On indefinite BV-integrals

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Abstract. We present an example of a locally BV-integrable function in the real line whose indefinite integral is not the sum of a locally absolutely continuous function and a function that is Lipschitz at all but countably many points.

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In 1986 Bruckner, Fleissner and Foran [2] obtained a descriptive definition of a minimal extension of the Lebesgue integral which integrates the derivative of any differentiable function. Recently, Bongiorno, Di Piazza and Preiss [1] showed that this minimal integral can be obtained from McShane's definition of the Lebesgue integral [4] by imposing a mild regularity condition on McShane's partitions.

The one-dimensional BV-integral defined in [5, Definition 13.4.2] lies properly in between the Lebesgue and Denjoy-Perron integrals [5, Theorem 11.4.5 and Example 12.3.5], and integrates the derivative (defined almost everywhere) of any function which is pointwise Lipschitz at all but countably many points [5, Theorem 12.2.5]. Moreover, in dimension one, the BV-integral is obtained from McShane's definition of the Lebesgue integral by using McShane's partitions consisting of finite unions of compact intervals, and imposing a regularity condition that is only slightly stronger than that employed in [1]. Thus it is natural to ask whether the BV-integral could be the minimal extension of the Lebesgue integral which integrates the derivative of any function that is Lipschitz at all but countably many points. We show by example the answer to this question is negative.

By $\mathbb{R}$ we denote the set of all real numbers equipped with its usual order and topology. The diameter and Lebesgue measure of a set $E \subset \mathbb{R}$ are denoted by $d(E)$ and $|E|$, respectively. Unless specified otherwise, all functions considered in this note are real-valued.

A *cell* is a compact nondegenerate subinterval of $\mathbb{R}$. A finite nonempty union of cells is called a *figure*. We say figures A and B *overlap* whenever $|A \cap B| > 0$. The *perimeter* of a figure A, denoted by $\|A\|$, is the number of the boundary points of A; clearly, $\|A\|$ equals twice the number of the connected components

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of A. The *regularity* of a pair (A, x) where A is a figure and $x \in \mathbb{R}$ is the number

$$r(A, x) = \frac{|A|}{d\big(A \cup \{x\}\big)\|A\|}\,.$$

If F is a function defined on a figure A and $B \subset A$ is a figure whose connected components are the cells $[a_1, b_1], \ldots, [a_p, b_p]$, we let

$$F(B) = \sum_{i=1}^{p} \big[F(b_i) - F(a_i)\big].$$

In this way, to each function F defined on A we associate a function defined on all subfigures of A, which is additive in the obvious way. With no danger of confusion, we denote both the function of points and the associated function of figures by the same symbol.

A *partition* is a collection (possibly empty) of pairs

$$P = \big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$$

where $A_1, \ldots, A_p$ are nonoverlapping figures, and $x_1, \ldots, x_p$ are points (not necessarily distinct) of $\mathbb{R}$. Given $\varepsilon > 0$ and a function δ defined on a set $E \subset \mathbb{R}$, we say the partition P is

- *ε-regular* if $r(A_i, x_i) > \varepsilon$ for $i = 1, \ldots, p$;
- *δ-fine* if $x_i \in E$ and $d\big(A_i \cup \{x_i\}\big) < \delta(x_i)$ for $i = 1, \ldots, p$.

A *gage* on a set $E \subset \mathbb{R}$ is a nonnegative function δ defined on E such that $\delta(x) > 0$ for all but countably many $x \in E$.

Definition 1. A function f defined on $\mathbb{R}$ is called *locally integrable* if there is a continuous function F defined on $\mathbb{R}$ satisfying the following condition: given $\varepsilon > 0$, we can find a gage δ on the interval $\big(-1/\varepsilon, 1/\varepsilon\big)$ so that

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each ε-regular δ-fine partition $\big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$.

The function F of Definition 1, which is uniquely determined by f up to an additive constant, is called the *indefinite integral* of f. A *primitive* is a function defined on $\mathbb{R}$ that is the indefinite integral of some locally integrable function.

As each one-dimensional BV set differs from a figure by a negligible set, using [5, Theorem 12.2.2] and [6, Corollary 4.4], it is easy to see that a function f defined on $\mathbb{R}$ is locally integrable if and only if for each cell A the restriction $f \restriction A$ is BV-integrable according to [5, Definition 13.4.2]. Thus it follows from [5, Theorem 12.3.2] that each primitive is differentiable almost everywhere, and it is the indefinite integral of its derivative extended arbitrarily to the whole of $\mathbb{R}$. Moreover, by [5, Theorems 11.4.5 and 12.2.5], locally absolutely continuous functions and functions that are Lipschitz at all but countably many points are primitives. Our goal is to show that not every primitive is the sum of these two types of functions.

If F is a continuous function defined on $\mathbb{R}$ and E is a subset of $\mathbb{R}$, let

$$V_*F(E) = \sup_\eta \inf_\delta \sup_P \sum_{i=1}^p \bigl|F(A_i)\bigr|$$

where η is a positive number, δ is a gage on E, and $P = \bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ is an η-regular δ-fine partition. The number $V_\# F(E)$ is defined analogously, except arbitrary gages δ on E are replaced by *positive gages* δ on E. According to [3, Lemma 4.9], the extended real-valued functions

$$V_*F : E \mapsto V_*F(E) \quad \text{and} \quad V_\# F : E \mapsto V_\# F(E)$$

are Borel regular measures in $\mathbb{R}$. Clearly $V_*F \le V_\# F$, and [7, Theorem 13] implies the next result.

Proposition 2. *A continuous function F defined on $\mathbb{R}$ is a primitive if and only if the measure V_*F is absolutely continuous.*

Choose a fixed positive $\alpha < 1$ so that $3^\alpha > 2$. Given a cell $K = [a, b]$, define a continuous function F_K on $\mathbb{R}$ by the formula

$$F_K(x) = (x - a)^\alpha \cos\left(\frac{\pi}{2} \cdot \frac{|K|^2}{x - a}\right)$$

if $a < x < a + |K|^2$, and let $F(x) = 0$ otherwise. Let D be the Cantor ternary set in the cell $[0, 1]$, and let $\mathcal{K}$ be the family of all cells whose interiors are connected components of the set $[0, 1] - D$. We show that the continuous function

$$F = \sum_{K \in \mathcal{K}} F_K \tag{1}$$

provides the desired example (see Theorem 9 below).

Lemma 3. *Let $K = [a, b]$ be a cell with $|K| < 1/2$, and let $\eta < 1/2$ and $\delta \le |K|/2$ be positive numbers. There is a constant $\kappa > 0$, depending only on η, such that*

$$\sum_{i=1}^p \bigl|F_K(A_i)\bigr| < \kappa \delta^\alpha$$

for each η-regular δ-fine partition $\bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ with

$$\{x_1, \dots, x_p\} \cap (a, b) = \emptyset\,.$$

In particular, the measure $V_{\#}F_K$ is absolutely continuous.

PROOF: With no loss of generality we may assume $a = 0$. Clearly, we only need to consider an η-regular δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ such that each figure A_i meets the interval $(0, b^2)$. As $b < 1/2$ and $\delta \le b/2$, this implies $x_i \le 0$ for $i = 1, \dots, p$. Moreover, if

$$A_i = \bigcup_{j=1}^{k_i} [a_i(j), b_i(j)]$$

where $a_i(1) < b_i(1) < \cdots < a_i(k_i) < b_i(k_i)$, then $b_i(k_i) > 0$. The regularity condition $r(A_i, x_i) > \eta$ implies $2k_i < 1/\eta$. Since it suffices to obtain separate estimates for the partitions

$$\{(A_i, x_i) : a_i(1) \ge 0\} \quad \text{and} \quad \{(A_i, x_i) : a_i(1) < 0\},$$

we can split the first part of the proof into the following two cases.

Case 1. Assume $a_i(1) \ge 0$ for $i = 1, \dots, p$. Reorder the figures $A_1, \dots, A_p$ so that

$$b_1(k_1) < b_2(k_2) < \cdots < b_p(k_p) < \delta, \tag{2}$$

and fix a positive integer $i \le p$. Since

$$\eta < r(A_i, x_i) \le r(A_i, 0) = \frac{1}{k_i} \sum_{j=1}^{k_i} \frac{b_i(j) - a_i(j)}{2b_i(k_i)},$$

there is a positive integer $j_i \le k_i$ such that

$$\eta < \frac{b_i(j_i) - a_i(j_i)}{2b_i(k_i)}.$$

Thus

$$2\eta b_i(k_i) < b_i(j_i) - a_i(j_i) \le b_i(k_i) - a_i(j_i), \tag{3}$$

and consequently

$$a_i(j_i) < (1 - 2\eta) b_i(k_i). \tag{4}$$

If $b_s(k_s) \le b_i(j_i)$ for each positive integer $s \le i$, let $s_i = i$; if $b_s(k_s) > b_i(j_i)$ for a positive integer $s \le i$, let s_i be the least such integer. By inequality (4),

$$b_s(k_s) \le a_i(j_i) < (1 - 2\eta) b_i(k_i) \tag{5}$$

whenever $1 \leq s < s_i$; indeed since $s < i$, the cells $[a_s(k_s), b_s(k_s)]$ and $[a_i(j_i), b_i(j_i)]$ do not overlap. If s is an integer with $s_i \leq s \leq i$, then

$$b_s(k_s) \geq b_{s_i}(k_{s_i}) \geq b_i(j_i)$$

according to inequality (2), and inequality (3) implies

$$b_s(j_s) - a_s(j_s) > 2\eta b_s(k_s) \geq 2\eta b_i(j_i) > 2\eta\big[2\eta b_i(k_i) + a_i(j_i)\big] \geq 4\eta^2 b_i(k_i).$$

As $[a_{s_i}(j_{s_i}), b_{s_i}(j_{s_i})], \dots, [a_i(j_i), b_i(j_i)]$ are disjoint subcells of $[0, b_i(k_i)]$, we have

$$b_i(k_i) \geq \sum_{s=s_i}^{i} \big[b_s(j_s) - a_s(j_s)\big] > (i - s_i + 1)\cdot 4\eta^2 b_i(k_i),$$

and consequently $i - s_i < 1/(4\eta^2) - 1$. It follows, there is a positive integer r, depending only on η, such that $i - r < s_i$. Inequality (5) yields

$$1 \leq s \leq i - r \Longrightarrow b_s(k_s) < (1 - 2\eta) b_i(k_i).$$

Thus letting $b_i = b_i(k_i)$ for $i = 1, \dots, p$, and $b_i = 0$ for $i = 0, -1, \dots,$ we obtain $b_{i-r} \leq (1 - 2\eta) b_i$ for all integers $i \leq p$. Proceeding inductively,

$$b_{p-kr} \leq (1 - 2\eta)^k b_p \tag{6}$$

for $k = 0, 1, \dots,$ and we calculate

$$\begin{aligned}
\sum_{i=1}^{p} \big|F_K(A_i)\big| &= \sum_{i=1}^{p} \left| \sum_{j=1}^{k_i} \Big[F_K\big(b_i(j)\big) - F_K\big(a_i(j)\big)\Big] \right| \\
&\leq \sum_{i=1}^{p} \sum_{j=1}^{k_i} \Big[\Big|F_K\big(b_i(j)\big)\Big| + \Big|F_K\big(a_i(j)\big)\Big|\Big] \\
&\leq \sum_{i=1}^{p} \sum_{j=1}^{k_i} \big[b_i(j)^\alpha + a_i(j)^\alpha\big] \leq \sum_{i=1}^{p} \sum_{j=1}^{k_i} 2 b_i^\alpha \qquad (7) \\
&< \frac{1}{\eta} \sum_{i=1}^{p} b_i^\alpha = \frac{1}{\eta} \sum_{k=0}^{\infty} \sum_{i=p-(k+1)r+1}^{p-kr} b_i^\alpha \\
&\leq \frac{1}{\eta} \sum_{k=0}^{\infty} r b_{p-kr}^\alpha < b_p^\alpha \frac{r}{\eta} \sum_{k=0}^{\infty} \big[(1 - 2\eta)^\alpha\big]^k < \beta \delta^\alpha
\end{aligned}$$

where $\beta > 0$ is a constant depending only on η.

Case 2. Assume $a_i(1) < 0$ for $i = 1, \ldots, p$. Since $b_i > 0$, there are unique figures $A_{i+} \subset [0, +\infty)$ and $A_{i-} \subset (-\infty, 0]$ whose union is A_i. After a suitable reordering, there is an integer q with $0 \leq q \leq p$ such that

$$r(A_{i+}, 0) \begin{cases} \leq \eta/2 & \text{for } i = 1, \ldots, q, \\ > \eta/2 & \text{for } i = q+1, \ldots, p. \end{cases}$$

According to Case 1 applied to the partition $\{(A_{(q+1)+}, 0), \ldots, (A_{p+}, 0)\}$,

$$\sum_{i=q+1}^{p} \left|F_K(A_{i+})\right| < \gamma \delta^\alpha$$

where $\gamma > 0$ is a constant depending only on η. As

$$\eta < r(A_i, x_i) \leq r(A_{i+}, 0) + r(A_{i-}, 0),$$

we see $r(A_{i-}, 0) > \eta/2$ for $i = 1, \ldots, q$. Reorder the figures $A_{1-}, \ldots, A_{q-}$ so that $a_q(1) < \cdots < a_1(1) < 0$, and let $a_i = a_i(1)$ for $i = 1, \ldots, q$ and $a_i = 0$ for $i = 0, -1, \ldots$. Applying Case 1 to the partition $\{(A^*_{1-}, 0), \ldots, (A^*_{q-}, 0)\}$ where $A^*_{i-} = \{-x : x \in A_{i-}\}$, find a positive integer r', depending only on η, so that inequalities analogous to (6) hold:

$$|a_{q-kr'}| \leq (1-\eta)^k |a_q|, \quad k = 0, 1, \ldots. \tag{8}$$

The inequality

$$\begin{aligned} \eta d\big(A_{i+} \cup \{0\}\big)\|A_{i+}\| &< \eta d\big(A_i \cup \{x_i\}\big)\|A_i\| < |A_i| = |A_{i+}| + |A_{i-}| \\ &< \frac{\eta}{2} d\big(A_{i+} \cup \{0\}\big)\|A_{i+}\| + |A_{i-}|, \end{aligned}$$

implies

$$\begin{aligned} \eta b_i = \eta d\big(A_{i+} \cup \{0\}\big) &\leq \frac{\eta}{2} d\big(A_{i+} \cup \{0\}\big)\|A_{i+}\| \\ &< |A_{i-}| \leq d\big(A_{i-} \cup \{0\}\big) = |a_i|. \end{aligned}$$

Calculating as in (7) and employing inequality (8), we obtain

$$\begin{aligned} \sum_{i=1}^{q} \left|F_K(A_{i+})\right| &< \frac{1}{\eta} \sum_{i=1}^{q} b_i^\alpha \leq \frac{1}{\eta^{\alpha+1}} \sum_{i=1}^{q} |a_i|^\alpha = \frac{1}{\eta^{\alpha+1}} \sum_{k=0}^{\infty} \sum_{i=q-(k+1)r'+1}^{q-kr'} |a_i|^\alpha \\ &\leq \frac{1}{\eta^{\alpha+1}} \sum_{k=0}^{\infty} r' |a_{q-kr'}|^\alpha < |a_q|^\alpha \frac{r'}{\eta^{\alpha+1}} \sum_{k=0}^{\infty} \big[(1-\eta)^\alpha\big]^k < \gamma' \delta^\alpha \end{aligned}$$

where $\gamma' > 0$ is a constant depending only on η. Now it suffices to let $\kappa = \gamma + \gamma'$.

The first part of the proof implies $V_\# F_K\big[\mathbb{R} - (a, b)\big] = 0$. Since the function F_K is locally absolutely continuous in (a, b), the measure $V_\# F_K$ is absolutely continuous in (a, b) by [7, Proposition 8]. The lemma follows. $\square$

Lemma 4. *Let $K = [a,b]$ be a cell with $|K| < 1/2$, and let $I_1, \dots, I_n$ be a sequence of nonoverlapping cells. Then*

$$\sum_{i=1}^{n} \bigl|F_K(I_i)\bigr| < \frac{2}{1-\alpha}\bigl(2+3n^{1-\alpha}\bigr)|K|^{2\alpha}.$$

Proof: With no loss of generality, we may assume that $a=0$ and that each I_i is a subcell of the cell $\bigl[0,b^2\bigr]$. Enlarging the number of the cells I_i by no more than $n+1$ cells and reordering them, we may further assume that $I_i = [c_{i-1}, c_i]$ where

$$0 = c_0 < c_1 < \cdots < c_m = b^2 \quad \text{and} \quad m \le 2n+1.$$

Let $a_0 = b^2$, and for $j = 1, 2, \dots$, let $a_j = b^2/(2j)$. Denote by j_1 the first positive integer with $a_{j_1} < c_1$, and let i_1 be the largest integer among $\{1, \dots, m\}$ with $c_{i_1} \le a_{j_1-1}$. If $j_1 = 1$, we stop. If $j_1 \ge 2$, denote by j_2 the first positive integer, necessarily smaller than j_1, with $a_{j_2} < c_{i_1+1}$, and let i_2 be the largest integer among $\{i_1+1, \dots, m\}$ with $c_{i_2} \le a_{j_2-1}$. Proceeding inductively, we obtain

$$\begin{aligned} 0 = c_0 < a_{j_1} < c_1 < \cdots c_{i_1} \le a_{j_1-1} < \cdots \\ < a_{j_2} < c_{i_1+1} < \cdots c_{i_2} \le a_{j_2-1} < \cdots \\ < \cdots < c_{i_p} = c_m = a_{j_p-1} = a_0 \end{aligned}$$

where $1 \le i_1 < \cdots < i_p = m$ and $1 = j_p < j_{p-1} < \cdots < j_1$. Thus $p \le m$ and $j_i \ge p-i+1$ for $i = 1, \dots, p$. As F_K is monotonic in each cell $[a_i, a_{i-1}]$, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \bigl|F_K(I_i)\bigr| \le \sum_{i=1}^{m} \bigl|F_K(I_i)\bigr| &\le \Bigl|F_K\bigl([0,a_{j_1}]\bigr)\Bigr| + \sum_{i=1}^{p} \Bigl|F_K\bigl([a_{j_i}, a_{j_i-1}]\bigr)\Bigr| \\ &\le a_{j_1}^{\alpha} + 2\sum_{i=1}^{p} a_{j_i-1}^{\alpha} \le a_p^{\alpha} + 2\sum_{i=1}^{p} a_{p-i}^{\alpha} \\ &< 2a_0^{\alpha} + 2\sum_{s=1}^{p} a_s^{\alpha} = 2b^{2\alpha}\left[1 + \sum_{s=1}^{p}\left(\frac{1}{2s}\right)^{\alpha}\right] \\ &< 2b^{2\alpha}\left(1 + \sum_{s=1}^{m} s^{-\alpha}\right) < 2b^{2\alpha}\left(2 + \int_1^m s^{-\alpha}\, ds\right) \\ &< \frac{2}{1-\alpha}\bigl(2+m^{1-\alpha}\bigr)|K|^{2\alpha} < \frac{2}{1-\alpha}\bigl(2+3n^{1-\alpha}\bigr)|K|^{2\alpha}. \end{aligned}$$

□

Lemma 5. *Let $K = [a,b]$ be a cell with $|K| < 1/2$, and let $\eta < 1/2$ be a positive number. There is a constant $\kappa' > 0$, depending only on η, such that*

$$\sum_{i=1}^{p} \bigl|F_K(A_i)\bigr| < \kappa' \,|K|^{\alpha}$$

for each η*-regular partition* $\{(A_1, x_1), \dots, (A_p, x_p)\}$ *such that*

$$\{x_1, \dots, x_p\} \cap (a, b) = \emptyset \quad \text{and} \quad \sum_{i=1}^{p} |A_i| \le 2.$$

Proof: After a suitable reordering, there is an integer q such that $0 \le q \le p$ and

$$d\big(A_i \cup \{x_i\}\big) \begin{cases} \ge \frac{1}{2}|K| & \text{for } i = 1, \dots, q, \\ < \frac{1}{2}|K| & \text{for } i = q+1, \dots, p. \end{cases}$$

As the partition $\{(A_{q+1}, x_{q+1}), \dots, (A_p, x_p)\}$ is $\big(|K|/2\big)$-fine, Lemma 3 yields

$$\sum_{i=q+1}^{p} \big|F_K(A_i)\big| < \kappa |K|^{\alpha} \tag{9}$$

where κ is a constant depending only on η. By our assumptions,

$$\sum_{i=1}^{q} \|A_i\| < \sum_{i=1}^{q} \frac{|A_i|}{\eta d\big(A_i \cup \{x_i\}\big)} \le \frac{2}{\eta|K|} \sum_{i=1}^{q} |A_i| \le \frac{4}{\eta|K|}\,.$$

It follows the figure $\bigcup_{i=1}^{q} A_i$ is the union of nonoverlapping intervals $I_1, \dots, I_n$ where $n \le 2/\big(\eta|K|\big)$. Observe $3^{\alpha} > 2$ implies $2\alpha > 1$, and use Lemma 4 to obtain

$$\begin{aligned} \sum_{i=1}^{q} \big|F_K(A_i)\big| \le \sum_{i=1}^{n} \big|F_K(I_i)\big| &< \frac{4}{1-\alpha}|K|^{2\alpha} + \frac{6}{1-\alpha}\left(\frac{2}{\eta|K|}\right)^{1-\alpha} |K|^{2\alpha} \\ &= \frac{2}{1-\alpha}\left[2|K|^{\alpha} + 3\left(\frac{2}{\eta}\right)^{1-\alpha} |K|^{2\alpha-1}\right] \cdot |K|^{\alpha} \\ &< \frac{2}{1-\alpha}\left[2 + 3\left(\frac{2}{\eta}\right)^{1-\alpha}\right] \cdot |K|^{\alpha}. \end{aligned} \tag{10}$$

The lemma follows by adding inequalities (9) and (10). □

Proposition 6. *The measure* $V_{\#}F$ *is absolutely continuous.*

Proof: It suffices to show $V_{\#}F(D) = 0$ (see the closing argument in the proof of Lemma 3). To this end, choose positive numbers $\eta < 1/2$ and $\varepsilon < 1$. If κ and κ' are the constants of Lemmas 3 and 5 corresponding to η, let

$$\beta = \max\{1, \kappa, \kappa'\}.$$

Select a positive integer N and $\delta > 0$ so that

$$2\beta \sum_{n=N+1}^{\infty} \left(\frac{2}{3^\alpha}\right)^n < \varepsilon \qquad \text{and} \qquad 2\delta^\alpha < \frac{\varepsilon}{3^N \beta},$$

and let $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ be an η-regular δ-fine partition such that $x_i \in D$ for $i = 1, \ldots, p$. As each figure A_i is contained in the cell $[-\delta, 1+\delta]$ and $\delta < 1/2$, the condition $\sum_{i=1}^{p} |A_i| \le 2$ of Lemma 5 is satisfied. For each integer $n \ge 1$, let $\mathcal{K}_n$ be the collection of 2^{n-1} cells whose length is 3^{-n} and whose interior is a connected component of $[0,1] - D$. Observe $\delta < |K|/2$ for each cell $K \in \mathcal{K}_n$ with $n \le N$. Applying Lemmas 3 and 5, we obtain

$$\begin{aligned}
\sum_{i=1}^{p} \bigl|F(A_i)\bigr| &\le \sum_{n=1}^{\infty} \sum_{K \in \mathcal{K}_n} \sum_{i=1}^{p} \bigl|F_K(A_i)\bigr| \\
&= \sum_{n=1}^{N} \sum_{K \in \mathcal{K}_n} \sum_{i=1}^{p} \bigl|F_K(A_i)\bigr| + \sum_{n=N+1}^{\infty} \sum_{K \in \mathcal{K}_n} \sum_{i=1}^{p} \bigl|F_K(A_i)\bigr| \\
&< \sum_{n=1}^{N} \sum_{K \in \mathcal{K}_n} \beta\delta^\alpha + \sum_{n=N+1}^{\infty} \sum_{K \in \mathcal{K}_n} \beta\bigl(3^{-n}\bigr)^\alpha \\
&= \beta\delta^\alpha \sum_{n=1}^{N} 2^{n-1} + \beta \sum_{n=N+1}^{\infty} 2^{n-1} 3^{-n\alpha} \\
&< \beta\delta^\alpha 2^N + \beta \sum_{n=N+1}^{\infty} \left(\frac{2}{3^\alpha}\right)^n < \beta\delta^\alpha 3^N + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

From the arbitrariness of ε and η, it is easy to infer $V_{\#}F(D) = 0$. □

A function G defined on $\mathbb{R}$ is called *Lipschitz at* $z \in \mathbb{R}$ if there are positive numbers c and η, depending on z, such that

$$\bigl|G(x) - G(z)\bigr| \le c|x - z|$$

for all $x \in \mathbb{R}$ with $|x - z| < \eta$. If H is a function defined on $\mathbb{R}$ and K is a cell, we denote by $VH(K)$ the *classical variation* of H in K.

Lemma 7. *Let $K = [a, b]$ be a cell with $|K| < 1/2$, let $\{z_i\}$ be a sequence in $(-\infty, a)$ converging to a, and select positive numbers c and η. Suppose G is a function defined on $\mathbb{R}$ such that for $i = 1, 2, \ldots,$*

$$\bigl|G(x) - G(z_i)\bigr| \le c|x - z_i|$$

whenever $|x - z_i| < \eta$. *If* $H = F_K - G$, *then* $VH(K) = \infty$.

PROOF: With no loss of generality, we may assume $a = 0$. Choose an $N > 0$, and select integers $s \geq r > 1/(4\eta)$ so that

$$(4r)^{1-\alpha} b^{2\alpha-2} \geq c + 1 \quad \text{and} \quad \sum_{k=r}^{s} \frac{b^2}{4k} > N + 1.$$

If $i \geq 1$ is an integer with $2(s - r + 1)cz_i > -1$ and $z_i > -\eta/2$, we obtain

$$\begin{aligned}
VH(K) &\geq \sum_{k=r}^{s} \left| H\left(\frac{b^2}{4k}\right) - H\left(\frac{b^2}{4k+2}\right) \right| \\
&= \sum_{k=r}^{s} \left| \left(\frac{b^2}{4k}\right)^{\alpha} + \left(\frac{b^2}{4k+2}\right)^{\alpha} - G\left(\frac{b^2}{4k}\right) + G\left(\frac{b^2}{4k+2}\right) \right| \\
&\geq \sum_{k=r}^{s} \left[\left(\frac{b^2}{4k}\right)^{\alpha} + \left(\frac{b^2}{4k+2}\right)^{\alpha} \right. \\
&\qquad \left. - \left| G\left(\frac{b^2}{4k}\right) - G(z_i) \right| - \left| G(z_i) - G\left(\frac{b^2}{4k+2}\right) \right| \right] \\
&\geq \sum_{k=r}^{s} \left[\left(\frac{b^2}{4k}\right)^{\alpha} + \left(\frac{b^2}{4k+2}\right)^{\alpha} \right. \\
&\qquad \left. - c\left| \frac{b^2}{4k} - z_i \right| - c\left| \frac{b^2}{4k+2} - z_i \right| \right] \\
&= \sum_{k=r}^{s} \left(\frac{b^2}{4k} \left[(4k)^{1-\alpha} b^{2\alpha-2} - c \right] \right. \\
&\qquad \left. + \frac{b^2}{4k+2} \left[(4k+2)^{1-\alpha} b^{2\alpha-2} - c \right] + 2cz_i \right) \\
&\geq \sum_{k=r}^{s} \left(\frac{b^2}{4k} + 2cz_i \right) = 2(s - r + 1)cz_i + \sum_{k=r}^{s} \frac{b^2}{4k} > N.
\end{aligned}$$

The lemma follows from the arbitrariness of N. □

Proposition 8. *Let* G *be a function defined on* $\mathbb{R}$ *that is Lipschitz at all but countably many points of the cell* $A = [0, 1]$. *If* $H = F - G$, *then* $VH(A) = \infty$.

PROOF: Since G is Lipschitz at all but countably many points of the Cantor ternary set D in A, a Baire category argument shows there are an open interval U with $D \cap U \neq \emptyset$, a set $C \subset D \cap U$ dense in $D \cap U$, and positive numbers c and η such that for each $z \in C$,

$$\left| G(x) - G(z) \right| \leq c|x - z|$$

whenever $|x - z| < \eta$. There is a connected component (a, b) of $A - D$ with $a \in D \cap U$. If $K = [a, b]$, Lemma 7 implies

$$VH(A) \geq VH(K) = V(F_K - G)(K) = \infty.$$

□

Theorem 9. *The function F defined by equation* (1) *is a primitive which is not the sum of a locally absolutely continuous function and a function that is Lipschitz at all but countably many points.*

Proof: Since $V_* F \leq V_\# F$, Propositions 2 and 6 show that F is a primitive. If G is a function defined on $\mathbb{R}$ that is Lipschitz at all but countably many points, then $F - G$ is not locally absolutely continuous by Proposition 8. □

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