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Some examples related to colorings

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Abstract. We complement the literature by proving that for a fixed-point free map $f : X \to X$ the statements (1) f admits a finite functionally closed cover $\mathcal{A}$ with $f[A] \cap A = \emptyset$ for all $A \in \mathcal{A}$ (i.e., a coloring) and (2) βf is fixed-point free are equivalent.

When functionally closed is weakened to closed, we show that normality is sufficient to prove equivalence, and give an example to show it cannot be omitted.

We also show that a theorem due to van Mill is sharp: for every $n \geq 2$ we construct a strongly zero-dimensional Tychonov space X and a fixed-point free map $f : X \to X$ such that f admits a closed coloring, but no coloring has cardinality less than n.

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All spaces are assumed Tychonov and all maps are continuous. For undefined notions we refer to [2].

Let $f: X \to X$ be a map. An element $x \in X$ is called a fixed point of f if $f(x) = x$. If f has no fixed points then f is called fixed-point free. If f is fixed-point free, then one naturally wonders whether its Čech-Stone extension $\beta f: \beta X \to \beta X$ is also fixed-point free.

Van Douwen [4] considered this question and used special covers that are now called colorings in the literature, named after their counterparts from graph theory (cf. [3], [1], [5], [6] and many others).

A coloring of a fixed-point free map $f: X \to X$ is a finite closed cover $\mathcal{A}$ of X such that for every $A \in \mathcal{A}$ we have $f[A] \cap A = \emptyset$. If $\mathcal{A}$ is an open / functionally open / functionally closed cover where $f[A] \cap A = \emptyset$ for every $A \in \mathcal{A}$ then $\mathcal{A}$ is called an open / etc. coloring of f.

In the literature (e.g., [1, p. 1052]) one now and then refers to van Douwen for the equivalence of the following statements: (1) f has a functionally closed coloring and (2) βf is fixed-point free. This statement was however not proved in van Douwen [4]. He restricted himself to closed maps. We fill in the gap in the literature and show that the statements are indeed equivalent.

If we weaken functionally closed to closed then, as to be expected, normality is needed for the equivalence of (1) and (2). We show that normality cannot be omitted by presenting a Tychonov space having a fixed-point free map that admits a finite closed coloring whereas its Čech-Stone extension has a fixed point.

Finally we modify this last example to show sharpness in a theorem due to van Mill [6]. He showed that if a normal space X has finite covering dimension n and f is a homeomorphism such that f admits a finite coloring, then a coloring of f with $n+3$ sets exists. (Aarts, Fokkink and Vermeer [1] did the same for metrizable spaces.) For every $n \geq 4$ we present a strongly zero-dimensional space with a fixed-point free homeomorphism having a finite closed coloring of n elements but not one with fewer elements.

1. Equivalence

We start with the following theorem.

Theorem 1. *Let X be a (normal) space and let $f\colon X \to X$ be a fixed-point free map. The following statements are equivalent:*

1. *βf is fixed-point free,*
2. *f admits a functionally open (open) coloring,*
3. *f admits a functionally closed (closed) coloring.*

Proof: First assume that X is an arbitrary space.

$1 \implies 2$: Let X be a space and let $f\colon X \to X$ be such that βf is fixed-point free. Let $x \in \beta X$. By normality of βX there exists a functionally open neighborhood U_x of x such that $\beta f[U_x] \cap U_x = \emptyset$. Now apply compactness and trace on X to obtain a functionally open coloring of f.

$2 \implies 3$: Let $\mathcal{A}$ be a functionally open coloring. By [2, 7.1.5] there exists a functionally closed shrinking $\mathcal{B}$ of $\mathcal{A}$. Clearly $\mathcal{B}$ is a functionally closed coloring.

$3 \implies 1$: Suppose $\mathcal{B} = \{B_1, \dots, B_n\}$ is a functionally closed coloring of X. Striving for a contradiction, assume $\beta f(p) = p$ for some $p \in \beta X$. Obviously, $p \in \beta X \setminus X$. Put $E = \{\, 1 \leq i \leq n : p \in \overline{B_i}\,\}$. These and all other closures are taken with respect to βX. Clearly $E \neq \emptyset$ as $\{\, \overline{B} : B \in \mathcal{B}\,\}$ covers βX. By known properties of the Čech-Stone compactification, it follows that for $G = \bigcap_{i\in E} B_i$ we have $\overline{G} = \bigcap_{i \in E} \overline{B_i}$. In particular, $p \in \overline{G}$. Hence,

$$\beta f(p) \in \beta f[\overline{G}] \subseteq \overline{\beta f[G]} = \overline{f[G]}.$$

Next observe that for all $i \in E$ we have $G \subseteq B_i$. Since B_i is a color this gives us that $B_i \cap f[G] = \emptyset$. But now since $\mathcal{B}$ is a cover we obtain $f[G] \subseteq \bigcup_{i \notin E} B_i$. Hence

$$p = \beta f(p) \in \overline{f[G]} \subseteq \overline{\bigcup_{i\notin E} B_i} = \bigcup_{i \notin E} \overline{B_i}.$$

So $p \in \overline{B_i}$ for some $i \notin E$ and so we have the desired contradiction.

It is clear that the same reasoning can be repeated for normal spaces, replacing functionally open/closed covers by open/closed covers. This is so because for a normal space X and closed subsets A, $B \subseteq X$ we have that $\overline{A} \cap \overline{B} = \overline{A \cap B}$. $\square$

2. A finite closed coloring exists, but βf has a fixed point

We now present an example showing that for non-normal spaces 'functionally closed' cannot be weakened to 'closed' in Theorem 1.

Example 1. Let X be the topological space obtained by identifying the two points ω_1 in the topological sum of two copies of ω_1+1 to a single point, say e. We identify one fixed copy of ω_1+1 in X with ω_1+1. For every $x\in X\setminus\{e\}$ we denote by $-x$ the corresponding element of the other copy of ω_1, and let $-e=e$. Moreover we define an ordering on X as follows:

$$0<1<2<\cdots<\omega<\cdots<\omega_1=e=-e$$
$$e=-e<\cdots<-\omega<\cdots<-2<-1<-0.$$

Put $Y=\mathbb{Z}\cup\{\infty,-\infty\}$ where as a subbase for the topology we put

$$\{\,\langle a,\infty]:a\in\mathbb{Z}\,\}\cup\{\,[-\infty,a\rangle:a\in\mathbb{Z}\,\},$$

i.e., Y is the two-point compactification of $\mathbb{Z}$. The map $h\colon Y\to Y$ defined by $h(y)=y+1$ $(y\in\mathbb{Z})$ and $h(\pm\infty)=\pm\infty$ is a homeomorphism.

Now consider the space $X\times Y$ and put $Z=(X\times Y)\setminus\{(e,\infty),(e,-\infty)\}$. See Figure 1 for a sketch of $X\times Y$ and Z.

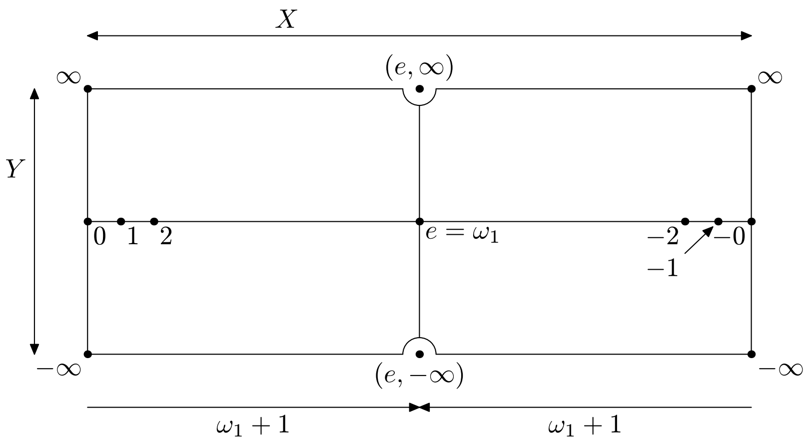

Figure 1. Four Tychonov planks put together, See Example 1

The reader readily checks that $\beta Z=X\times Y$. (cf. [2, 3.12.20(e)] and [2, Corollary 3.6.9])

Next we define $f\colon X\times Y\to X\times Y$ as follows

$$f\big((x,y)\big)=(-x,h(y)).$$

The map f is obviously a homeomorphism. One readily checks that (e,∞) and $(e,-\infty)$ are the only fixed points of f. So $f \restriction Z$ is fixed-point free. Since $\beta(f \restriction Z) = f$ it follows that f is not colorable.

We will now show that f admits a finite closed coloring. To this end we first put $A = \{\, x \in X : x \in \omega_1 + 1 \,\}$ and $B = -A$. Furthermore let $C = \{0, \pm 2, \pm 4, \dots\}$ and $D = \{\pm 1, \pm 3, \pm 5, \dots\}$. Obviously

$$\{A \times (Y \setminus C), A \times (Y \setminus D), B \times (Y \setminus C), B \times (Y \setminus D)\}$$

is a finite closed cover of $X \times Y$. We leave it to the reader to verify that by tracing this cover to Z we obtain a closed coloring of f.

3. Finite closed colorings of arbitrary size

Now that we have shown that there exists a zero-dimensional Tychonov space and a homeomorphism with a finite coloring (of 4 elements) one is interested to know whether an upper bound on the minimal number of colors can be found.

Van Mill [6] showed that for homeomorphisms on finite dimensional (in the sense of dim) *normal* spaces every finite coloring induces a coloring of cardinality dimension plus three. By modifying our example from the previous section we show that for every $n \geq 4$ there exists a strongly zero-dimensional space with a fixed-point free homeomorphism having a finite closed coloring of n elements but not one with fewer elements.

Example 2. For all $m \in \omega$, we will construct by induction spaces Z_m and homeomorphisms $h_m\colon Z_m \to Z_m$ such that the following conditions are satisfied for all m.

1. $|Z_m| \leq \aleph_{m+1}$,
2. Z_m is strongly zero-dimensional,
3. $|\beta Z_m \setminus Z_m| = 2$,
4. $h_m\colon Z_m \to Z_m$ is a fixed-point free homeomorphism,
5. βh_m has two fixed points,
6. h_m admits a finite coloring,
7. for every $z \in \beta Z_m \setminus Z_m$, every open neighborhood $U \ni z$ and every closed coloring $\mathcal{A}$ of h_m there exist $m+2$ distinct elements $A_1, \dots, A_{m+2}$ of $\mathcal{A}$ such that $U \cap A_i \neq \emptyset$ (for $1 \leq i \leq m+2$). In particular $|\mathcal{A}| \geq m+2$.

Fons van Engelen pointed out that with a modification of the argument the number $m+2$ for h_m can easily be raised.

We start the construction with $Z_0 = Z$ and $h_0 = f$ as in Example 1. It obviously satisfies all conditions.

We proceed by induction. Assume Z_{m-1} and h_{m-1} have been constructed as specified above. For notational convenience we put $Z = Z_{m-1}$ and $h = h_{m-1}$. We fix ∞ and $-\infty$ such that $\beta Z \setminus Z = \{\infty, -\infty\}$. In a similar way as in Example 1 we identify in two copies of $\omega_{m+1}+1$ the points ω_{m+1} to a single point. We put

$X = \omega_{m+1} \cup \{e = \omega_{m+1}\} \cup -\omega_{m+1}$. We define $-x$ and the ordering as before. Again X is compact.

Define $f\colon X \times \beta Z \to X \times \beta Z$ by $f\big((x,y)\big) = (-x, h(y))$. It is easy to see that f has two fixed-points: (e,∞) and $(e,-\infty)$. Let D be the set of all non-fixed points of f. We claim that $Z_m = D$ and $h_m = f \restriction D$ are as required.

One readily observes that $|Z_m| \le \aleph_{m+1}$. It is well-known that $\beta\omega_{m+1} = \omega_{m+1} + 1$, so it follows that $\beta(\omega_{m+1} \times Z) = (\omega_{m+1} + 1) \times Z$. This implies that $\beta D = X \times \beta Z$. Since βD is strongly zero-dimensional it follows that $\dim D = 0$ as well.

We will now show that $f \restriction D$ admits a closed coloring. Let $\mathcal{B} = \{B_1, \dots, B_n\}$ be a closed coloring of h. Then

$$\Big\{\overline{(\omega_{m+1} \cup \{e\}) \times B_1}^{(\beta Z_m)}, \dots, \overline{(\omega_{m+1} \cup \{e\}) \times B_n}^{(\beta Z_m)},$$
$$\overline{(-\omega_{m+1} \cup \{e\}) \times B_1}^{(\beta Z_m)}, \dots, \overline{(-\omega_{m+1} \cup \{e\}) \times B_n}^{(\beta Z_m)}\Big\}$$

is a closed cover of βZ. One readily checks that the intersections with Z yield a closed coloring of f.

To finish the construction we need to check condition (7). By symmetry it suffices to prove it for one of the fixed points. Let $U = U_0 \times U_1$ be a basic open neighborhood of the fixed point (e,∞). Consult Figure 2 for more information.

Let $\mathcal{A}$ be a coloring of f. For $z \in Z$, we consider the sets $B_z = \{\,(y,z) : z \in Z,\ y \in \omega_{m+1}\,\}$. From the fact that $\mathcal{A}$ is finite it follows that there exists an $A_z \in \mathcal{A}$ and a closed unbounded set $C_z \subseteq \omega_{m+1}$ such that $C_z \times \{z\} \subseteq A_z$. As A_z is closed, we have $(e,z) \in A_z$.

Now $\{\,A_z \cap (\{e\} \times Z) : z \in Z\,\}$ corresponds to a coloring of h and hence there exist at least $(m-1)+2 = m+1$ distinct such A_z denoted $A^1, \dots, A^{m+1}$. Moreover these have non-empty intersection with U.

Actually, for every open set U_2 with $\infty \in U_2 \subseteq U_1$ we have that $m+1$ such sets exist. Without loss of generality we may assume that for every open set U_2 with $\infty \in U_2 \subseteq U_1$ and every $i \le m+1$ we have $A^i \cap U_2 \ne \emptyset$ (where $i \le m+1$). Define the map $\xi\colon Z \to \{1, \dots, m+1\}$ by $\xi(z) = i$ if and only if $A_z = A^i$.

For a moment fix i and consider the set A^i. The set $\{\,z : \xi(z) = i\,\}$ has cardinality $\le |Z_m| \le \aleph_m$. So the intersection

$$C^i = \bigcap\{\,C_z : \xi(z) = i\,\}$$

is an intersection of at most $\aleph_m$ closed unbounded subsets of ω_{m+1}. Hence C^i is closed unbounded as well (by regularity of ω_{m+1}).

We claim that for every $x \in C^i$, we have $(x,\infty) \in A^i$. This is not complicated. Let $V = V_0 \times V_1$ be a neighborhood of (x,∞) in Z. As V_1 is a neighborhood of ∞ in Z, it follows from (7) that there exists a $z \in V_1$ such that $A_z = A^i$. In particular, $(x,z) \in V$. So $(x,\infty) \in \overline{A^i} = A^i$, as A^i is closed.

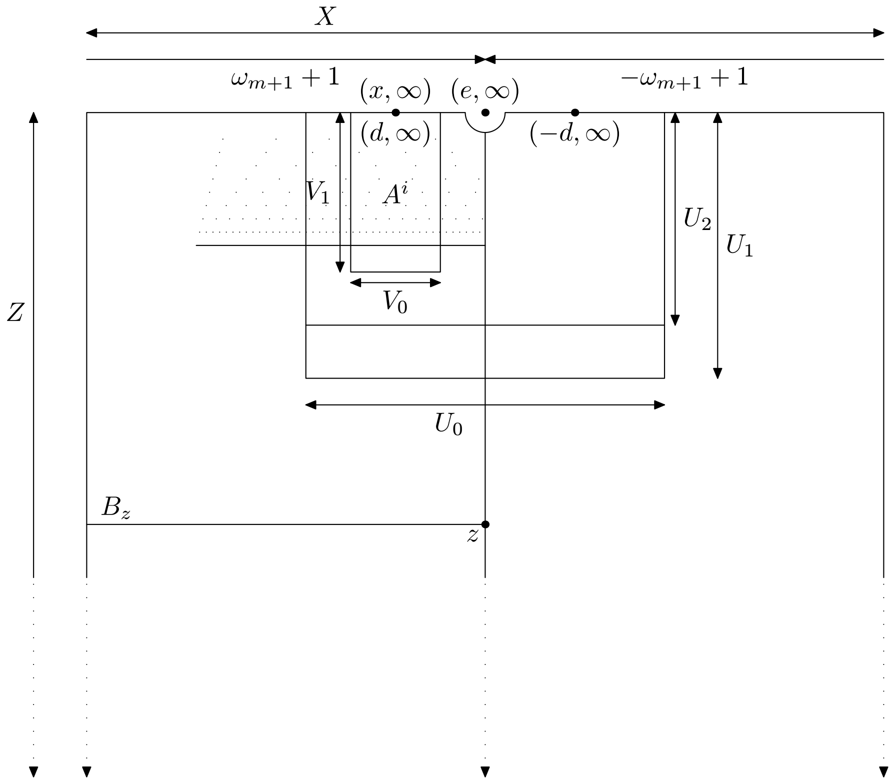

FIGURE 2. Figure for Example 2

Fix $c \in U_0$ such that $-c \in U_0$ as well. For every i, the set C^i is closed unbounded, so there exists a $d > c$ such that $d \in \bigcap_i C^i$. Obviously $(d,\infty) \in A^1 \cap \cdots \cap A^{m+1}$. Since $\mathcal{A}$ is a coloring it follows that $f\big((d,\infty)\big) \notin A^1 \cup \cdots \cup A^{m+1}$. Hence there exists an $A \in \mathcal{A} \setminus \{A^1, \ldots, A^{m+1}\}$ with $(-d,\infty) \in A \cap U$. This completes the construction.

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