Hopf algebras of smooth functions on compact Lie groups

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Abstract. A C^∞-Hopf algebra is a C^∞-algebra which is also a convenient Hopf algebra with respect to the structure induced by the evaluations of smooth functions. We characterize those C^∞-Hopf algebras which are given by the algebra $C^\infty(G)$ of smooth functions on some compact Lie group G, thus obtaining an anti-isomorphism of the category of compact Lie groups with a subcategory of convenient Hopf algebras.

Keywords: C^∞-Hopf-algebras, algebras of smooth functions on compact Lie groups, duality theorem

Classification: 16W30, 22D35, 22E15, 46E25

1. Introduction

The model example of a Hopf algebra being associated with a group structure is the group algebra of a finite group. It is the space $\mathcal{F}(G)$ of functions on G with pointwise multiplication, whereas dualizing the group multiplication μ yields the comultiplication $\mu^* : \mathcal{F}(G) \to \mathcal{F}(G \times G) \cong \mathcal{F}(G) \otimes \mathcal{F}(G)$.

If G is not finite, the associated function space $\mathcal{F}(G)$ will be infinite dimensional and the space $\mathcal{F}(G \times G)$ can no longer be identified with $\mathcal{F}(G) \otimes \mathcal{F}(G)$. This is why in the general case one has to restrict to the subalgebra of representative functions (see [1]) and thus obtains a functor from the category of groups to the category of Hopf algebras. The subcategories of the latter corresponding (by antiequivalence) to particular subcategories of groups such as the finite ones (cf. [13]), compact topological groups (see [14], [15], [5]), compact Lie groups or affine algebraic groups, are characterized completely. For the case of compact Lie groups, this is a consequence of the Tannaka duality theorem (cf. [1]).

In case of a connected Lie group G, one usually considers the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ (see [2]).

However, if dealing with a smooth group, i.e. a group with additional smooth structure (among which we will be interested in characterizing the compact Lie groups of class C^∞), there is a very natural alternative. The "right" function space here is the space $C^\infty(G)$ of smooth functions on G. We will use the setting of smooth spaces and convenient vector spaces developed in [4] by Frölicher and Kriegl. The latter offers a category in which for each smooth space the space of smooth functions is an object and which is symmetric monoidally closed, i.e.

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admits an appropriate tensor product which coincides with the completed projective tensor product for Fréchet spaces. We therefore modify the definition of a Hopf algebra and consider so-called convenient Hopf algebras with the convenient tensor product $\tilde{\otimes}$ replacing the algebraic tensor product in the definition of comultiplication. Finally, we introduce the notion of a C^∞-Hopf algebra, which reflects the additional C^∞-algebra structure of our generic example $C^\infty(G)$.

In Section 2 we present fundamental aspects and results of the theory of smooth spaces and convenient vector spaces needed later on. In Section 3, we show the existence of a functor from the category of convenient Hopf algebras to the category of smooth groups. A generalization of the notion of a representative function to arbitrary convenient Hopf algebras is defined via a natural action of the $\mathbb{R}$-valued convenient algebra homomorphisms.

Section 4 contains our main theorem 4.2 which is a smooth version of the Tannaka duality theorem: We show that the constructed functor induces an anti-isomorphism between the category of C^∞-Hopf algebras which admit a gauge and are finitely generated by "representative elements" and the category of compact Lie groups.

In [3], the duality of compactological groups in terms of formal projective limits of systems of commutative C^*-algebras with unit is investigated. A characterization of finite dimensional separable smooth manifolds in terms of C^∞-algebras has been given in [10].

2. Smooth spaces and convenient vector spaces

The notions and methods we shall use in this paper are almost entirely based on the smooth calculus as developed in [4]. For the latest exposition of this theory, see [8].

2.1 Smooth spaces. A *smooth structure* on a set X is a set $\mathcal{C}$ of curves $\mathbb{R} \to X$ (its *structure curves*) and a set $\mathcal{F}$ of functions $X \to \mathbb{R}$ (its *structure functions*) which are maximal sets with the property $\mathcal{F} \circ \mathcal{C} \subseteq C^\infty(\mathbb{R}, \mathbb{R})$. A set X together with a smooth structure is called a *smooth space*. On $\mathbb{R}^n$ (in particular $\mathbb{R}$), the C^∞-curves and C^∞-functions in the usual sense give a smooth structure and we will always refer to this structure, whenever we assume a smooth structure on $\mathbb{R}^n$ to be given. A map between smooth spaces is said to be smooth if it preserves the structure curves or equivalently the structure functions. Given a set X and a set $\mathcal{F}$ of functions $X \to \mathbb{R}$, one obtains a smooth structure on X by saturating $\mathcal{F}$ in an obvious sense. Then $\mathcal{F}$ is an initial source in the category $\underline{C^\infty}$ of smooth maps between smooth spaces with respect to the forgetful functor $\underline{C^\infty} \to \underline{Set}$ and we say that it generates the smooth structure of X. More generally, given a smooth space X, a set Y and a map $g : Y \to X$, we obtain the initial smooth structure on Y with respect to g by endowing Y with the smooth structure generated by the family $g^*(\mathcal{F})$, where $\mathcal{F}$ denotes the set of structure functions of X. The embedding $U \hookrightarrow \mathbb{R}^n$ of an open subset then induces the usual smooth structure.

The finite cartesian product of smooth spaces together with the smooth structure induced by the projections yields a product in the category of smooth spaces.

A *smooth (semi-)group* is a smooth space G together with a compatible (semi-)group structure on its underlying set, i.e. the respective structure maps are smooth. The respective categories of smooth (semi-)group homomorphisms between smooth (semi-)groups are denoted by $\underline{C^\infty\text{-}Gr}$ ($\underline{C^\infty\text{-}SemiGr}$, respectively).

2.2 Convenient vector spaces. A curve $c : \mathbb{R} \to E$ into a locally convex space is said to be *smooth* or C^∞ if all derivatives exist. If E is c^∞-complete (i.e. locally complete) then the smooth curves form indeed the structure curves for a smooth structure on E, namely the one induced by its bounded linear functionals (cf. [4, 4.1]). A locally convex vector space which is c^∞-complete is called *convenient*. Multilinear mappings between convenient vector spaces are smooth if and only if they are bounded. In particular the smooth structure of a convenient vector space determines its locally convex topology up to bornological isomorphism. The category of smooth linear maps between convenient vector spaces is denoted by $\underline{Con}$. We will denote by E' the space of all bounded linear functionals on the convenient vector space E.

2.3 Spaces of smooth mappings. Given smooth spaces X, Y and a convenient vector space E, the space $C^\infty(Y, E)$ is again a convenient vector space with the locally convex topology of convergence of compositions with smooth curves in Y, uniformly on compact intervals, in all derivatives separately, and we have the exponential law

$$C^\infty(X, C^\infty(Y, E)) \cong C^\infty(X \times Y, E).$$

For Xan open subset of $\mathbb{R}^n$ or more generally a finite dimensional smooth separable manifold, the above topology is bornologically isomorphic (and hence $\underline{Con}$-isomorphic) to the usual nuclear Fréchet topology on $C^\infty(X, \mathbb{R})$.

Another important property is the differentiable uniform boundedness principle stating that $\mathrm{ev}_x : C^\infty(X, E) \to E$ $(x \in X)$ is an initial source with respect to the forgetful functor $\underline{Con} \to \underline{VS}$ to the category of vector spaces. Instead of $C^\infty(X, \mathbb{R})$, we will write $C^\infty(X)$.

2.4 Multilinear maps and tensor products. If $E_1, \dots, E_n, F$ are convenient vector spaces, then the space $L(E_1, \dots, E_n; F)$ of smooth n-linear maps from $\prod_{i=1}^n E_i$ to F, endowed with the locally convex topology of uniform convergence on bounded sets, is a convenient vector space. Its smooth structure is the initial one with respect to the inclusion into $C^\infty(\prod E_i, F)$ and the multilinear versions of the exponential law and of the uniform bounded principle hold.

The category of smooth linear maps between convenient vector spaces is symmetric monoidally closed, i.e. it admits a unique tensor product $\tilde{\otimes}$, called the convenient tensor product and natural isomorphisms of convenient vector spaces $E\tilde{\otimes}\mathbb{R} \cong E$, $E\tilde{\otimes}F \cong F\tilde{\otimes}E$, $(E\tilde{\otimes}F)\tilde{\otimes}G \cong E\tilde{\otimes}(F\tilde{\otimes}G)$ and $L(E\tilde{\otimes}F, G) \cong L(E, L(F, G))$. It can be constructed by c^∞-completion of the bornological tensor product. For Fréchet spaces convenient and projective tensor product coincide (see [6, 11.1.6]) and for smooth finite dimensional separable manifolds M we have

$$C^\infty(M \times M) \cong C^\infty(M)\tilde{\otimes}C^\infty(M)$$

as a consequence of the corresponding statement for open subsets of a finite dimensional vector space ([6, 21.6]).

3. C^∞-Hopf algebras

3.1 (cf. [4, 5.2.1]). A (*commutative*) *convenient algebra* A is a convenient vector space (also denoted by A) together with an associative (and commutative) smooth linear map $m : A\tilde{\otimes}A \to A$ called *multiplication* and a smooth linear unit $u : \mathbb{R} \to A$ with respect to m. The category $\underline{ConAlg}$ has as objects the convenient algebras and the $\underline{Con}$-morphisms which preserve multiplication and unit. Given a convenient algebra A, the convenient vector space $A\tilde{\otimes}A$ is a convenient algebra with multiplication $(m\tilde{\otimes}m) \circ s_{23}$ and unit $u\tilde{\otimes}u$, where s_{23} denotes the map which commutes the second and the third factor of the tensor product and where we omit (as we will occasionally do in the sequel) the natural isomorphism $\mathbb{R}\tilde{\otimes}\mathbb{R} \cong \mathbb{R}$.

A *convenient coalgebra* C is a convenient vector space (also denoted by C) together with a coassociative smooth linear map $\Delta : C \to C\tilde{\otimes}C$ called *comultiplication* and a smooth linear counit $\varepsilon : C \to \mathbb{R}$ with respect to Δ. The corresponding category is denoted by $\underline{ConCoAlg}$. Given a convenient coalgebra C, the convenient vector space $C\tilde{\otimes}C$ is a convenient coalgebra with comultiplication $s_{23} \circ (\Delta\tilde{\otimes}\Delta)$ and counit $\varepsilon\tilde{\otimes}\varepsilon$.

3.2. A *convenient bialgebra* B is a convenient vector space which is both a convenient algebra and a convenient coalgebra such that the algebra structure maps are $\underline{ConCoAlg}$-morphisms or equivalently that the coalgebra structure maps are $\underline{ConAlg}$-morphisms. The category of convenient bialgebras and structure preserving $\underline{Con}$-morphisms is denoted by $\underline{ConBiAlg}$. A convenient bialgebra H admitting an *antipode*, i.e. a smooth linear map $T : H \to H$ with the property that

$$m \circ (\mathrm{id}\,\tilde{\otimes}T) \circ \Delta = m \circ (T\tilde{\otimes}\,\mathrm{id}) \circ \Delta = u \circ \varepsilon,$$

is called a *convenient Hopf algebra*. The full subcategory of $\underline{ConBiAlg}$ consisting of convenient Hopf algebras is denoted by $\underline{ConHopfAlg}$.

3.3. A C^∞*-algebra* A in the sense of [12] is a product preserving functor A from the category of finite-dimensional real vector spaces and C^∞-mappings to the category $\underline{Set}$ of sets, where $A(\mathbb{R})$ is said to be the underlying vector space and identified with A itself. We thus obtain a structure map $\mathrm{e}^A : C^\infty(\mathbb{R}^n, \mathbb{R}) \times A^n \to A$ given by $\mathrm{e}^A(f, a) = A(f)(a)$. The simplest example of such a C^∞-algebra is $\mathbb{R}$ together with the usual evaluation of C^∞-functions. In [7], each C^∞-algebra is endowed with the finest locally convex topology such that all associated mappings $\mathrm{e}^A_a := \mathrm{e}^A(\_, a) : C^\infty(\mathbb{R}^n) \to A$, $a \in A^n$, $n \in \mathbb{N}$ are continuous, where

$C^\infty(\mathbb{R}^n)$ carries its usual Fréchet topology, as explained in 2.3. This topology on A is called its *natural topology* and coincides with the above topology for $A = C^\infty(\mathbb{R}^n)$ by [7, 3.2]. This is true also for $C^\infty(M)$, where M is a finite dimensional smooth separable manifold: This is a consequence of the open mapping theorem as the natural topology of a finitely generated C^∞-algebra (i.e. the quotient of the free C^∞-algebra $C^\infty(\mathbb{R}^n)$ for some n) is (nuclear) Fréchet by [7, 4.2] and finer than the usual one, since the structure maps are continuous with respect to the latter. A *convenient* C^∞*-algebra* is then a C^∞-algebra with the property that the associated natural topology is separated and c^∞-complete. By [7, 2.4 Theorem], the category of structure map preserving maps between convenient C^∞-algebras is a full subcategory of $\underline{ConAlg}$.

A C^∞*-bialgebra* B is a C^∞-algebra (also denoted by B) which is also a convenient bialgebra with the same underlying convenient algebra structure. A C^∞*-Hopf algebra* is a C^∞-bialgebra with an antipode. A C^∞-Hopf algebra is said to be *commutative* if its underlying convenient algebra is commutative. Given a finite dimensional separable smooth Lie group, the convenient vector space $C^\infty(G)$ carries its natural topology according to 3.3 and hence is a C^∞-Hopf algebra in a natural way.

3.4 Proposition. *The Homfunctor lifts to functors*

$$\underline{ConBiAlg}^{op} \to \underline{C^\infty\text{-}SemiGr},$$

and

$$\underline{ConHopfAlg}^{op} \to \underline{C^\infty\text{-}Gr}.$$

Given a convenient bialgebra B, the composition of two elements $\varphi_1, \varphi_2 \in \underline{ConAlg}(B, \mathbb{R})$ given by $\varphi_1\varphi_2 := (\varphi_1 \tilde{\otimes} \varphi_2) \circ \Delta$ defines an associative smooth map (with respect to the smooth structure induced by the inclusion $\underline{ConAlg}(B, \mathbb{R}) \to B'$) $\underline{ConAlg}(B, \mathbb{R}) \times \underline{ConAlg}(B, \mathbb{R}) \to \underline{ConAlg}(B, \mathbb{R})$ with unit element the counit ε of B. If H is a convenient Hopf algebra with antipode T, then the map T^* is an inversion.

Proof: The composition of $\underline{ConAlg}$-morphisms specified above is well defined since $\Delta : B \to B \tilde{\otimes} B$ is a morphism of convenient algebras and so is

$$\varphi_1 \tilde{\otimes} \varphi_2 : B \tilde{\otimes} B \to \mathbb{R} \tilde{\otimes} \mathbb{R} \cong \mathbb{R},$$

for any two $\underline{ConAlg}$-morphisms $\varphi_1, \varphi_2 : B \to \mathbb{R}$. The composition may be viewed as restriction of the smooth map $\Delta^* \circ (\_\tilde{\otimes}\_) : B' \times B' \to B'$, where $(\_\tilde{\otimes}\_) : B' \times B' \to L^2(B, B; \mathbb{R})$ is the canonical bilinear inclusion associated with the functor $\tilde{\otimes}$. It is smooth by the uniform boundedness principle 2.4 since its composition with evaluation $\mathrm{ev}_{(b_1, b_2)}$ in an arbitrary element $(b_1, b_2) \in B_1 \times B_2$ is smooth.

In order to show associativity, consider $\varphi_1, \varphi_2, \varphi_3 \in \underline{ConAlg}(B, \mathbb{R})$: Then

$$\begin{aligned}(\varphi_1\varphi_2)\varphi_3 &= \left[\left((\varphi_1 \tilde{\otimes} \varphi_2) \circ \Delta\right) \tilde{\otimes} \varphi_3\right] \circ \Delta = (\varphi_1 \tilde{\otimes} \varphi_2 \tilde{\otimes} \varphi_3) \circ (\Delta \tilde{\otimes} \mathrm{id}_B) \circ \Delta \\ &= (\varphi_1 \tilde{\otimes} \varphi_2 \tilde{\otimes} \varphi_3) \circ (\mathrm{id}_B \tilde{\otimes} \Delta) \circ \Delta = \left[\varphi_1 \tilde{\otimes} \left((\varphi_2 \tilde{\otimes} \varphi_3) \circ \Delta\right)\right] \circ \Delta = \varphi_1(\varphi_2\varphi_3),\end{aligned}$$

by coassociativity of Δ. Further, if $\varphi \in \underline{ConAlg}(B,\mathbb{R})$ and ε the counit of B, then $\varepsilon\tilde{\otimes}\varphi = \varphi \circ (\varepsilon\tilde{\otimes}\,\mathrm{id}_B)$ by linearity and hence $\varepsilon\varphi = \varphi \circ (\varepsilon\tilde{\otimes}\,\mathrm{id}_B)\circ\Delta = \varphi$. Similarly, $\varphi\varepsilon = \varphi$. As φ was arbitrary, ε is a unit. Finally, given a convenient Hopf algebra H with antipode $T : H \to H$, we obtain

$$(\varphi\circ T)\varphi = ((\varphi\circ T)\tilde{\otimes}\varphi)\circ\Delta = (\varphi\tilde{\otimes}\varphi)\circ(T\tilde{\otimes}\,\mathrm{id}_H)\circ\Delta = \varphi\circ u\circ\varepsilon = \varepsilon,$$

where $u : \mathbb{R} \to H$ denotes the unit of H. In the same manner it follows that $\varphi\circ(\varphi\circ T) = \varepsilon$ so that T^* is indeed the inversion map. In particular it is smooth as restriction of the bounded linear map $T^* : H' \to H'$ and we are done. $\square$

In the following, we will denote by $ConAlg(H,\mathbb{R})$ the set $\underline{ConAlg}(H,\mathbb{R})$ of $\underline{ConAlg}$-morphisms from the convenient Hopf algebra H to $\mathbb{R}$ together with the smooth group structure specified in Proposition 3.4.

3.5 Proposition. *Given a convenient Hopf algebra H and a pair*

$$(\varphi,h)\in ConAlg(H,\mathbb{R})\times H,$$

the assignment

$$\lambda : (\varphi,h)\mapsto(\mathrm{id}_H\tilde{\otimes}\varphi)(\Delta(h))$$

defines a smooth left action $ConAlg(H,\mathbb{R})\times H \to H$. Similarly, H becomes a smooth right (and even a smooth two-sided) $ConAlg(H,\mathbb{R})$-module via the smooth right action $\rho(h,\varphi) := (\varphi\tilde{\otimes}\,\mathrm{id}_H)(\Delta(h))$.

Proof: Let $\varphi_1,\varphi_2 \in ConAlg(H,\mathbb{R})$ and $h\in H$. Then

$$\begin{aligned}\lambda((\varphi_1\varphi_2),h) &= (\mathrm{id}_H\tilde{\otimes}\varphi_1\varphi_2)\circ\Delta = \big(\mathrm{id}_H\tilde{\otimes}((\varphi_1\tilde{\otimes}\varphi_2)\circ\Delta)\big)\circ\Delta\\ &= (\mathrm{id}_H\tilde{\otimes}\varphi_1\tilde{\otimes}\,\mathrm{id}_{\mathbb{R}})\circ(\mathrm{id}_H\tilde{\otimes}\,\mathrm{id}_H\tilde{\otimes}\varphi_2)\circ(\mathrm{id}_H\tilde{\otimes}\Delta)\circ\Delta\\ &= (\mathrm{id}_H\tilde{\otimes}\varphi_1\tilde{\otimes}\,\mathrm{id}_{\mathbb{R}})\circ(\mathrm{id}_H\tilde{\otimes}\,\mathrm{id}_H\tilde{\otimes}\varphi_2)\circ(\Delta\tilde{\otimes}\,\mathrm{id}_H)\circ\Delta\\ &= (\mathrm{id}_H\tilde{\otimes}\varphi_1\tilde{\otimes}\,\mathrm{id}_{\mathbb{R}})\circ(\Delta\tilde{\otimes}\,\mathrm{id}_{\mathbb{R}})\circ(\mathrm{id}_H\tilde{\otimes}\varphi_2)\circ\Delta = \lambda(\varphi_1,\lambda(\varphi_2,h)).\end{aligned}$$

Smoothness is clear (e.g. by the exponential law). That left and right action commute follows again by coassociativity. $\square$

3.6 Proposition. *Let H be a convenient Hopf algebra. Then the evaluation map* ev *constitutes a $\underline{ConAlg}$-morphism*

$$H \to C^\infty(ConAlg(H,\mathbb{R})),$$

if we endow $C^\infty(ConAlg(H,\mathbb{R}))$ with pointwise multiplication. Composition with elements of $C^\infty(\mathbb{R})$ gives the only possible C^∞-algebra structure on the algebra $C^\infty(ConAlg(H,\mathbb{R}))$ (cf. [7, 6.7. Lemma]) *and makes* ev *into a C^∞-algebra morphism if H is a C^∞-Hopf algebra.*

Proof: The map $\mathrm{ev} : H \to C^\infty(ConAlg(H,\mathbb{R}))$ is smooth by the differentiable uniform boundedness principle 2.3. It remains to show that ev is a morphism of C^∞-algebras. For $(h_1,\dots,h_n)\in H^n$ consider the structure map

$$\mathrm{e}^H_{(h_1,\dots,h_n)} : C^\infty(\mathbb{R}^n)\to H.$$

We have to show that

$$\mathrm{ev} \circ \mathrm{e}^{H}_{(h_1,\dots,h_n)} = \mathrm{e}^{C^\infty(\underline{ConAlg}(H,\mathbb{R}))}_{(\mathrm{ev}(h_1),\dots,\mathrm{ev}(h_n))}.$$

For $f \in C^\infty(\mathbb{R}^n)$ and $\varphi \in \underline{ConAlg(H,\mathbb{R})}$ arbitrary, we have

$$\begin{aligned}(\mathrm{ev} \circ \mathrm{e}^{H}_{(h_1,\dots,h_n)})(f)(\varphi) &= \varphi(\mathrm{e}^{H}_{(h_1,\dots,h_n)}(f)) = \mathrm{e}^{\mathbb{R}}_{(\varphi(h_1),\dots,\varphi(h_n))}(f) \\ &= f(\varphi(h_1),\dots\varphi(h_n)) = \mathrm{e}^{C^\infty(\underline{ConAlg}(H,\mathbb{R}))}_{(\mathrm{ev}(h_1),\dots,\mathrm{ev}(h_n))}(f)(\varphi),\end{aligned}$$

where the second identity holds again by [7, 2.4 Theorem] as any $\underline{ConAlg}$-morphism between C^∞-Hopf algebras is a C^∞-algebra morphism.

□

3.7 Definition. A convenient Hopf algebra H is said to be *reduced* if $\mathrm{ev} : H \to C^\infty(\underline{ConAlg}(H,\mathbb{R}))$ is an injection.

Clearly, each reduced convenient Hopf algebra is commutative.

If G is a smooth semigroup, then the convenient C^∞-algebra $C^\infty(G)$ admits a twosided smooth action of G via $(xfy)(z) = f(yzx)$. Thus a reduced convenient Hopf algebra H may be viewed as an algebraic $\underline{ConAlg}(H,\mathbb{R})$-submodule of $C^\infty(\underline{ConAlg}(H,\mathbb{R}))$. The following version of Theorem 2.2.6 from [1] will be useful. We sketch its proof for the sake of completeness:

3.8 Proposition and Definition. *Let G be a smooth semigroup with multiplication μ and identity e. Then for $f \in C^\infty(G)$ the following are equivalent:*

(1) $\mu^*(f) \in C^\infty(G) \otimes C^\infty(G) \subseteq C^\infty(G \times G)$.
(2) *The left G-submodule of $C^\infty(G)$ generated by f is finite-dimensional.*
(3) *The right G-submodule of $C^\infty(G)$ generated by f is finite-dimensional.*
(4) *The two-sided G-submodule of $C^\infty(G)$ generated by f is finite-dimensional.*

An element $f \in C^\infty(G)$ satisfying the above equivalent conditions is called a representative function.

Proof: The implications $(1) \Rightarrow (2),(3)$ being trivial, we proceed to show the implication $(2) \Rightarrow (3)$: Let $\{f_1,\dots,f_n\}$ be a basis for the left G-submodule generated by f. Then $xf = \sum_{i=1}^n g_i(x) f_i$ with $g_i(x) \in \mathbb{R}$ for all $x \in G$. Choose $l_j \in C^\infty(G)'$ with the property that $l_j(f_i) = \delta_{ij}$. As the action of G is smooth, the function $g_j : x \mapsto l_j(xf)$ is smooth so that $fy = \sum_{i=1}^n f_i(y) g_i$ with $g_i \in C^\infty(G)$, which is (3). The converse implication $(3) \Rightarrow (2)$ is shown in the same manner. Note that in particular $f(xy) = \sum_{i=1}^n f_i(x) g_i(y)$ so that we have shown $(2) \Rightarrow (1)$. It remains to prove $(2) \Rightarrow (4)$: Let again $\{f_1,\dots,f_n\}$ be a basis for the left G-submodule generated by f. Then for each $i = 1,\dots,n$ the left and, by what has been already proved, also the right G-submodule generated by f_i is finite-dimensional. But the two sided G-submodule generated by f is the linear span of the union of the finitely many right G-submodules generated by the basis elements f_i and hence itself finite-dimensional. □

Corollary. *Let H be a reduced convenient Hopf algebra and $ConAlg(H,\mathbb{R})$ the smooth group of its $\mathbb{R}$-valued $\underline{ConAlg}$-morphisms. Then for an arbitrary element $h \in H$ the following are equivalent:*

(1) $\mathrm{ev}(h)$ *is a representative function on $ConAlg(H,\mathbb{R})$.*
(2) *The left $ConAlg(H,\mathbb{R})$-submodule of H generated by h is finite dimensional.*
(3) *The right $ConAlg(H,\mathbb{R})$-submodule of H generated by h is finite dimensional.*
(4) *The two-sided $ConAlg(H,\mathbb{R})$-submodule of H generated by h is finite dimensional.*

We will denote by H_R the subset of H consisting of elements satisfying the above equivalent conditions. Clearly, any element $h \in H$ with the property that $\Delta(h) \in H \otimes H \subseteq H \tilde{\otimes} H$ belongs to H_R.

4. A smooth duality theorem

4.1 Definition. A *gauge* on a convenient Hopf algebra over $\mathbb{R}$ is an element $I \in H'$ such that

(1) $(I \tilde{\otimes} \mathrm{id}_H) \circ \Delta = u \circ I$,
(2) $I(h^2) > 0$ for each $h \in H, h \neq 0$.

4.2 Theorem. *Let H be a reduced C^∞-Hopf algebra with gauge, finitely generated as a C^∞-algebra by elements of H_R. Then the smooth group $\underline{ConAlg}(H,\mathbb{R})$ carries the structure of a compact Lie group in a natural way and* ev *induces an isomorphism of C^∞-Hopf algebras*

$$H \cong C^\infty(ConAlg(H,\mathbb{R})).$$

4.3 Lemma. *Let H be a C^∞-Hopf algebra admitting a gauge and generated as a C^∞-algebra by H_R. Then the topology induced on $\underline{ConAlg}(H,\mathbb{R})$ by the family* $\mathrm{ev}(H_R)$ *is compact.*

Proof: Note first that, for $\varphi \in \underline{ConAlg}(H,\mathbb{R})$, we have

$$I \circ (\mathrm{id}_H \tilde{\otimes} \varphi) \circ \Delta = (I \tilde{\otimes} \varphi) \circ \Delta = \varphi \circ (I \tilde{\otimes} \mathrm{id}_H) \circ \Delta = \varphi \circ u \circ I = I,$$

i.e., I is $ConAlg(H,\mathbb{R})$-invariant. For $h \in H_R$, the linear subspace V of H generated by $\{\lambda(\varphi,h) : \varphi \in \underline{ConAlg}(H,\mathbb{R})\}$ is finite dimensional by definition of H_R. Let $\{h_1,\ldots,h_n\}$ be an orthonormal base of V with respect to the inner product given on H by the smooth positive definite bilinear form $I \circ m$. Then $\lambda(\varphi,h) = \sum_{i=1}^n \varphi_i h_i$ with real coefficients φ_i depending on φ. Furthermore, $I(h^2) = I((\lambda(\varphi,h))^2) = \sum_{i=1}^n \varphi_n^2$ so that the latter sum is independent of φ. On the other hand,

$$\varphi = \varepsilon\varphi = (\varepsilon \tilde{\otimes} \varphi) \circ \Delta = \varepsilon \circ (\mathrm{id}_H \tilde{\otimes} \varphi) \circ \Delta$$

and hence $\varphi(h) = \varepsilon(\lambda(\varphi, h)) = \sum_{i=1}^n \varphi_i \varepsilon(h_i)$. It follows that $ConAlg(H, \mathbb{R})$ is $\mathrm{ev}(H_R)$-bounding, i.e. the image of $ConAlg(H, \mathbb{R})$ under its canonical inclusion into $\prod_{H_R} \mathbb{R}$ given by $\varphi \mapsto (\varphi(h))_h$is relatively compact. It is closed as by [7, 2.4] for convenient C^∞-algebras $\underline{ConAlg}$-morphisms are the same as C^∞-algebra morphisms and pointwise limits of C^∞-algebra morphisms are again C^∞-algebra morphisms, as can be easily verified. Since H_R generates the C^∞-algebra H and the trace of $\prod_{H_R} \mathbb{R}$ on $\underline{ConAlg}(H, \mathbb{R})$ is Hausdorff. □

Proof of 4.2: Let $\Lambda \subseteq H_R$ be a (finite) set of generators for H and $\{h_1, \dots, h_n\}$ be base of the $ConAlg(H, \mathbb{R})$-submodule generated by Λ, orthonormal with respect to $I \circ m$. Then the assignment $\varphi \mapsto (I(h_i \lambda(\varphi, h_j)))_{ij}$ gives a faithful representation of $\underline{ConAlg(H, \mathbb{R})}$ onto a subgroup of $O(n)$. It is continuous with respect to the initial topology induced on $\underline{ConAlg}(H, \mathbb{R})$ by the family $\mathrm{ev}(H_R)$: The compact Lie group $O(n)$ is a submanifold of $\mathbb{R}^{n^2}$ and hence its topology is induced by the functions pr_{ij}, $i, j = 1, \dots n$. The restriction of the smooth linear functional on H' given by $h' \mapsto I(h_i((\mathrm{id}_H \tilde{\otimes} h')(\Delta h_j)))$ to $\underline{ConAlg}(H, \mathbb{R})$ gives an element of $\mathrm{ev}(H_R)$, by reflexivity of the nuclear Fréchet space H (see 3.3), which corresponds to the restriction of the function pr_{ij} to the image. Hence the image of $\underline{ConAlg}(H, \mathbb{R})$ is a compact Lie subgroup and a submanifold of $O(n)$, which we will denote by G.

The identity $ConAlg(H, \mathbb{R}) \to G$ is obviously smooth and hence induces an injective C^∞-algebra morphism $\mathrm{id}^* : C^\infty(G) \to C^\infty(ConAlg(H, \mathbb{R}))$. We claim that $\mathrm{id}^*(C^\infty(G)) = \mathrm{ev}(H)$: The C^∞-algebra $C^\infty(G)$ is generated by the subset $\{\mathrm{pr}_{ij} : i, j = 1, \dots, n\}$ so that $\mathrm{id}^*(C^\infty(G)) \subseteq \mathrm{ev}(H)$ by what we have already shown. For the converse, let $h \in \Lambda$ and $\varphi \in \underline{ConAlg}(H, \mathbb{R})$. Then $h = \sum_{i=1}^n \lambda_i h_i$ and

$$\varphi(h) = \varepsilon(\lambda(\varphi, h)) = \sum_{i=1}^n \lambda_i I(h_j \lambda(\varphi, h_i)) \varepsilon(h_j),$$

which corresponds to the function $\sum_{i=1}^n \lambda_i \varepsilon(h_j) \,\mathrm{pr}_{ji} \in C^\infty(G)$. Hence $C^\infty(G) \cong H$ as convenient C^∞-algebras and, by definition of the comultiplication on $C^\infty(G)$, also as C^∞-Hopf algebras. But then also $G \to ConAlg(H, \mathbb{R})$ is smooth since elements of $\mathrm{ev}(H)$ are smooth on G, i.e. G is diffeomorphic with $ConAlg(H, \mathbb{R})$.
□

4.4 Corollary. *The functor* $\underline{ConHopf}^{op} \to \underline{C^\infty\text{-}Gr}$ *explained in* 3.4 *induces an anti-isomorphism between the category of reduced* C^∞*-Hopf algebras* H *with gauge, which are finitely generated by* H_R *and the category of compact smooth Lie groups.*

Proof: The claim follows by Milnor and Stasheff's exercise for finite dimensional separable smooth manifolds (see [11]). □

Open question. Is it true that a C^∞-Hopf algebra satisfying the conditions of Theorem 4.2 except for existence of a gauge is given by the algebra of smooth functions on a finite dimensional separable smooth Lie group?

4.5 Compact smooth groups which are not Lie groups. The following example was pointed out to me by Peter Michor: Consider the "infinite dimensional torus" T, which is the countable product of copies of the unit circle S^1, endowed with the product smooth structure, i.e. the initial smooth structure with respect to the projections (see 2.1). The topology induced on T by the smooth functions is the product topology and as such compact. However, it is not an infinite dimensional Lie group. By [9] T is smoothly realcompact, in particular each convenient algebra homomorphism $C^\infty(T) \to \mathbb{R}$ is given by evaluations in T. Moreover, each smooth function on T factors over a finite number of coordinates so that the convenient C^∞-algebra $C^\infty(T)$ is the strict regular inductive limit of its complemented subalgebras $C^\infty((S^1)^n)$ for each $n \in \mathbb{N}$. This is the natural topology defined in 3.3 of the C^∞-algebra $C^\infty(T)$. As the bornological tensor product preserves colimits, we obtain that $C^\infty(T)$ is a convenient C^∞-Hopf algebra in a natural way.

We do not know whether this can be generalized to arbitrary smoothly compact groups, that is, smooth groups which are compact with respect to the initial topology induced on them by their smooth functions: Does the functor given in 4.3 induce an anti-isomorphism between the category of smoothly compact groups and the category of C^∞-Hopf algebras H which admit a gauge and are generated by H_F?

Note that the functor C^∞ does not give a functor $\underline{C^\infty\text{-}Gr} \to \underline{ConHopf}^{op}$: Consider e.g. the convenient vector space $\mathbb{R}^\mathbb{N}$ with its smooth additive group structure. Then $C^\infty(\mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N}) \not\cong C^\infty(\mathbb{R}^\mathbb{N})\tilde{\otimes}C^\infty(\mathbb{R}^\mathbb{N})$ (see [4, 7.4.5]).

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