Products in almost f-algebras

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Abstract. Let A be a uniformly complete almost f-algebra and a natural number $p \in \{3, 4, \dots\}$. Then $\Pi_p(A) = \{a_1 \dots a_p;\ a_k \in A,\ k = 1, \dots, p\}$ is a uniformly complete semiprime f-algebra under the ordering and multiplication inherited from A with $\Sigma_p(A) = \{a^p;\ 0 \leq a \in A\}$ as positive cone.

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1. Introduction

It is shown by Buskes and van Rooij in [5; Theorem 9] that if A is a uniformly complete almost f-algebra, then the set $\Pi_3(A) = \{fgh; f, g, h \in A\}$ is a uniformly complete semiprime f-algebra with respect to the multiplication and ordering inherited from A. However, their proof relies heavily on the representation theory, so Zorn's lemma (i.e., the axiom of choice) is (unnecessarily) involved.

In this work, we present a generalization of the latter result to the case of an arbitrary natural number $p \geq 3$ using only algebraic methods, thus avoiding Zorn's lemma as well as representation theorems.

One of the results shown in this paper is a generalization of a theorem about homogeneous polynomials on f-algebras (due to Beukers and Huijsmans) to the case of almost f-algebras. The theorem in question is the following: if A is a uniformly complete semiprime f-algebra and $F \in \mathbb{R}^+[X_1, \dots, X_n]$ is a homogeneous polynomial of degree p ($p \in \mathbb{N}$) then there exists $a \in A^+$ such that $a^p = F(a_1, \dots, a_n)$ for every $a_1, \dots, a_n \in A^+$ ([3; Theorem 5]).

For terminology and properties of vector lattices not explained or proven in this paper we refer to [7] and for elementary almost f-algebras, d-algebras and f-algebras theories, we refer to [1], [2], [8], and [9].

2. Preliminaries, some results in almost f-algebras

All (real) vector lattices and (real) lattice ordered algebras under consideration are supposed to be Archimedean and the only topology we consider in this paper is the (relatively) uniform topology ([7; Sections 16 and 63]).

Let A be a vector lattice and $0 \leq e \in A$. The principal o-ideal (order ideal) of A generated by e is denoted by A_e and it is a sublattice of A with e as a strong order unit.

The vector lattice A is said to be a *lattice ordered algebra* (or *l-algebra*) if there exists an associative multiplication in A with the usual algebraic properties such that $ab \in A^+$ for all $a, b \in A^+$. The l-algebra A is called an *f-algebra* whenever $a \wedge b = 0$, and $0 \leq c \in A$ imply $ac \wedge b = ca \wedge b = 0$. The l-algebra A is called an *almost f-algebra* if $a \wedge b = 0$ implies $ab = 0$. The l-algebra A is called a *d-algebra* whenever $a \wedge b = 0$, and $0 \leq c \in A$ imply $ac \wedge bc = ca \wedge cb = 0$.

Any f-algebra is an almost f-algebra and a d-algebra but not conversely. Almost f-algebras need not be d-algebras and vice versa. Archimedean (almost) f-algebras are automatically commutative (and even associativity follows in case of f-algebras) and have positive squares. Archimedean d-algebras need not be commutative nor have positive squares. Archimedean d-algebras which are commutative or do have positive squares are almost f-algebras.

For any l-algebra A, we denote by $N(A)$ the set of all nilpotent elements of A. We said that A is *semiprime* if $N(A) = \{0\}$. If A is an f-algebra then

$$N(A) = \{a \in A;\ a^2 = 0\} = \{a \in A;\ ab = 0 \text{ for all } b \in A\}.$$

If A is an almost f-algebra then

$$N(A) = \{a \in A;\ a^3 = 0\} = \{a \in A;\ abc = 0 \text{ for all } b, c \in A\}$$

and the quotient $A/N(A)$ is an Archimedean semiprime f-algebra. The equivalence class of the element $a \in A$ in $A/N(A)$ is denoted by $[a]$. Any almost f-algebra (or d-algebra) with multiplication unit $0 < e$ is semiprime and any semiprime almost f-algebra (or d-algebra) is an f-algebra.

Let A be an l-algebra and $p \in \{1, 2, \dots\}$. Throughout this paper we will keep the following notations:

(i) $\Pi_p(A) = \{a_1 \dots a_p;\ a_k \in A,\ k = 1, \dots, p\}$;
(ii) $\Sigma_p(A) = \{a^p;\ 0 \leq a \in A\}$.

By agreement, we put $a^0 b = ba^0 = b$ for all $a, b \in A$.

Let A be a vector lattice. The order bonded operator π of A is called *orthomorphism* if $|a| \wedge |b| = 0$ implies $|\pi(a)| \wedge |b| = 0$. The collection $Orth(A)$ of all orthomorphisms on A is, with respect to the usual vector spaces operations and composition as multiplication, an Archimedean f-algebra with the identity mapping I_A on A as a unit element. Moreover, if A is uniformly complete, so is $Orth(A)$ (for more information about orthomorphisms, refer to [8; Chapter 20]).

Let A and B be vector lattices and $p \in \{2, 3, \dots\}$. The p-linear map $\Psi : A^p \to B$ is said to be *positive* if $\Psi(a_1, \dots, a_p) \in B^+$ for all $a_1, \dots, a_p \in A^+$. The positive p-linear map $\Psi : A^p \to B$ is said to *have the property* (AF) if $a_i \wedge a_j = 0$ for some $i, j \in \{1, \dots, p\}$ implies $\Psi(a_1, \dots, a_p) = 0$.

The proof of commutativity of Archimedean almost f-algebras, given by Bernau and Huijsmans in [2; Theorem 2.15], does not make use of associativity. In fact, Bernau and Huijsmans showed the following theorem.

Theorem 1. *Let A and B be Archimedean vector lattices. Then every bilinear map $\Psi : A \times A \to B$ having the property (AF) is symmetrical.*

The following theorem is a generalization of the previous one.

Theorem 2. *Let A and B be Archimedean vector lattices, $p \in \{2, 3, \dots\}$, Ψ a p-linear map from A^p into B having the property (AF), and a permutation $\sigma \in S(p)$. Then $\Psi\left(a_1, .., a_p\right) = \Psi\left(a_{\sigma(1)}, .., a_{\sigma(p)}\right)$ for all $a_1, \dots, a_p \in A$.*

PROOF: Since $S(p)$ is generated by transpositions, it suffices to prove that if $i \neq j$ then $\Psi\left(.., a_i, .., a_j, ..\right) = \Psi\left(.., a_j, .., a_i, ..\right)$ for all $a_1, \dots, a_p \in A$.

We begin by the case $0 \leq a_1, \dots, a_p$. Let $i \neq j \in \{1, \dots, p\}$ and define

$$\begin{array}{rcl} \Phi: \quad A^2 & \rightarrow & B \\ (u, v) & \mapsto & \Psi\left(a_1, .., \overset{i}{u}, .., \overset{j}{v}, .., a_p\right). \end{array}$$

Evidently, Φ is a bilinear map with the property (AF), therefore it is symmetrical (Theorem 1). Hence

$$\Psi\left(.., a_i, .., a_j, ..\right) = \Phi\left(a_i, a_j\right) = \Phi\left(a_j, a_i\right) = \Psi\left(.., a_j, .., a_i, ..\right).$$

Assume now that $a_1, \dots, a_p \in A$ (no necessarily positive). We denote by D_p the set of all sequences $\varepsilon = \left(\varepsilon_1, \dots, \varepsilon_p\right)$ of length p consisting only of 1 and -1 and by $\Pi(\varepsilon)$ the product $\varepsilon_1 \dots \varepsilon_p \in \{-1, 1\}$. For every $a \in A$, put $a(1) = a^+$ and $a(-1) = a^-$. Then $\Psi\left(a_1, .., a_i, .., a_j .., a_p\right) = U - V$ with

$$U = \sum_{\substack{\varepsilon = (\varepsilon_1, \dots, \varepsilon_p) \\ \Pi(\varepsilon) = 1}} \Psi\left(a_1(\varepsilon_1), .., a_i\left(\varepsilon_i\right), .., a_j\left(\varepsilon_j\right), .., a_p\left(\varepsilon_p\right)\right)$$

and

$$V = \sum_{\substack{\varepsilon = (\varepsilon_1, \dots, \varepsilon_p) \\ \Pi(\varepsilon) = -1}} \Psi\left(a_1(\varepsilon_1), .., a_i\left(\varepsilon_i\right), .., a_j\left(\varepsilon_j\right), .., a_p\left(\varepsilon_p\right)\right).$$

Similarly, $\Psi\left(a_1, .., a_j, .., a_i .., a_p\right) = U' - V'$ with

$$U' = \sum_{\substack{\varepsilon = (\varepsilon_1, \dots, \varepsilon_p) \\ \Pi(\varepsilon) = 1}} \Psi\left(a_1(\varepsilon_1), .., a_j\left(\varepsilon_j\right), .., a_i\left(\varepsilon_i\right), .., a_p\left(\varepsilon_p\right)\right)$$

and

$$V' = \sum_{\substack{\varepsilon = (\varepsilon_1, \dots, \varepsilon_p) \\ \Pi(\varepsilon) = -1}} \Psi\left(a_1(\varepsilon_1), .., a_j\left(\varepsilon_j\right), .., a_i\left(\varepsilon_i\right), .., a_p\left(\varepsilon_p\right)\right).$$

Using the previous case, we get $U = U'$ and $V = V'$, which gives the desired result. □

The previous theorem allows to prove the following proposition which will turn out to be useful later.

Proposition 1. *Let A and B be Archimedean vector lattices, $p \in \{2, 3, \dots\}$, Ψ a p-linear map from A^p into B having the property (AF), and π an orthomorphism on A. Then, for every $i \neq j \in \{1, \dots, p\}$*

$$\Psi\left(a_1, .., \pi\left(a_i\right), .., a_j, .., a_p\right) = \Psi\left(a_1, .., a_i, .., \pi\left(a_j\right), .., a_p\right)$$

for all $a_1, \dots, a_p \in A$.

Proof: It suffices to prove this result for $0 \leq \pi$. Then let $i \neq j \in \{1, \dots, p\}$ and define

$$\begin{array}{rccc} \Phi: & A^p & \rightarrow & B \\ & \left(u_1, .., u_p\right) & \mapsto & \Psi\left(u_1, .., \pi\left(u_i\right), .., u_p\right). \end{array}$$

It is straightforward to show that Φ is a p-linear map with the property (AF). Consider now the transposition $\tau = (i, j)$. Applying Theorem 2 to Φ, we obtain $\Phi\left(a_1, .., a_p\right) = \Phi\left(a_{\tau(1)}, .., a_{\tau(p)}\right)$. Hence, again by Theorem 2 applied to Ψ,

$$\Psi\left(a_1, .., \pi\left(a_i\right), .., a_j, .., a_p\right) = \Psi\left(a_1, .., a_i, .., \pi\left(a_j\right), .., a_p\right)$$

as required. □

It is shown by Beukers and Huijsmans [3; Theorem 5] that if A is a uniformly complete semiprime f-algebra and $F \in \mathbb{R}^+\left[X_1, \dots, X_n\right]$ a homogeneous polynomial of degree $p \in \mathbb{N}$ then, for all $a_1, \dots, a_n \in A^+$, there exists (unique) $0 \leq a \in A$ such that $a^p = F\left(a_1, \dots, a_n\right)$. At the end of this section we will show that this result subsists in the case of uniformly complete almost f-algebras (evidently, we lose uniqueness). In order to hit this mark, we need the following proposition.

Proposition 2. *Let A be a uniformly complete almost f-algebra and a natural number $p \geq 2$. Then*

(i) *for every $0 \leq a_1, .., a_p \in A$, there exists $0 \leq u \in A$ such that $u^p = a_1 \dots a_p$;*
(ii) *for every $0 \leq a, b \in A$, there exists $0 \leq u \in A$ such that $u^p = a^p + b^p$.*

Proof: (i) Let $0 \leq a_1, .., a_p \in A$ and put $e = a_1 + \dots + a_p$. Consider

$$\begin{array}{rccc} \Psi: & \left(A_e\right)^p & \rightarrow & A \\ & \left(u_1, .., u_p\right) & \mapsto & u_1 \dots u_p. \end{array}$$

Obviously, Ψ is a p-linear mapping with the property (AF). Moreover, for every $k \in \{1, \dots, p\}$, there exists $0 \leq \pi_k \in Orth(A_e)$ such that $a_k = \pi_k(e)$ ([4; Theorem 2.6]). Hence

$$\begin{aligned} a_1 \dots a_p &= \Psi\left(a_1, \dots, a_p\right) \\ &= \Psi\left(\pi_1\left(e\right), .., \pi_p\left(e\right)\right) \\ &= \Psi\left(e, .., \left(\pi_1 .. \pi_p\right)\left(e\right)\right) \end{aligned}$$

(where we use Proposition 1). Now, applying [3; Theorem 5] to $Orth(A_e)$ which is a uniformly complete f-algebra with unit (therefore semiprime), there exists a positive orthomorphism π on A_e such that $\pi^p = \pi_1..\pi_p$. Consequently

$$\begin{aligned} a_1 \dots a_p &= \Psi\left(e, .., \pi^p\left(e\right)\right) \\ &= \Psi\left(\pi\left(e\right), .., \pi\left(e\right)\right) \\ &= \pi\left(e\right)^p . \end{aligned}$$

This gives the desired result.

The second assertion is obtained likewise. □

Now, we will state the main result of this section which is a simple inference of the previous proposition.

Theorem 3 ([3; Theorem 5]). *Let A be a uniformly complete almost f-algebra, a natural number $p \in \{1, 2, \dots\}$, and a homogeneous polynomial F of degree p in $\mathbb{R}^+[X_1, \dots, X_n]$. Then, for every $a_1, \dots, a_n \in A^+$, there exists $0 \leq a \in A$ such that $a^p = F(a_1, \dots, a_n)$.*

3. The l-algebra $\Pi_P(A)$

The main topic of this section is to investigate order and algebra structures of the set $\Pi_p(A)$ where A is a uniformly complete almost f-algebra and $p \in \{2, 3, \dots\}$.

In the next theorem we will show that if $p \geq 3$ then $\Pi_p(A)$ is a vector lattice under the ordering inherited from A.

Theorem 4. *Let A be a uniformly complete almost f-algebra and $p \in \{3, 4, \dots\}$. Then $\Pi_p(A)$ is a vector lattice under the ordering inherited from A with $\Sigma_p(A)$ as positive cone in its own right with the following supremum and infimum*

$$a^p \wedge_p b^p = (a \wedge b)^p \quad \textit{and} \quad a^p \vee_p b^p = (a \vee b)^p \quad \textit{for all} \;\; 0 \leq a, b \in A.$$

Proof: At first, we prove that $\Pi_p(A)$ is an order vector subspace of A with $\Pi_p(A)^+ = \Sigma_p(A)$. It is a straight deduction from Proposition 2(ii) that $\Sigma_p(A)$ is a positive cone in A. Therefore $\Sigma_p(A) - \Sigma_p(A)$ is an order vector subspace of A with $\left(\Sigma_p(A) - \Sigma_p(A)\right)^+ = \Sigma_p(A)$.

Let $0 \leq a, b \in A$. We have $a^p - b^p = (a - b)\left(\sum_{k=0}^{p-1} a^k b^{p-1-k}\right)$. Consider $F(X, Y) = \sum_{k=0}^{p-1} X^k Y^{p-1-k}$; F is a homogeneous polynomial of degree $p - 1$ in $\mathbb{R}^+[X, Y]$. Hence, by Theorem 3, there exists $0 \leq u \in A$ such that $u^{p-1} = \sum_{k=0}^{p-1} a^k b^{p-1-k}$. Therefore $a^p - b^p = u^{p-1}(a - b)$ which implies that $\Sigma_p(A) - \Sigma_p(A) \subset \Pi_p(A)$.

Conversely, let $a_1, \dots a_p \in A$ and keep the same notations as previously used in the proof of Theorem 2. We get

$$\begin{aligned} a_1 \dots a_p &= \left(a_1^+ - a_1^-\right) \dots \left(a_p^+ - a_p^-\right) \\ &= \sum_{\substack{\varepsilon=(\varepsilon_1,\dots,\varepsilon_p) \\ \Pi(\varepsilon)=1}} a_1(\varepsilon_1) \dots a_p\left(\varepsilon_p\right) \\ &\quad - \sum_{\substack{\varepsilon=(\varepsilon_1,\dots,\varepsilon_p) \\ \Pi(\varepsilon)=-1}} u_1(\varepsilon_1) \dots u_p\left(\varepsilon_p\right). \end{aligned}$$

Moreover, by Proposition 2(i), for every $\varepsilon = \left(\varepsilon_1, \dots, \varepsilon_p\right) \in D_p$, there exists $a\left(\varepsilon\right) \in A^+$ such that $a\left(\varepsilon\right)^p = a_1(\varepsilon_1) \dots a_p\left(\varepsilon_p\right)$. Therefore

$$a_1 \dots a_p = \sum_{\varepsilon, \Pi(\varepsilon)=1} a(\varepsilon)^p - \sum_{\varepsilon, \Pi(\varepsilon)=-1} a(\varepsilon)^p.$$

Put now $n = Card\left\{\varepsilon \in D_p ;\, \Pi\left(\varepsilon\right) = 1\right\}$ and $R\left(X_1, \dots, X_n\right) = \sum_{k=1}^n X_k^p$. Since R is a homogeneous polynomial of degree p in $\mathbb{R}^+\left[X_1, \dots, X_n\right]$, there exists $0 \leq u, v \in A$ such that

$$u^p = \sum_{\varepsilon, \Pi(\varepsilon)=1} a(\varepsilon)^p \quad \text{and} \quad v^p = \sum_{\varepsilon, \Pi(\varepsilon)=-1} a(\varepsilon)^p.$$

Hence $\Pi_p\left(A\right) \subset \Sigma_p\left(A\right) - \Sigma_p\left(A\right)$ and thus $\Pi_p\left(A\right) = \Sigma_p\left(A\right) - \Sigma_p\left(A\right)$. We deduce that $\Pi_p\left(A\right)$ is an order vector subspace of A such that $\Pi_p\left(A\right)^+ = \Sigma_p\left(A\right)$.

Now, let $a, b, c \in A^+$. The inequalities $(a \vee b)^p \geq a^p$ and $(a \vee b)^p \geq b^p$ being clear, assume that $c^p \geq a^p$ and $c^p \geq b^p$. Therefore $[c]^p \geq [a]^p$ and $[c]^p \geq [b]^p$ in $A/N\left(A\right)$ which is a semiprime f-algebra. Then, by [3; Proposition 2], $[c] \geq [a]$ and $[c] \geq [b]$. Hence $[c - a] \geq 0$, $[c - b] \geq 0$ and thus $(c - a)^-, (c - b)^- \in N\left(A\right)$. Since $N\left(A\right)$ is an o-ideal, we get $(c - (a \vee b))^- = (c - a)^- \vee (c - b)^- \in N\left(A\right)$. This implies that

$$(c - (a \vee b))^- \left(\sum_{k=0}^{p-1} c^k \left(a \vee b\right)^{p-1-k}\right) = 0,$$

and thus

$$\begin{aligned} c^p - (a \vee b)^p &= (c - (a \vee b)) \left(\sum_{k=0}^{p-1} c^k \left(a \vee b\right)^{p-1-k}\right) \\ &= (c - (a \vee b))^+ \left(\sum_{k=0}^{p-1} c^k \left(a \vee b\right)^{p-1-k}\right). \end{aligned}$$

Finally $c^p \geq (a \vee b)^p$. Therefore $\sup\{a^p, b^p\}$ exists in $\Pi_p(A)$ and satisfies

$$\sup\{a^p, b^p\} = a^p \vee_p b^p = (a \vee b)^p .$$

Analogously, $\inf\{a^p, b^p\}$ exists in $\Pi_p(A)$ and satisfies

$$\inf\{a^p, b^p\} = a^p \wedge_p b^p = (a \wedge b)^p .$$

We conclude that $\Pi_p(A)$ is a vector lattice. $\square$

The next example shows that the previous result need not be true in the case $p = 2$.

Example 1. *Consider $A = C([0,1])$ with the pointwise addition, scalar multiplication and partial ordering. For $f, g \in A$, define*

$$(f * g)(x) = \begin{cases} 0 & \left(0 \leq x \leq \frac{1}{2}\right) \\ \displaystyle\int_0^{x-\frac{1}{2}} f(s)g(s)\, ds & \left(\frac{1}{2} < x \leq 1\right) \end{cases} .$$

A straightforward computation shows that A is a uniformly complete almost f-algebra under the multiplication $*$ and that $h \in \Pi_2(A)$ if and only if $h(x) = 0$ for all $x \in [0, \frac{1}{2}]$ and the restriction of h to $[\frac{1}{2}, 1]$ belongs to $C^1\left([\frac{1}{2}, 1]\right)$. Hence $\Pi_2(A)$ cannot be a vector lattice under the order inherited from A.

It will be shown in the next proposition that if $w = a_1 \dots a_p \in \Pi_p(A)$ and if the absolute value of w in $\Pi_p(A)$ is denoted by $|w|_p$ then $|w|_p = |a_1| \dots |a_p|$. In order to prove this equality, we state the following lemma.

Lemma 1. *Let A be an uniformly complete almost f-algebra, a natural number $p \geq 2$ and $u, v \in A$ such that $u \wedge v = 0$. Then $(a^p u) \wedge_{p+1} (b^p v) = 0$ in $\Pi_{p+1}(A)$ for all $0 \leq a, b \in A$.*

PROOF: Using Proposition 2(i), there exists $0 \leq x, y \in A$ such that $x^{p+1} = a^p u$ and $y^{p+1} = b^p v$. Moreover $u \wedge v = 0$ implies $[u] \wedge [v] = 0$ in $A/N(A)$. Since $A/N(A)$ is an f-algebra, we obtain $([a]^p [u]) \wedge ([b]^p [v]) = 0$. Hence, by [3; Proposition 1]

$$\begin{aligned} ([x] \wedge [y])^{p+1} &= [x]^{p+1} \wedge [y]^{p+1} \\ &= \left[x^{p+1}\right] \wedge \left[\, y^{p+1}\right] \\ &= [a^p u\ ] \wedge [b^p v] \\ &= ([a]^p [u]) \wedge ([b]^p [v]) = 0. \end{aligned}$$

By the fact that $A/N(A)$ is semiprime, we get $[x \wedge y] = [x] \wedge [y] = 0$. Therefore $x \wedge y \in N(A)$ and $(x \wedge y)^{p+1} = 0$ (because $p \geq 2$). We conclude that $(a^p u) \wedge_{p+1} (b^p v) = x^{p+1} \wedge_{p+1} y^{p+1} = (x \wedge y)^{p+1} = 0$.

The proof is complete. $\square$

Proposition 3. *Let A be a uniformly complete almost f-algebra, a natural number $p \geq 3$ and $w = a_1 \dots a_p$ in the vector lattice $\Pi_p(A)$. Then $|w|_p = |a_1| \dots |a_p|$.*

Proof: We will keep the same notations as already used in the proof of Theorem 2. Then

$$a_1 \dots a_p = \left(a_1^+ - a_1^-\right) \dots \left(a_p^+ - a_p^-\right) = U - V$$

where

$$U = \sum_{\substack{\varepsilon=(\varepsilon_1,\dots,\varepsilon_p)\\ \Pi(\varepsilon)=1}} a_1(\varepsilon_1) \dots a_p\left(\varepsilon_p\right)$$

and

$$V = \sum_{\substack{\varepsilon=(\varepsilon_1,\dots,\varepsilon_p)\\ \Pi(\varepsilon)=-1}} a_1(\varepsilon_1) \dots a_p\left(\varepsilon_p\right).$$

Let $\varepsilon = \left(\varepsilon_1, \dots, \varepsilon_p\right)$, $\varepsilon' = \left(\varepsilon_1', \dots, \varepsilon_p'\right) \in D_p$ such that $\varepsilon \neq \varepsilon'$. Without restriction, assume that $\varepsilon_1 \neq \varepsilon_1'$. Hence $a_1\left(\varepsilon_1\right) \wedge a_1\left(\varepsilon_1'\right) = 0$. By Proposition 2(i) and Lemma 1, $\left(a_1(\varepsilon_1) \dots a_p\left(\varepsilon_p\right)\right) \wedge_p \left(a_1(\varepsilon_1') \dots a_p\left(\varepsilon_p'\right)\right) = 0$. We infer that $U \wedge_p V = 0$. Therefore $U = \left(a_1 \dots a_p\right)^+$ and $V = \left(a_1 \dots a_p\right)^-$ in the vector lattice $\Pi_p(A)$. Consequently $\left|a_1 \dots a_p\right|_p = U + V = |a_1| \dots \left|a_p\right|$ as required. □

At this point, we will show that if A is a uniformly complete almost f-algebra and $p \geq 3$ then $\Pi_p(A)$ is uniformly complete. To this effect, we need the following lemma.

Lemma 2. *Let A be an almost f-algebra and natural numbers n, m, p such that $p \geq 3$, $m \neq 0$ and $n + m = p$. Then $(a + x)^n b_1 \dots b_m = a^n b_1 \dots b_m$ for all $a, b_1, \dots, b_m \in A^+$ and $x \in N(A)$.*

Proof: The method of proof is by induction on n.

The case $n = 0$ being clear, suppose that $(a + x)^n b_1 \dots b_m = a^n b_1 \dots b_m$ for all natural numbers n, m, p such that $p \geq 3$, $m \neq 0$ and $n + m = p$; $a, b_1, \dots, b_m \in A^+$ and $x \in N(A)$.

Give natural numbers n, m, p such that $p \geq 3$, $m \neq 0$ and $n + 1 + m = p$, $a, b_1, \dots, b_m \in A^+$ and $x \in N(A)$. Then

$$\begin{aligned}(a + x)^{n+1} b_1 \dots b_m &= (a + x)(a + x)^n b_1 \dots b_m \\ &= a(a + x)^n b_1 \dots b_m + x(a + x)^n b_1 \dots b_m \\ &= a(a + x)^n b_1 \dots b_m + 0,\end{aligned}$$

(where we use that $n + m = p - 1$). It follows from the induction hypothesis that

$$\begin{aligned}a(a + x)^n b_1 \dots b_m &= a a^n b_1 \dots b_m \\ &= a^{n+1} b_1 \dots b_m,\end{aligned}$$

which completes the induction step. □

Theorem 5. *Let A be a uniformly complete almost f-algebra and a natural number $p \geq 3$. Then the vector lattice $\Pi_p(A)$ is uniformly complete.*

Proof: Let $0 \leq u \in A$ and $(X_n)_{n=1}^{\infty}$ a u^p-uniform Cauchy sequence in $\Pi_p(A)$. According to the Birkoff inequality [7; Theorem 12.4], we can suppose that $0 \leq X_n$ in $\Pi_p(A)$ for all $n \in \{1, 2, \dots\}$. Therefore, for every $n \in \{1, 2, \dots\}$, there exists $0 \leq a_n \in A$ such that $X_n = a_n^p$. Observe that for every $W \in \Pi_p(A)$, $|W| \leq |W|_p$. Thus $\left(a_n^p\right)_{n=1}^{\infty}$ a u^p-uniform Cauchy sequence in A and then $\left(\left[a_n^p\right]\right)_{n=1}^{\infty}$ is a $[u]^p$-uniform Cauchy sequence in $A/N(A)$. Since $A/N(A)$ is semiprime, $([a_n])_{n=1}^{\infty}$ is an $[u]$-uniform Cauchy sequence in $A/N(A)$ ([3; Corollary 3]). Moreover, $A/N(A)$ is uniformly complete. Then there exists $0 \leq a \in A$ such that $([a_n])_{n=1}^{\infty}$ converge $[u]$-uniformly to $[a]$ in $A/N(A)$. Let $0 \leq \varepsilon \leq 1$. There exists $N_\varepsilon \in \{1, 2, \dots\}$ such that $|[a_n] - [a]| \leq \varepsilon [u]$ for all $n \geq N_\varepsilon$. Hence $|[a_n - a]| \leq [\varepsilon u]$. Using [7; Theorem 59.1], there exists $x \in N(A)$ such that $|a_n - a - x| \leq \varepsilon u$ (note that x depends on ε and n). Therefore

$$0 \leq a_n \leq \varepsilon u + |a + x| . \tag{$*$}$$

Furthermore

$$\begin{aligned}
\left|a_n^p - a^p\right|_p &= \left|(a_n - a) \sum_{k=1}^{p-1} a_n^k a^{p-k-1}\right|_p \\
&= \left|(a_n - a - x) \sum_{k=1}^{p-1} a_n^k a^{p-k-1}\right|_p .
\end{aligned}$$

Hence, by Proposition 3 and $(*)$

$$\begin{aligned}
\left|a_n^p - a^p\right|_p &= |a_n - a - x| \sum_{k=1}^{p-1} a_n^k a^{p-k-1} \\
&\leq \varepsilon u \sum_{k=1}^{p-1} \left(\varepsilon u + |a + x|\right)^k a^{p-k-1} \\
&= \varepsilon \sum_{k=1}^{p-1} \sum_{s=0}^{k} \varepsilon^s u^{s+1} \left(|a + x|\right)^{k-s} a^{p-k-1} \\
&= \varepsilon \sum_{k=1}^{p-1} \sum_{s=0}^{k} \varepsilon^s \left|u^{s+1} (a + x)^{k-s} a^{p-k-1}\right|_p .
\end{aligned}$$

Then, by Lemma 2

$$
\begin{aligned}
|a_n^p - a^p|_p &\leq \varepsilon \sum_{k=1}^{p-1} \sum_{s=0}^{k} \varepsilon^s u^{s+1} a^{k-s} a^{p-k-1} \\
&= \varepsilon \sum_{k=1}^{p-1} (\varepsilon u + a)^k \, u a^{p-k-1} \\
&\leq \varepsilon \left(u \sum_{k=1}^{p-1} (u + a)^k \, a^{p-k-1} \right) \\
&= \varepsilon \left((u + a)^p - a^p \right),
\end{aligned}
$$

which completes the proof. □

Generally, $\Pi_p(A)$ $(p \geq 3)$ needs not be a vector sublattice of A as it is shown in the following example.

Example 2. *Take $A = C([-1,1])$ with the pointwise addition, scalar multiplication and partial ordering, and define $\omega \in A$ by*

$$
\varpi(x) = \begin{cases} -x & (-1 \leq x \leq 0) \\ 0 & (0 \leq x \leq 1) \end{cases}.
$$

For $f, g \in A$, define

$$
(f * g)(x) = \begin{cases} \varpi(x) f(x) g(x) & (-1 \leq x \leq 0) \\ \displaystyle\int_{-x}^{0} f(s) g(s) \, ds & (0 \leq x \leq 1) \end{cases}.
$$

A straightforward computation shows that A is a uniformly complete almost f-algebra under the multiplication $*$. Define $\alpha \in A$ by

$$
\alpha(x) = 2x + 1 \quad \textit{for all} \quad x \in [-1, 1].
$$

By a simple calculation we get

$$
|\alpha * \alpha * \alpha|(1) = \frac{1}{10} \neq |\alpha * \alpha * \alpha|_3(1) = (|\alpha| * |\alpha| * |\alpha|)(1) = \frac{1}{8}.
$$

Assume now that A is, in addition, a d-algebra (i.e. A is a uniformly complete commutative d-algebra), then the situation improves. In this case $\Pi_p(A)$ is a vector sublattice of A. In order to prove this result, we need the following proposition which is a generalization of [3; Proposition 1] to the case of commutative d-algebras.

Proposition 4 ([3; Proposition 1])**.** *Let A be a commutative d-algebra and $p \in \{1, 2, \dots\}$. Then*

$$(a \vee b)^p = a^p \vee b^p \quad \text{and} \quad (a \wedge b)^p = a^p \wedge b^p$$

for all $a, b \in A^+$.

PROOF: The case $p = 1$ being evident, assume that $p \geq 2$. We have

$$\begin{aligned} 0 &\leq a^p \wedge b^p - (a \wedge b)^p \\ &= (a^p - (a \wedge b)^p) \wedge (b^p - (a \wedge b)^p) . \end{aligned}$$

Observe now that

$$\begin{aligned} 0 &\leq a^p - (a \wedge b)^p \\ &= (a - (a \wedge b)) \left(\sum_{k=0}^{p-1} a^k (a \wedge b)^{p-1-k} \right) \\ &\leq p (a + b)^{p-1} (a - (a \wedge b)) . \end{aligned}$$

Similarly

$$\begin{aligned} 0 &\leq b^p - (a \wedge b)^p \\ &\leq p (a + b)^{p-1} (b - (a \wedge b)) . \end{aligned}$$

Hence

$$\begin{aligned} 0 &\leq a^p \wedge b^p - (a \wedge b)^p \\ &\leq p^2 (a + b)^{p-1} \{(a - (a \wedge b)) \wedge (b - (a \wedge b))\} = 0. \end{aligned}$$

Analogously for the supremum. □

Corollary 1. *Let A be a uniformly complete commutative d-algebra and a natural number $p \geq 3$. Then $\Pi_p (A)$ is a uniformly complete vector sublattice of A.*

Now, we will show that if A is a uniformly complete almost f-algebra and $p \in \{2, 3, \dots\}$ then $\Pi_p(A)$ is a semiprime f-algebra under the multiplication inherited from A.

Theorem 6. *Let A be a uniformly complete almost f-algebra and a natural number $p \geq 3$. Then $\Pi_p (A)$ is a semiprime f-algebra under the multiplication inherited from A.*

PROOF: Obviously, $\Pi_p (A)$ is an l-algebra under the multiplication inherited from A.

Let $0 \leq a, b \in A$ such that $a^p \wedge_p b^p = 0$. Hence $(a \wedge b)^p = 0$ and thus $a \wedge b \in N(A)$. Moreover, by [1; Proposition 1.0] or [2; Proposition 1.13],

$$ab = (a \wedge b)(a \vee b).$$

We get

$$\begin{aligned} a^p b^p &= (ab)^p \\ &= (a \wedge b)(a \vee b)\left[(ab)^{p-2}\right] = 0. \end{aligned}$$

Thus, $\Pi_p(A)$ is an almost f-algebra. Therefore, it suffices to show that it is semiprime. Let $0 \leq a \in A$ such that $a^p \in N\left(\Pi_p(A)\right)$. There exists $n \in \{1, 2, \dots\}$ such that

$$(a^p)^n = a^{pn} = 0.$$

As a consequence $a \in N(A)$. Since $p \geq 3$, we obtain $a^p = 0$ and thus $N\left(\Pi_p(A)\right) = \{0\}$ as required. □

Corollary 2. *Let A be a uniformly complete commutative d-algebra and a natural number $p \geq 3$. Then $\Pi_p(A)$ is a uniformly complete semiprime sub f-algebra of A.*

Note that if A is a uniformly complete commutative d-algebra then $\Pi_2(A)$ is a sub f-algebra of A. Indeed, it is shown in [4; Corollary 3.7] that $\Pi_2(A)$ is a sub d-algebra of A. Moreover, if $a, b, c \in A^+$ such that $a^2 \wedge b^2 = 0$ then

$$0 \leq \left(c^2 a^2\right) \wedge b^2 = ((ca) \wedge b)^2 \leq cab = c(a \vee b)(a \wedge b).$$

Now $(a \wedge b)^2 = a^2 \wedge b^2 = 0$. This implies that $a \wedge b \in N(A)$ and thus $c(a \vee b)(a \wedge b) = 0$. Finally $\left(c^2 a^2\right) \wedge b^2 = 0$. Therefore $\Pi_2(A)$ is an f-algebra, but not necessarily semiprime as it is shown in the following example.

Example 3. *Let A be the coordinatewise ordered space $\mathbb{R}^3$ with the following multiplication defined by:*

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} cc' \\ bb' \\ 0 \end{pmatrix}.$$

It is clear that A is a uniformly complete commutative d-algebra and

$$\Pi_2(A) = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix};\ x, y \in \mathbb{R} \right\}.$$

Obviously, $\Pi_2(A)$ is a sub f-algebra of A. However, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^2 = 0$ and thus $\Pi_2(A)$ is not semiprime.

At last, we give the following corollary.

Corollary 3. *Let A be a uniformly complete f-algebra and a natural number $p \geq 2$. Then $\Pi_p(A)$ is a uniformly complete semiprime sub f-algebra of A.*

PROOF: It is shown in [4; Corollary 3.7] that $\Pi_2\,(A)$ is a semiprime sub f-algebra of A. Furthermore, the "uniformly completion" property can be obtained using the same method of Theorem 5.

The case $p \geq 3$ is an immediate inference from the Theorems 5 and 6. □

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