# Three-and-more set theorems

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*Abstract.* In this paper we generalize classical 3-set theorem related to stable partitions of arbitrary mappings due to Erdős-de Bruijn, Katětov and Kasteleyn. We consider a structural generalization of this result to partitions preserving sets of inequalities and characterize all finite sets of such inequalities which can be preserved by a "small" coloring. These results are also related to graph homomorphisms and (oriented) colorings.

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## 1. Introduction

Given a mapping $f: X \to X$ without fixpoints, there exists a partition $X = X_1 \cup X_2 \cup X_3$ such that for every $i = 1, 2, 3$, $f(X_i) \cap X_i = \emptyset$. This is a particular case of the celebrated 3-set theorem of Erdős and de Bruijn [2] which has been rediscovered in a different context (set topology) by M. Katětov [8] and Kasteleyn (as quoted by Katětov). This discovery led Z. Frolík to a surprisingly easy proof of non-homogeneity of Stone-Čech Compactification $\beta(\mathbb{N})$ ([4], [5]). In the topological setting this result has a remarkable history, see e.g. [1], [5], [9].

However, Erdős and de Bruijn were motivated in their paper by a pure combinatorial problem: given a relation $R \subseteq X \times X$ which has no loops (for every $x \in X$, $(x, x) \notin R$) and is such that the out-degree $d^+(x) = |\{y; (x, y) \in R\}|$ of every vertex $x$ is bounded by some fixed $k$, determine the chromatic number $\chi(X, R)$. In doing so they rediscovered the compactness property of the chromatic number and proved:

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**Theorem 1** ([2]). *For every loopless relation $(X,R)$ all of whose vertices have $k$-bounded out-degree, the inequality $\chi(X,R) \le 2k+1$ holds.*

By considering a loopless relation whose corresponding graph is a tournament on $2k+1$ vertices with out-degree $k$ it can be seen that this bound is tight.

A relation $(X,R)$ is said to be $k$-bounded if $d^+(x) \le k$ for every $x \in X$. We can clearly consider every $k$-bounded relation $(X,R)$ either as a (multi) mapping $f:\ X \to \mathcal{P}(X)$ where for every $x$, $f(x)$ is a subset of $X$ of size at most $k$, or as a union of $k$ partial mappings $f_i:\ X \to X$, $i = 1,2,\ldots,k$.

On the other hand, the 3-set theorem can be interpreted as a homomorphism $\varphi$ of the relation $(X,R)$ into a complete (loopless) relation with 3 elements. Recall that a homomorphism $\varphi$ of a relation $(X,R)$ to a relation $(T,S)$ is a mapping $\varphi:\ X \to T$ which preserves the relations: $(\varphi(x),\varphi(y)) \in S$ whenever $(x,y) \in R$.

Denote by $\mathcal{R}_n$ the complete antireflexive relation on the set $\{1,2,\ldots,n\}$. The 3-set theorem can then be rephrased by saying that for every mapping $f:\ X \to X$ there exists a homomorphism $\varphi:\ (X,f) \to \mathcal{R}_3$. Similarly, Theorem 1 may be rephrased by saying that for every $k$-bounded relation $(X,R)$ there exists a homomorphism $\varphi:\ (X,R) \to \mathcal{R}_{2k+1}$.

As we are assuming that both $(X,f)$ and $(X,R)$ have no loops we can also say that the homomorphism $\varphi$ preserves the inequality $f(x) \neq x$, that is $f(x) \neq x \Longrightarrow \varphi(f(x)) \neq \varphi(x)$. In this paper we are motivated by this approach and we completely characterize the inequalities which can be preserved by homomorphisms to a target bounded side.

It appears that the inequalities which can be preserved are exactly all initial inequalities which are defined as follows:

Let $f_1, f_2, \ldots, f_k$ be partial mappings from $X$ to $X$. A $(p,\ell)$-initial inequality, $p > \ell \ge 0$, is any inequality of the form

$$f_{i_p} \circ f_{i_{p-1}} \circ \cdots \circ f_{i_1}(x) \ \neq\ f_{i_\ell} \circ f_{i_{\ell-1}} \circ \cdots \circ f_{i_1}(x)$$

where $x \in X$ and $i_1, \ldots i_p$ are (not necessary distinct) indices $1,\ldots,k$. For $\ell = 0$ this should be understood as inequality

$$f_{i_p} \circ \cdots \circ f_{i_2} \circ f_{i_1}(x) \neq x.$$

Thus for $k = 2$, $\ell = 1$ $(2,1)$-inequalities are

$f_1 \circ f_1(x) \neq f_1(x)$, $f_2 \circ f_1(x) \neq f_1(x)$

$f_1 \circ f_2(x) \neq f_2(x)$, $f_2 \circ f_2(x) \neq f_2(x)$ (for $l = 1$)

and $(2,0)$-initial inequalities include for example $f_1 \circ f_2(x) \neq x$, $f_2 \circ f_1(x) \neq x$. We say that a homomorphism $\varphi$ preserves an initial inequality if for every $x \in X$ we have

$$\begin{aligned} &f_{i_p} \circ \cdots \circ f_{i_2} \circ f_{i_1}(x) \neq f_{i_\ell} \circ \cdots \circ f_{i_2} \circ f_{i_1}(x) \\ \Longrightarrow\ &\varphi(f_{i_p} \circ \cdots \circ f_{i_2} \circ f_{i_1}(x)) \neq \varphi(f_{i_\ell} \circ \cdots \circ f_{i_2} \circ f_{i_1}(x)). \end{aligned}$$

We prove the following:

**Theorem 2.** *Given $k$ and $p$ there exists a relation $(T,S)$ with $|T| \leq 1+2k\cdot\frac{k^p-1}{k-1}$ (or $|T| \leq 2p+1$ if $k=1$), such that for every $k$ partial mappings $f_1, f_2, \ldots, f_k$ from $X$ to $X$, there exists a homomorphism $\varphi:\ (X, f_1 \cup f_2 \cup \cdots \cup f_k) \to (T,S)$ such that $\varphi$ preserves all the $(p',\ell)$-initial inequalities, $p \geq p' > \ell \geq 0$.*

We shall see that this is a consequence of Theorem 1. What is perhaps more interesting is that a similar theorem does not hold for other than initial inequalities. This will be stated below as Theorem 3 (after introducing necessary notions).

This bound on the size of $T$ is tight for arbitrary $k$ and $p=1$ (as shown by regular tournament with $2k+1$ vertices) and for arbitrary $p$ and $k=1$ (as shown by the odd cycle of length $2p+1$). For other values this seems to be a difficult combinatorial problem similar to oriented Moore graphs see e.g. [11] and proceedings [12].

In fact we can demand somewhat stronger property, namely that the homomorphism $\varphi : (X, f_1 \cup f_2 \cup \cdots \cup f_k) \to (T,S)$ not only preserves the inequalities but that $S$ itself satisfies these forbidden inequalities. More precisely we have:

**Theorem 3.** *For every $p,k>0$ there exists $t(k,p)$ with the following property: if $f_1, f_2, \ldots, f_k$ are partial mappings from $X$ to $X$ such that for every $x \in X$ and every $\ell$, $0 \leq \ell < p$,*

$$f_{i_p} \circ f_{i_{p-1}} \circ \cdots \circ f_{i_1}(x) \ \neq\ f_{i_\ell} \circ f_{i_{\ell-1}} \circ \cdots \circ f_{i_1}(x),$$

*then there exists a set $T$, $|T| \leq t(k,p)$, and partial mappings $g_1, g_2, \ldots, g_q$ from $T$ to $T$ and a homomorphism*
*$\varphi:\ (X, F = f_1 \cup f_2 \cup \cdots \cup f_k) \to (T, S = g_1 \cup g_2 \cup \cdots \cup g_q)$, such that:*

1. *$\varphi$ preserves all inequalities*

$$f_{i_p} \circ f_{i_{p-1}} \circ \cdots \circ f_{i_1}(x) \ \neq\ f_{i_\ell} \circ f_{i_{\ell-1}} \circ \cdots \circ f_{i_1}(x)$$

   *for every $\ell$, $0 \leq \ell < p$;*
2. *for every $t \in T$, every choice of indexes $j_1, \ldots, j_p$ and every $\ell$, $0 \leq \ell < p$, it holds*

$$g_{j_p} \circ g_{j_{p-1}} \circ \cdots \circ g_{j_1}(t) \ \neq\ g_{j_\ell} \circ g_{j_{\ell-1}} \circ \cdots \circ g_{j_1}(t).$$

In the proof of the second statement of Theorem 2 we shall exhibit specific relations $(X,F)$ which also satisfy the requirements of Theorem 3. This proves that for Theorem 3 also, no other type of inequalities can be preserved.

After introducing the necessary notions in Section 2, the theorems will be proved in Section 3. In Section 4 we provide some further strengthenings and open questions.

## 2. Preliminaries and statement of results

Let $f_1, f_2, \ldots, f_k$ be partial mappings from $X$ to $X$ (we think of $f_i$ as a subset of $X \times X$). We denote by $F$ the set $F = \bigcup_{i=1}^{k} f_i$ (this is a union of relations). We say that the mappings $f_1, f_2, \ldots, f_k$ satisfy the inequality $(i_1, i_2, \ldots, i_p) \neq (j_1, j_2, \ldots, j_q)$ if for every $x \in X$ we have

$$f_{i_p} \circ \cdots \circ f_{i_2} \circ f_{i_1}(x) \neq f_{j_q} \circ \cdots \circ f_{j_2} \circ f_{j_1}(x).$$

Moreover, we say that the mappings $f_1, f_2, \ldots, f_k$ satisfy the inequality $(i_1, i_2, \ldots, i_p) \neq \varepsilon$ if for every $x \in X$ we have

$$f_{i_p} \circ \cdots \circ f_{i_2} \circ f_{i_1}(x) \neq x.$$

If $\varphi : (X, F) \to (T, S)$ is a homomorphism then we say that $\varphi$ preserves the inequality $(i_1, i_2, \ldots, i_p) \neq (j_1, j_2, \ldots, j_q)$ if for every $x \in X$ we have

$$\begin{aligned} &f_{i_p} \circ \cdots \circ f_{i_2} \circ f_{i_1}(x) \neq f_{j_q} \circ \cdots \circ f_{j_2} \circ f_{j_1}(x) \\ \Longrightarrow\ &\varphi(f_{i_p} \circ \cdots \circ f_{i_2} \circ f_{i_1}(x)) \neq \varphi(f_{j_q} \circ \cdots \circ f_{j_2} \circ f_{j_1}(x)). \end{aligned}$$

Similarly, we say that $\varphi$ preserves the inequality $(i_1, i_2, \ldots, i_p) \neq \varepsilon$ if for every $x \in X$ we have

$$f_{i_p} \circ \cdots \circ f_{i_2} \circ f_{i_1}(x) \neq x \Longrightarrow \varphi(f_{i_p} \circ \cdots \circ f_{i_2} \circ f_{i_1}(x)) \neq \varphi(x).$$

All this notation will be preserved in the sequel.
Clearly $(i_1, i_2, \ldots, i_p) \neq (i_1, i_2, \ldots, i_\ell)$ for every $\ell$, $0 \leq \ell < p$, is an initial $(p, l)$-inequality introduced in Section 1.

The Erdős-de Bruijn result Theorem 1 may be now formulated as follows:

**Theorem 1′.** *For every $k > 0$, there exists a relation $(T, S)$ such that for every loopless relation $(X, F)$, $F = \bigcup_{i=1}^{k} f_i$, there exists a homomorphism $\varphi : (X, F) \to (T, S)$ which preserves the inequalities $(i) \neq \varepsilon$ for every $i$, $1 \leq i \leq k$. Moreover, we have $|T| \leq 2k + 1$.*

Observe that in this new setting we do no longer need to assume that the relation $(X, F)$ has no fixpoint. Theorem 1′ still holds for relations $(X, F)$ having fixpoints by simply taking as a target relation the reflexive closure of $(T, S)$.

We shall prove the following lemmas:

**Lemma 4.** *For every $k > 0$, $p > 0$, there exists a finite relation $(T_{k,p}, S)$ with $|T_{k,p}| \leq 1 + 2k \cdot \frac{k^p - 1}{k - 1}$ (or $|T_{k,p}| \leq 2p + 1$ if $k = 1$), such that for every $(X, F)$, $F = \bigcup_{i=1}^{k} f_i$, there exists a homomorphism $\varphi : (X, F) \to (T_{k,p}, S)$ which preserves the initial $(p', l)$-inequalities $0 \leq \ell < p' \leq p$.*

**Lemma 5.** *Let $(i_1, i_2, \ldots, i_p) \neq (j_1, j_2, \ldots, j_q)$ be any non-initial inequality. Then for every $n > 0$ and $k \geq 2$, there exist partial mappings $f_1, f_2, \ldots, f_k$, such that any relation $(T, S)$ with a homomorphism $\varphi : (X, F) \to (T, S)$, $F = \bigcup_{i=1}^{k} f_i$, which preserves the inequality $(i_1, i_2, \ldots, i_p) \neq (j_1, j_2, \ldots, j_q)$ satisfies $|T| \geq n$.*

These results thus characterize all the inequalities which can be demanded to be preserved into finite targets relations. Lemmas 4 and 5 together clearly imply our Theorems 2, 3.

A result similar to our Lemma 4 has been proved in [10]. Using our terminology, it states the following: if we assume that the relation $(X, F)$ satisfies all inequalities $(i_1, i_2) \neq \varepsilon$ then for every $p > 0$ there exists a relation $(T'_{k,p}, S')$ which also satisfies all inequalities $(i_1, i_2) \neq \varepsilon$ and a homomorphism $\varphi : (X, F) \to (T'_{k,p}, S')$ which preserves all inequalities $(i_1, i_2, \ldots, i_p) \neq (i_1, i_2, \ldots, i_\ell)$ for every $\ell$, $0 \leq \ell < p$.

The upper bound in Lemma 4 is tight either for $k = 1$ (and $p$ arbitrary) as shown by oriented cycle of length $2p + 1$ or for $p = 1$ (and $k$ arbitrary) as shown by Erdős and de Bruijn (by the regular tournament with $2k+1$ vertices). For the remaining cases the tightness of the bound in Lemma 4 is a difficult combinatorial problem.

For proving Theorem 3 it seems more convenient to deal with the (directed) graphs of the relations. In this setting, our Theorem 3 can be rephrased as follows:

**Theorem 3′.** *For every $k > 0$, $p > 0$, there exists a digraph $H_{k,p}$ with no directed cycle of length less than $p+1$ such that every digraph $G$ with out-degree at most $k$ and with no directed cycle of length less than $p + 1$ homomorphically maps to $H_{k,p}$.*

Related results and extensions of this theorem (in the context of A-mote graphs) are given in [7].

For $k \geq 2$, our proof will be an adaptation of a proof given in [6] and [3] where it is shown that for every fixed finite family of connected graphs (or digraphs) $\mathcal{A}$, there exists a graph (or digraph) $H_{\mathcal{A}}$ such that (i) there is no homomorphism of a member of $\mathcal{A}$ to $H_{\mathcal{A}}$ and (ii) every graph (or digraph) $G$ with *degree* at most $b$, and such that there is no homomorphism of a member of $\mathcal{A}$ to $G$, maps homomorphically to $H_{\mathcal{A}}$. Our result thus states that if the family $\mathcal{A}$ is the family of directed cycles of length at most $p$ then it suffices to consider *out-degrees* only.

## 3. Proof of theorems

We start by proving Lemmas 4 and 5 which together imply Theorem 2.

Proof of Lemma 4: We first observe that a homomorphism preserves all inequalities $(i_1, i_2, \ldots, i_p) \neq (i_1, i_2, \ldots, i_\ell)$ for every $\ell$, $0 \leq \ell < p' \leq p$, if and only if it preserves all inequalities $(i_{\ell+1}, i_{\ell+2}, \ldots, i_p) \neq \varepsilon$ for every $\ell$, $0 \leq \ell < p' \leq p$. From $(X, F)$, we define a new relation $(X, F_p)$ given by $(x, y) \in F_p$ if and only if $y = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_{p'}}(x)$ for some $p'$, $0 < p' \leq p$. Every element $x$ has out-degree at most $q = k\frac{k^p-1}{k-1}$ (or $q = p$ if $k = 1$) in $F_p$. Therefore, $F_p$ can be viewed as the union of $q$ partial mappings $g_i$, $1 \leq i \leq q$. By Theorem 1′ we know that there exists a relation $(T, S)$ with $|T| \leq 2q + 1$ and a homomorphism $\varphi : (X, F_p) \to (T, S)$ which preserves the inequalities $(i) \neq \varepsilon$ for every $i$, $1 \leq i \leq q$. Clearly, $\varphi$ is also a homomorphism from $(X, F)$ to $(T, S)$ which preserves all inequalities $(i_{\ell+1}, i_{\ell+2}, \ldots, i_p) \neq \varepsilon$ for every $\ell$, $0 \leq \ell < p$, as required.

□

Proof of Lemma 5: Let $(i_1, i_2, \ldots, i_p) \neq (j_1, j_2, \ldots, j_q)$ be any non-initial inequality. Explicitly, $p, q \geq 1$ and there exists $\ell$, $1 \leq \ell \leq min(p, q)$, such that $i_\ell \neq j_\ell$ while $i_m = j_m$ for every $m$, $1 \leq m < \ell$.
We assume without loss of generality that $q \geq p$. Let

$$\begin{aligned} X = & \{x_1, x_2, \ldots x_n\} \\ & \cup \ \{y_c^{a,b} : 1 \leq a < b \leq n,\ c \in \{i_1, i_2, \ldots, i_p, j_\ell, j_{\ell+1}, \ldots, j_q\}\}. \end{aligned}$$

($y_c^{a,b}$ are supposed to be mutually distinct and distinct from elements $x_i$).
Elements $x_1, x_2, \ldots, x_n$ will be called *main elements*. We then define the partial mappings $f_1, f_2, \ldots, f_k$ as follows. For every $a, b$, $1 \leq a < b \leq n$, $\alpha \in \{1, 2, \ldots, p-1\}$, $\beta \in \{1, 2, \ldots, q-1\}$, let

- $f_{i_\alpha}(y_{i_\alpha}^{a,b}) = y_{i_{\alpha+1}}^{a,b}$,
- $f_{j_\beta}(y_{i_\beta}^{a,b}) = y_{i_{\beta+1}}^{a,b}$ if $\beta \leq \ell - 1$,
- $f_{j_\ell}(y_{i_\ell}^{a,b}) = y_{j_{\ell+1}}^{a,b}$,
- $f_{j_\beta}(y_{j_\beta}^{a,b}) = y_{j_{\beta+1}}^{a,b}$ if $\beta \geq \ell + 1$,
- $f_{i_p}(y_{i_p}^{a,b}) = x_a$,
- $f_{j_q}(y_{j_q}^{a,b}) = x_b$.

In other words, for every two main elements $x_a$ and $x_b$, $a < b$, there is an element $y_{i_1}^{a,b}$ such that

$$f_{i_p} \circ f_{i_{p-1}} \circ \cdots \circ f_{i_1}(y_{i_1}^{a,b}) = x_a$$

and

$$f_{j_q} \circ f_{j_{q-1}} \circ \cdots \circ f_{j_1}(y_{i_1}^{a,b}) = x_b.$$

Therefore, if there exists a relation $(T, S)$ and a homomorphism $\varphi : (X, F) \to (T, S)$ which preserves the inequality $(i_1, i_2, \ldots, i_p) \neq (j_1, j_2, \ldots, j_q)$ then all the main elements have to be mapped to distinct elements of $T$ and thus $|T| \geq n$.

□

Before the proof of Theorem 3′ let us introduce the key construction. For any digraph $G$ we denote by $\vec{d}_G(x, y)$ the oriented distance of $x$ to $y$ in $G$, that is the minimal length of a directed path from $x$ to $y$ (provided that such a path exists).

Assume that $k \geq 2$. The digraph $H_{k,p}$ is constructed as follows. Let $V$ be a fixed set of $2\frac{k^{p+1}-1}{k-1} - 1$ elements. The vertices of $H_{k,p}$ are all possible tuples of the form $(a; A_1, A_2, \dots, A_p)$, such that:

(i) $a \in V$,
(ii) $A_i \subseteq V \setminus \{a\}$ for every $i$, $1 \leq i \leq p$.

If $(a; A_1, A_2, \dots, A_p)$ and $(b; B_1, B_2, \dots, B_p)$ are two vertices in $H_{k,p}$ then there is an arc from $(a; A_1, A_2, \dots, A_p)$ to $(b; B_1, B_2, \dots, B_p)$ if and only if:

(iii) $b \in A_1$,
(iv) $B_i \subseteq A_{i+1}$ for every $i$, $1 \leq i < p$.

We now prove that the digraph $H_{k,p}$ satisfies the required property:

**Lemma 6.** *The digraph $H_{k,p}$ contains no directed cycle of length less than $p+1$.*

Proof: Suppose that $(a^1; A_1^1, A_2^1, \dots, A_p^1)$, $\dots$ , $(a^q; A_1^q, A_2^q, \dots, A_p^q)$, $(a^1; A_1^1, A_2^1, \dots, A_p^1)$ is a directed cycle in $H_{k,p}$ of length $q \leq p$. By condition (iii) we have $a^1 \in A_1^q$ and by condition (iv) we have $A_1^q \subseteq A_2^{q-1} \subseteq \dots \subseteq A_q^1$. We thus get $a^1 \in A_q^1$, in contradiction to condition (ii). $\square$

We can now prove Theorem 3′.

Proof of Theorem 3′: If $k \geq 2$ we use the digraph $H_{k,p}$ previously constructed. Let $G$ be any digraph with out-degree at most $k$ and no directed cycle of length less than $p+1$. The $p$-th power $G^p$ of $G$ is the digraph with same vertex set as $G$ and such that there is an arc from $x$ to $y$ in $G^p$ if and only if $0 < \vec{d}_G(x, y) \leq p$. The digraph $G^p$ has out-degree at most $t = \frac{k^{p+1}-1}{k-1} - 1$ and its underlying undirected graph $Und(G^p)$ is therefore $(2t+1)$-colorable (to see that, simply observe that every subgraph of $Und(G^p)$ has to contain a vertex of degree at most $2t$). Let us denote by $c$ such a $(2t+1)$-coloring.

We now define a homomorphism $\varphi : G \to H_{k,p}$ as follows: for every $x \in V(G)$, let $\varphi(x) = (c(x); X_1, X_2, \dots, X_p)$ where for every $i$, $1 \leq i \leq p$, $X_i$ is the set of all colors $c(y_i)$ such that there is a directed path in $G$ from $x$ to $y_i$ of length $i$. From the definition of $c$ we get that $c(x) \notin X_i$ for every $i$, $1 \leq i \leq p$. Therefore $\varphi(x)$ is indeed a vertex in $H_{k,p}$. Moreover, if $(x, y)$ is an arc in $G$, $(\varphi(x), \varphi(y))$ is clearly an arc in $H_{k,p}$ since every directed path of length $i$ starting at $y$ can be extended to a directed path of length $i+1$ starting at $x$.

If $k = 1$, every digraph $G$ with out-degree at most 1 and no directed cycle of length less than p+1 has clearly a homomorphism to the digraph $T_{1,p}$ obtained from a collection of $p$ directed cycles of respective lengths $p+1$, $p+2$, $\dots$, $2p+1$, containing respectively a vertex $x_{p+1}$, $x_{p+2}$, $\dots$, $x_{2p+1}$, by identifying these vertices into a unique vertex $x$. $\square$

## 4. Discussion

The bound we gave in Theorem 2 is tight. For Theorem 3, our construction leads to a value of the bound $t(k,p)$ of order $k^p \times 2^{kp^2}$. It would be interesting to have a better estimation of this upper bound.

Our Theorem 2 says that there exists a relation $(T, S = \bigcup_{j=1}^{q} g_j)$ such that for every relation $(X, F = \bigcup_{i=1}^{k} f_i)$ there exists a homomorphism $\varphi$ from $(X,F)$ to $(T,S)$ such that every initial inequality is preserved by $\varphi$ whenever it is satisfied by $F$. Here, the relation $(T,S)$ cannot be required itself to satisfy the initial inequalities. Our Theorem 3 says that if we only consider relations $(X,F)$ that satisfy all the initial inequalities then one can construct a target relation $(T,S)$ which also satisfies these inequalities. In fact, by slightly modifying the proof of Theorem $3'$, one can in some sense generalize Theorem 2 by constructing a target relation $(T,S)$ such that for every $k$-bounded relation $(X,F)$ there exists a homomorphism $\varphi : (X,F) \to (T,S)$ such that if all initial inequalities are satisfied by $F$ at some $x \in X$ then they are also satisfied by $S$ at $\varphi(x)$. More formally we have:

**Theorem 7.** *Let $f_1, f_2, \ldots, f_k$ be partial mappings from $X$ to $X$. For every $p > 0$, there exist a finite set $T$, partial mappings $g_1, g_2, \ldots, g_q$ from $T$ to $T$, and a homomorphism $\varphi :\ (X, F = f_1 \cup f_2 \cup \cdots \cup f_k) \to (T, S = g_1 \cup g_2 \cup \cdots \cup g_q)$ such that*

1. *$\varphi$ preserves all $(p',\ell)$-initial inequalities for every $\ell$, $0 \le \ell < p' \le p$,*
2. *for every $x \in X$, if we have*

$$f_{i_p} \circ f_{i_{p-1}} \circ \cdots \circ f_{i_1}(x) \neq f_{i_\ell} \circ f_{i_{\ell-1}} \circ \cdots \circ f_{i_1}(x)$$

*for every $\ell$, $0 \le \ell < p$, then we also have*

$$g_{j_p} \circ g_{j_{p-1}} \circ \cdots \circ g_{j_1}(x) \neq g_{j_{\ell'}} \circ g_{j_{\ell'-1}} \circ \cdots \circ g_{j_1}(x)$$

*for every $\ell'$, $0 \le \ell' < p$.*

To see that, it suffices to replace the condition (ii) in the definition of the target graph $H_{k,p}$ by the following condition:

(ii$'$) $A_i \subseteq V$ for every $i$, $1 \le i \le p$, and $a \notin A_1$.

We then get a new target graph $H'_{k,p}$ having short cycles. More explicitly, every vertex $(a; A_1, A_2, \ldots, A_p)$ of $H'_{k,p}$ belongs to a directed cycle of length $\ell \le p$ if and only if $a \in A_\ell$. Therefore the homomorphism $\varphi$ we used in the proof of Theorem $3'$ is such that every vertex not belonging to a directed cycle of length $\ell$ is mapped to a vertex not belonging to a directed cycle of length $\ell$.

One can also consider several variations of this problem. One of them is the following: given disjoint mappings $f, g : X \to X$, considered as relations, can we find disjoint finite relations $R_f$ and $R_g$ on some set $T$ and a mapping $\varphi : X \to T$

which is a homomorphism for both $(X, f) \to (T, R_f)$ and $(X, g) \to (T, R_g)$. The answer to this question is no as provided by the following example. Let $X = \mathbb{N} \cup \binom{\mathbb{N}}{2}$, $f(i,j) = i$ and $g(i,j) = j$ whenever $i < j$. Then every homomorphism $\varphi : X \to Y$ satisfies that $\varphi$ restricted to $\mathbb{N}$ is injective. This is a particular example involved in the proof of Lemma 5.

Another variant of the problem is obtained if we allow inverse mappings (even in inequalities). Also in this case the answers become very quickly negative. For suppose that $f, g$ are mappings from $\binom{\mathbb{N}}{2}$ to $\mathbb{N}$ defined as above. Put $(X, R)$ where $R = f \cup g^{-1}$. Clearly $R \cap R^{-1} = \emptyset$. Then any homomorphism $\varphi : (X, R) \to (T, S)$ where $S \cap S^{-1} = \emptyset$ satisfies that $\varphi$ restricted to $\mathbb{N}$ is injective.

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