

(i) $\{U(t)\}_{t \ge 0}$ *is a compact semigroup;*
(ii) $0 \in D$, F *is compact and* $1 \in \varrho(U(T))$.

PROOF: Let $\delta > 0$ and consider a closed, convex and bounded subset $D_\delta := \{x \in E : d(x, D) \leq \delta\}$ with nonvoid interior. Extend F to an usc map $F_\delta : [0,T] \times D_\delta \to 2^E \setminus \emptyset$ by $F_\delta(t,x) := F(t, r(x))$. Clearly, we have that $F_\delta(t,x) \cap T_{D_\delta}(x) \neq \emptyset$ and $U(t)D_\delta \subset D_\delta$ since $\omega \leq 0$. Hence, an application of Theorem 9 (iii) and Theorem 10 yield the existence of a periodic solution x_δ to $y' \in Ay + F_\delta(t,y)$ in both cases.

Now let $\delta_n \to 0$. The usual compactness argument shows that w.l.o.g. $x_{\delta_n} \to x \in C([0,T], E)$. Clearly, $x(t) \in D$ for every $t \in [0,T]$ and one shows that x is in fact a solution to (10). □

Let us note that in case X is uniformly convex each closed, convex and bounded set possesses the metric retraction.

Finally we turn to the case where D is compact.

The following lemma gives a positive answer to open problem 13.1 in [9].

Lemma 14. *Let E be a Banach space, $D \subset E$ compact, convex and let $F : [0,T] \times D \to 2^E \setminus \emptyset$ be usc with compact, convex values such that* (4) *holds. Then the periodic problem $x'(t) \in F(t, x(t))$, $x(0) = x(T)$ has a strong solution.*

PROOF: The relevant sets D and $K := F([0,T] \times D)$ are compact and thus $E_0 := \overline{\operatorname{span}}\, D \cup K$ is a separable subspace of E. Therefore this subspace can be given an equivalent strictly convex norm and, hence, we may assume that E is a strictly convex Banach space. But then recall, that the metric retraction r on D exists and we can define D_δ and F_δ as in the proof of Theorem 13. Denote for $y_0 \in D$ by $S_\delta(y_0)$ the set of strong solutions to $y' \in F_\delta(t,y), y(0) = y_0$. Then $S_\delta : D \to 2^{C([0,T], D_\delta)} \setminus \emptyset$ is an usc map with R_δ-values. Let $g := r \circ e_T : C([0,T], D_\delta) \to D$. By Proposition 8 there exists a fixed point of the map $g \circ S_\delta$, i.e. there exists a function $x_\delta : [0,T] \to D_\delta$ having $x_\delta(0) \in D$, $x'_\delta(t) \in F_\delta(t, x_\delta(t))$ a.e. on $[0,T]$ and and $r \circ x_\delta(T) = x_\delta(0)$. Letting $\delta_n \to 0$ we obtain w.l.o.g. $x_{\delta_n} \to x \in C([0,T], E)$. Then $x(t) \in D$ for every $t \in [0,T]$ and x is a strong solution to $y' \in F(t,y)$. Moreover, $x(T) = r(x(T)) = r(\lim x_{\delta_n}(T)) = \lim r(x_{\delta_n}(T)) = \lim x_{\delta_n}(0) = x(0)$ which proves the lemma. □

Theorem 15. *Let E be a Banach space and $D \subset E$ compact, convex and let A be the generator of a C_0-semigroup such that $U(t)D \subset D$ for each $t \geq 0$. Assume that $F : [0,\infty) \times D \to 2^E \setminus \emptyset$ is usc, T-periodic with compact, convex values such that* (4) *holds. Then the periodic problem* (10) *has a solution.*

PROOF: For $\lambda > 0$ sufficiently small the Yosida approximation $A_\lambda = \lambda^{-1}((I - \lambda A)^{-1} - I)$ is well-defined and $A_\lambda \in \mathcal{L}(E)$. Let $G_\lambda : [0,T] \times D \to 2^E \setminus \emptyset$ be given by $G_\lambda(t,x) := A_\lambda x + F(t,x)$ and consider the periodic problem

$$(+_\lambda) \quad \begin{cases} x'(t) \in G_\lambda(t, x(t)), \\ x(0) = x(T). \end{cases}$$

We wish to show that $G_\lambda(t,x) \cap T_D(x) \neq \emptyset$ for each $t \in [0,T]$ and $x \in D$. By (4) there is $y \in F(t,x)$ such that $y \in T_D(x)$. Now $A_\lambda x + y \in G_\lambda(t,x)$ and given

$x^* \in E^*$, $|x^*| = 1$ and $\operatorname{Re} x^*(x) = \sup_D \operatorname{Re} x^*(z)$, we see that $\operatorname{Re} x^*(A_\lambda x + y) \leq \operatorname{Re} x^*(A_\lambda x) = \operatorname{Re} x^*(\lambda^{-1}((I - \lambda A)^{-1} x - x)) = \lambda^{-1}(\operatorname{Re} x^*((I - \lambda A)^{-1} x) - \operatorname{Re} x^*(x)) \leq 0$ since $(I - \lambda A)^{-1} x \in D$ by Lemma 1. Hence, it follows from Lemma 14 that problem $(+_\lambda)$ has a strong solution u_λ in D. Clearly u_λ is then also a mild solution and we have

$$u_\lambda(t) = \exp(tA_\lambda)x_0 + \int_0^t \exp((t-s)A_\lambda) f_\lambda(s)\, ds \text{ for every } t \in [0, T], \tag{15}$$

where $f_\lambda \in N_F(u_\lambda)$.

Let $\lambda_n \to 0$ and let x_{λ_n} be a solution of $(+_{\lambda_n})$. Taking into account that $\exp(\tau A_{\lambda_n})x \to U(\tau)x$ uniformly on compact subsets of E and $\tau \in [0, T]$ it follows from (15) that $\{x_{\lambda_n} : n \geq 1\}$ is equicontinuous. Hence, w.l.o.g. $x_{\lambda_n} \to x$ and x is a mild solution to (10). □

5. Existence of equilibria

Let $F : D \to 2^E \setminus \emptyset$. In this section we will consider the autonomous system

$$x'(t) \in Ax(t) + F(x(t)). \tag{16}$$

A stationary solution to (16), i.e. a point $x_0 \in D(A) \cap D$ satisfying $0 \in Ax_0 + F(x_0)$ is called an equilibrium of (16).

Theorem 16. *Let E be a Banach space and $D \subset E$ closed, convex and bounded. Let A be the generator of a C_0-semigroup of type $(1, \omega)$ such that $U(t)D \subset D$ for each $t \geq 0$. Assume that $F : D \to 2^E \setminus \emptyset$ is usc with compact, convex values such that*

$$F(x) \cap T_D(x) \neq \emptyset \text{ for each } x \in D \tag{17}$$

holds and consider the conditions

- (a) *$\operatorname{int} D \neq \emptyset$;*
- (b) *the metric retraction on D exists and $\{U(t)\}_{t \geq 0}$ is C_0-semigroup of type $(1, 0)$.*

Then (16) *has an equilibrium provided*

- (i) (a) *or* (b) *holds and $\{U(t)\}_{t \geq 0}$ is a compact semigroup,*
- (ii) (a) *or* (b) *holds, $0 \in D$, $\omega \leq 0$, F is compact and there exists t_0 such that $1 \in \varrho(U(t))$ for each $0 < t < t_0$,*
- (iii) (a) *holds, F is a k-set-contraction, $t \mapsto U(t)$ is continuous with respect to the norm in $\mathcal{L}(E)$ for $t > 0$ and $\omega T + 4k < 0$,*
- (iv) (a) *holds, E is separable and $\omega T + k < 0$.*

Proof: Choose $0 < T < t_0$. It follows from the results of the previous section that in all cases we get a T_n-periodic solution x_n to (16) for every $n \geq 1$, where

$T_n := 2^{-n}T$. The usual compactness argument shows that w.l.o.g $x_n \to x \in C([0,T],E)$, x is a solution to (16) and it is easily seen that $x(t) \equiv x_0 \in D$. Thus we have obtained

$$x_0 = U(t)x_0 + \int_0^t U(t-s)y_0\, ds \tag{18}$$

with some $y_0 \in F(x_0)$. From (18) we get for $t > 0$ that

$$\frac{1}{t}(U(t)x_0 - x_0) = -\left(\frac{1}{t}\int_0^t U(t-s)y_0\, ds\right) = -\left(\frac{1}{t}\int_0^t U(s)y_0\, ds\right).$$

Hence, letting $t \to 0, t > 0$, we see that $x_0 \in D(A)$ and $Ax_0 = -y_0$ showing that x_0 is a solution to (16). □

An evaluation of Theorem 15 gives:

Theorem 17. *Let E be a Banach space and $D \subset E$ compact, convex and let A be the generator of a C_0-semigroup such that $U(t)D \subset D$ for each $t \geq 0$. Assume that $F : D \to 2^E \setminus \emptyset$ is usc, with closed, convex values such that* (17) *holds. Then* (16) *has an equilibrium.*

6. Application to a problem of controlled heat transfer in an isotropic rigid body

We will be concerned with a periodic control problem described by a semilinear parabolic equation.

Let the quantity $x(t,y)$ describe the temperature in the point $y \in [0,1]$ at time t. The following differential system expresses the controlled process of heat transfer in an isotropic rigid body (see [2], [7]):

$$\begin{cases} \frac{\partial}{\partial t}x(t,y) & = \frac{\partial}{\partial y}\left[p(y)\frac{\partial}{\partial y}x(t,y)\right] + \sum_{i=1}^m w_i(t)\Delta_i(y,x(t,y)), \\ & \quad y \in [0,1],\ t \in [0,a], \\ x(t,0) & = x(t,1) = 0, \quad t \in [0,a], \\ x(0,y) & = x(a,y), \quad y \in [0,1]. \end{cases} \tag{19}$$

We assume that the coefficient of heat conductivity $p : [0,1] \to (0,\infty)$ is continuously differentiable and the functions $\Delta_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ $i = 1,\dots m$, which model the densities of the m heating sources, satisfy conditions as in [7], i.e.:

(Δ1) $\Delta_i(\cdot,z) : [0,1] \to \mathbb{R}$ is measurable for all $z \in \mathbb{R}$,
(Δ2) $|\Delta_i(y,z)| \leq \alpha_i(y)$ for all $z \in \mathbb{R}$ where $\alpha_i \in L^2([0,1])$,
(Δ3) $|\Delta_i(y,z_1) - \Delta_i(y,z_2)| \leq \beta_i|z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{R}$.

The control functions $w_i : [0,a] \to \mathbb{R},\ i = 1, \dots m$, are supposed to be measurable and subject to the feedback condition

$$(w_1(t), \dots , w_m(t)) \in W(t, x(t, \cdot)),\ t \in [0,a],$$

where $W : [0,a] \times L^2([0,1]) \to 2^V \setminus \emptyset$. Here $V \subset \mathbb{R}^m$ is compact and in order to avoid trivialities let us put $0 \notin V$.

We have the following

Theorem 18. *Let $W : [0,a] \times L^2([0,1]) \to 2^V \setminus \emptyset$ be usc with compact, convex values and such that for every $t \in [0,a]$ and $u \in L^2([0,1])$*

$$W(t,u) \cap \{v \in \mathbb{R}^m : \sum_{i=1}^{m} v_i = 1 \text{ and } v_i \geq 0 \text{ for each } i = 1, \dots , m\} \neq \emptyset. \tag{20}$$

Let for each $i = 1, \dots , m$ the conditions ($\Delta 1$), ($\Delta 2$) *and* ($\Delta 3$) *hold and assume also that there exists $C > 0$ such that*

$$\Delta_i(y, C) \leq 0,\ \Delta_i(y, -C) \geq 0 \text{ for every } y \in [0,1]. \tag{$\Delta 4$}$$

Then problem (19) *has a mild periodic solution, i.e. there exist measurable controls $w_i : [0,a] \to \mathbb{R},\ \ i = 1, \dots m$, satisfying $(w_1(t), \dots , w_m(t)) \in W(t, x(t, \cdot))$, $t \in [0,a]$, and x is a mild solution to* (19).

In case W is independent of t we obtain the existence of controls $w_1, \dots , w_m \in V$ such that (19) *has an equilibrium.*

Proof: Let E denote the Hilbert space $L^2([0,1])$ and define on $D(A) := \{f \in E : f, f'$ are absolutely continuous, $f(0) = f(1) = 0, f'' \in E\}$ the differential operator $Af := (pf')'$. It is well-known that A is the generator of a compact, analytic semigroup $\{U(t)\}_{t \geq 0}$ in E (see [15, p. 305]). Next consider $f_i : E \to E,\ i = 1, \dots , m$, given by

$$f_i(u)(y) := \Delta_i(y, u(y)),\ y \in [0,1].$$

Clearly, in view of our assumptions ($\Delta 1$), ($\Delta 2$) and ($\Delta 3$), the maps f_i are well-defined and continuous.

With these definitions we arrive at the abstract formulation of problem (19)

$$\begin{cases} x'(t) = Ax(t) + \sum_{i=1}^{m} w_i(t) f_i(x(t)), \\ x(0) = x(a), \end{cases} \tag{21}$$

where we look for a measurable $w : [0,a] \to \mathbb{R}^m$ such that $w(t) \in W(t, x(t))$, $t \in [0,a]$, and x is a mild solution to (21).

In order to solve (21) let us introduce the set-valued map $F : [0,a] \times E \to 2^E \setminus \emptyset$,

$$F(t,u) := \left\{ v \in E : v = \sum_{i=1}^{m} w_i f_i(u) \text{ where } w = (w_1, \dots , w_m) \in W(t,u) \right\}.$$

It follows that F is an usc, bounded map with compact convex values. Moreover, let us also introduce the closed, convex and bounded subset $D := \{v \in E : -C \leq v(s) \leq C$ for a.e. $s \in [0,1]\}$ of E.

One easily shows that $U(t)D \subset D$ for all $t \geq 0$ and that, in view of our assumption $(\Delta 4)$, $f_i(u) \in T_D(u)$ for each $u \in D$ and $i = 1, \dots, m$. The latter implies $F(t,u) \cap T_D(u) \neq \emptyset$, since, by (20), take $w \in W(t,u)$ such that $\sum_{i=1}^m w_i = 1$, $w_i \geq 0$ for $i = 1, \dots, m$, and then $\sum_{i=1}^m w_i f_i(u) \in T_D(u)$ follows from the convexity of $T_D(u)$.

We are in a position to apply Theorem 13 (i) and we see that $x' \in Ax + F(t,x)$, $x(0) = x(a)$ has a mild solution, i.e. there exists $x : [0,a] \to E$, $x(0) = x(a)$ and such that

$$x(t) = U(t)x(0) + \int_0^t U(t-s)g(s)\,ds, \quad t \in [0,a]$$

where $g \in N_F(x)$. But from the latter we infer by Filippov's theorem (see [1, p. 316]) the existence of a measurable map $w : [0,a] \to \mathbb{R}^m$ such that $w(t) \in W(t,x(t))$, $t \in [0,a]$, and $g(t) = \sum_{i=1}^m w_i(t) f_i(x(t))$ a.e. on $[0,a]$, i.e. x is a mild solution to (21). □

References

[1] Aubin J.-P., Frankowska H., *Set-valued Analysis*, Birkhäuser, 1990.

[2] Anichini G., Zecca P., *Multivalued differential equations in Banach spaces. An application in control theory*, J. Optim. Theory and Appl. **21** (1977), 477–486.

[3] Bader R., *Fixed point theorems for compositions of set-valued maps with single-valued maps*, Annales Universitatis Mariae Curie-Skłodowska, Vol. LI.2, Sectio A, Lublin, 1997, pp. 29–41.

[4] Bader R., *The periodic problem for semilinear differential inclusions in Banach spaces*, Comment. Math. Univ. Carolinae **39** (1998), 671–684.

[5] Ben-El-Mechaiekh H., Kryszewski W., *Equilibria of set-valued maps on nonconvex domains*, Trans. Amer. Math. Soc. **349** (1997), 4159–4179.

[6] Bothe D., *Multivalued differential equations on graphs and applications*, Ph. D. dissertation, Universität Paderborn, 1992.

[7] Conti G., Obukhovskii V., Zecca P., *On the topological structure of the solution set for a semilinear functional-differential inclusion in a Banach space*, in: Topology in Nonlinear Analysis, K. Geba and L. Górniewicz (eds.), Polish Academy of Sciences, Institute of Mathematics, Banach Center Publications **35**, Warszawa, 1996, pp. 159–169.

[8] Deimling K., *Periodic solutions of differential equations in Banach spaces*, Manuscripta Math. **24** (1978), 31–44.

[9] Deimling K., *Multivalued Differential Equations*, de Gruyter, Berlin-New York, 1992.

[10] Diestel J., *Remarks on weak compactness in $L_1(\mu, X)$*, Glasgow Math. J. **18** (1977), 87–91.

[11] Górniewicz L., *Topological approach to differential inclusions*, in: Topological methods in differential equations and inclusions, A. Granas and M. Frigon (eds.), NATO ASI Series C **472**, Kluwer Academic Publishers, 1995, pp. 129–190.

[12] Hu S., Papageorgiou N.S., *On the topological regularity of the solution set of differential inclusions with constraints*, J. Differential Equations **107** (1994), 280–290.

[13] Hyman D.M., *On decreasing sequences of compact absolute retracts*, Fund. Math. **64** (1969), 91–97.

[14] Kamenskii M., Obukhovskii V., Zecca P., *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, de Gruyter, to appear.

[15] Martin R., *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley, New York, 1976.
[16] Pavel N., *Invariant sets for a class of semi-linear equations of evolution*, Nonlinear Anal. **1** (1977), 187–196.
[17] Prüss, J., *Periodic solutions of semilinear evolution equations*, Nonlinear Anal. **3** (1979), 601–612.
[18] Shuzhong Shi., *Viability theorems for a class of differential-operator inclusions*, J. Differential Equations **79** (1989), 232–257.

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