Projections from $L(X,Y)$ onto $K(X,Y)$

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Abstract. Generalization of certain results in [Sap] and simplification of the proofs are given. We observe e.g.: Let X and Y be Banach spaces such that X is weakly compactly generated Asplund space and X^* has the approximation property (respectively Y is weakly compactly generated Asplund space and Y^* has the approximation property). Suppose that $L(X,Y) \neq K(X,Y)$ and let $1 < \lambda < 2$. Then X (respectively Y) can be equivalently renormed so that any projection P of $L(X,Y)$ onto $K(X,Y)$ has the sup-norm greater or equal to λ.

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Let $K(X,Y)$ (resp. $L(X,Y)$) denote the space of all compact (resp. bounded) linear operators from the Banach space X to the Banach space Y. The question whether $K(X,Y)$ is an uncomplemented subspace of $L(X,Y)$ whenever $K(X,Y) \neq L(X,Y)$ is long-standing ([AtWi], [Ku], [Th], [To], [ToWi]). The positive answer was given e.g. if X or Y has unconditional basis ([DM], [Em1], [Fe1], [Fe2], [J1], [Ka], [Jo], [Ru]). More generally the question has positive answer if $c_0 \subset K(X,Y)$ as it was independently shown in [Em2] and [Jo2]. In [EJ] it was observed that under some geometric assumptions on the spaces X and Y there are no norm one projections P from $L(X,Y)$ onto $K(X,Y)$.

An other step forward to the general solution was made in [Sap]. The author using the notion of the Godun set (see Definition 2) proves e.g.:

(S) *Suppose that* $1 < \lambda < 2$ *and* $L(X,Y) \neq K(X,Y)$. *If* Y^* *is separable and has the approximation property then* Y *can be equivalently renormed so that any projection* P *of* $L(X,Y)$ *onto* $K(X,Y)$ *has the sup-norm greater or equal to* λ.

Saphar [Sap] actually proves more. He proves a general lemma (Lemma 2.2) telling that if λ is in the Godun set $G(E,M)$ of E relative to $M \subset E^{**}$ then any projection P from M onto E has the sup norm $\geq \lambda$. Next he shows that under the assumptions of (S) we have $\lambda \in G(K(X,Y), L(X,Y))$. The result (S) then follows.

Our paper was inspired by these results of P.D. Saphar. We follow his ideas and observe that his estimates of the norm of the projection P may be obtained very

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easily without the reference to the notion of the Godun set. Of course, the idea (inequality (2) bellow) is contained in [Sap]: Suppose that P is a projection from a space $M \subset E^{**}$ onto the space E and if an element T, $\|T\| = 1$ from $P^{-1}(0)$ may be w* approximated by elements $T_\alpha \in E$ in such a way that $\|T - \lambda T_\alpha\| \leq \|T\|$. Then we obtain $\|\lambda T_\alpha\| = \|PT - \lambda PT_\alpha\| \leq \|P\|$. Now $1 = \|T\| \leq \limsup \|T_\alpha\|$ because $T_\alpha \xrightarrow{w^*} T$. We get immediately $\lambda \leq \|P\|$.

Moreover our simplification gives generalizations of certain results in [Sap]. We prove e.g. the above mentioned result when the assumptions on Y are pushed to the space X (Corollary 2). We also show that the norm of the projection P in question is $\geq \lambda$ if X or Y is reflexive and has the approximation property. The results concerning the Godun set namely that e.g. $\lambda \in G(K(X,Y), L(X,Y))$ are also possible in our cases (Remark 2).

All operators in this paper are linear and all Banach spaces are real. If Z is a Banach space we denote by Id_Z the identity operator in Z. Following Kalton [Ka] we will denote by w' the linear topology on $L(X,Y)$ which is generated by the functionals $x^{**} \otimes y^* \in X^{**} \otimes Y^*$. Thus $T_\alpha \xrightarrow{w'} T$ means that $y^{**}(T^*_\alpha y^*) \longrightarrow y^{**}(T^* y^*)$ for all $x^{**} \in X^{**}$ and all $y^* \in Y^*$. We will also use the following result due to [Ka]:

(K) *In $K(X,Y)$ coincides the w^* convergence of sequences and the convergence of sequences in the weak topology of the Banach space $K(X,Y)$.*

Definition 1. Let us denote by K_λ the class of all Banach spaces Z such that there is a net $\{k_\alpha\}$ of compact operators in Z such that

(i) $k_\alpha(z) \longrightarrow z$ weakly for all $z \in Z$

and

(ii) $\limsup_\alpha \|Id_Z - \lambda k_\alpha\| \leq 1$.

If moreover $\limsup_\alpha \|k_\alpha\| \leq 1$ we will speak about the class K^1_λ.
Evidently $K^1_\lambda \subset K_\lambda$.

Proposition 1. *Let the Banach space X or the Banach space Y belong to K_λ and suppose that $L(X,Y) \neq K(X,Y)$. Then any projection P of $L(X,Y)$ onto $K(X,Y)$ has the sup-norm greater or equal to λ.*

Proof: Suppose that $X \in K_\lambda$ and let $\{k_\alpha\} \subset K(X)$ be the net of compact operators in X satisfying (i) and (ii) from the Definition 1. Set $T_\alpha = Tk_\alpha$. Similarly we set $T_\alpha = k_\alpha T$ if $Y \in K_\lambda$ and if $\{k_\alpha\} \subset K(Y)$ is the sequence of compact operators in Y satisfying (i) and (ii) from the Definition 1.

For any $\epsilon > 0$ and for suitable $x \in X$, $\|x\| \leq 1$ and suitable $y^* \in Y^*$ $\|y\| \leq 1$ we have $\|T\| \leq y^*(Tx) + \epsilon = \lim y^*(T_\alpha x) + \epsilon \leq \liminf \|T_\alpha\| + \epsilon$. The $\epsilon > 0$ being arbitrary we get

$$\|T\| \leq \liminf \|T_\alpha\|. \tag{1}$$

Let us choose $T \in P^{-1}(0)$, $T \neq 0$. Then

$$\begin{aligned} \limsup_\alpha \|\lambda T_\alpha\| &= \limsup_\alpha \|P(T-\lambda T_\alpha)\| \le \|P\| \limsup_\alpha \|(T-\lambda T_\alpha)\| \\ &\le \|P\| \limsup_\alpha \|(Id_X - k_\alpha)\| \|T\| \le \|P\| \|T\|. \end{aligned} \tag{2}$$

From (1) and (2) we conclude that $\lambda \le \|P\|$. □

Proposition 2. *Let E be a Banach space such that its dual is separable and has the approximation property. Let λ be a scalar with $1 < \lambda < 2$. Then there is an equivalent norm $|||\cdot|||$ on E such that $(E, |||\cdot|||) \in K^1_\lambda$.*

Proof: Similarly as in [Sap] we choose by a result of [Zip] a Banach space $E_1 \supset E$ such that E_1 has a shrinking basis. Let $\{k_n\}$ be the projections in E_1 given by the shrinking basis. Following again Saphar's paper we use [CasKa, Lemma 3.4] to get an equivalent norm $|||\cdot|||$ on E_1 such that $|||Id_{E_1} - \lambda k_n||| = 1$ and $|||k_n||| = 1$. It is well known that if E^* has the (metric) approximation property [LT] and if E^* is separable then there is a shrinking approximating sequence in E (cf. e.g. [Sin, Remark 9.13]). This means that there is a sequence of finite-dimensional operators in E such that $h_n \xrightarrow{w'} Id_E$. Evidently $k_n \xrightarrow{w'} Id_{E_1}$, so that $k_{/E} = k_n i \xrightarrow{w'} i$ where i is the imbedding of E into E_1. Let us set

$$l_n = h_n - k_{n/E} : E \longrightarrow E_1.$$

Easily we observe that $l_n \xrightarrow{w'} 0$ which means by (K) that $\{l_n\}$ converges to 0 in the weak topology of $K(E, E_1)$. This implies that certain convex combinations $\{l'_p\}$ of $\{l_n\}$ converge to 0 in the norm topology of $K(E, E_1)$. Let $\{h'_p\}$ (resp. $\{k'_p\}$) be the same convex combinations of $\{h_n\}$ (resp. $\{k_n\}$). Let us consider on E the equivalent norm $|||\cdot|||$ induced from E_1. Then

$$\limsup_p |||Id_E - \lambda h'_p||| = \limsup_p |||Id_E - \lambda k'_{p/E}||| \le 1$$

and similarly

$$\lim_p |||h'_p||| = \lim_p |||k'_{p/E}||| = 1.$$

Observing that $h'_p \xrightarrow{w'} Id_E$ finishes the proof. □

Propositions 1 and 2 have the following immediate corollaries the first of which was proved in [Sap] and the second is new:

Corollary 1 (Saphar)**.** *Let the Banach space X and Y be Banach spaces such that Y^* is separable and has the approximation property. Let λ be a scalar with $1 < \lambda < 2$ and suppose that $L(X,Y) \neq K(X,Y)$. Then Y can be equivalently renormed so that any projection P of $L(X,Y)$ onto $K(X,Y)$ has the sup-norm greater or equal to λ.*

Corollary 2. *Let X and Y be Banach spaces such that X^* is separable and has the approximation property. Let λ be a scalar with $1 < \lambda < 2$ and suppose that $L(X,Y) \neq K(X,Y)$. Then X can be equivalently renormed so that any projection P of $L(X,Y)$ onto $K(X,Y)$ has the sup-norm greater or equal to λ.*

The Proposition 3 generalizes certain results from [Sap].

Proposition 3. *Let X and Y be Banach spaces such that X is reflexive and has the approximation property (resp. Y is reflexive and has the approximation property). Suppose that $L(X,Y) \neq K(X,Y)$ and let $1 < \lambda < 2$. Then X (resp. Y) can be equivalently renormed so that any projection P of $L(X,Y)$ onto $K(X,Y)$ has the sup-norm greater or equal to λ.*

Proof: First suppose that there is a norm one projection Q in X (resp. in Y) which has the separable range and a noncompact operator $T \in L(X,Y)$ such that

(a) $0 \neq T \in P^{-1}(0)$,
(b) $TQ = T$ (resp. $T = QT$),

where P is the bounded projection of $L(X,Y)$ onto $K(X,Y)$.
Having in mind that the Banach space X^* (resp. Y^*) has the metric approximation property [LT] we see that also the range $Q^*X^* = (QX)^*$ (resp. $Q^*Y^* = (QY)^*$) of norm one projection Q^* has the metric approximation property. Let us denote by $\|\cdot\|_1$ the initial norm on X (resp. on Y) so that Q has the norm one with respect to these norms. The Proposition 2 tells that there is an equivalent norm $|||\cdot|||$ on QX (resp. on QY) so that $QX \in K_\lambda$ (resp. $QY \in K_\lambda$) in the norm $|||\cdot|||$. Now we proceed as in the proof of the Proposition 1. Let $\{k_\alpha\} \subset K(QX)$ (resp. $K(QY)$) be a sequence of compact operators in $QX \subset X$ (resp. $QY \subset Y$) such that $k_\alpha(z) \longrightarrow z$ weakly for all $z \in QX$ (resp. $z \in QY$) and $\limsup_\alpha |||Id_{QX} - \lambda k_\alpha||| \leq 1$ (resp. $\limsup_\alpha |||Id_{QY} - \lambda k_\alpha||| \leq 1$). Let us extend this equivalent norm on QX (resp. on QY) to an equivalent norm $\|\cdot\|$ on the whole X (resp. Y) in such a way that $\|Q\| = 1$ again. We may put e.g. $\|x\| = |||Qx||| + \|(Id - Q)x\|_1$. Set $T_\alpha = Tk_\alpha Q$ (resp. $T_\alpha = k_\alpha QT$). Again we have (1) and (2) and thus $\lambda \leq \|P\|$.

It remains to observe that there are a projection Q in X (resp. in Y) which has the separable range and $T \in L(X,Y)$ such that (a) and (b) hold. Consider the set S of all $T \in L(X,Y)$ such that T has the separable range. Evidently $K(X,Y) \subset S$ and S is a linear subspace of $L(X,Y)$. Let us choose a noncompact $T_1 \in L(X,Y)$. As in [Sap] we use that the noncompactness of T_1 is separable property. We choose a sequence $\{x_n\} \subset X$, $\|x_n\|_1 = 1$ such that sequence $\{T_1 x_n\} \subset Y$ has no convergent subsequences. Now if X (resp. Y) is reflexive there is a projection Q_1 in X (resp. Y) such that Q_1 has separable range containing $\{x_n\}$ (resp. $\{T_1 x_n\}$). Then $T_2 = T_1 Q$ (resp. $T_2 = QT_1$) is a noncompact operator with a separable range. Thus $K(X,Y) \subset S$, $K(X,Y) \neq S$ and the projection P is invariant on $S \subset L(X,Y)$. Let us consider the restriction $P_{/S}$ of P on S. We may choose $0 \neq T \in P_{/S}^{-1}(0)$. Now if Y is reflexive we chose by [Lin, Proposition 1] a projection Q in Y, Q having a separable range QY which contains the range of T and thus

$T = QT$. The separability of TX implies the w*-separability of $T^*Y^* \subset X^*$ (cf. e.g. [AmLin, Lemma 5] which works also for linear operators]). Now if X is reflexive T^*Y^* is weakly separable and thus separable. Using again [Lin] we get a projection Q in X such that the range of Q^* contains T^*Y^*. Thus $T^* = Q^*T^*$ which means that $T = TQ$. □

Remark 1. With slightly more care it can be seen that the assumption of the reflexivity of the Banach space X (resp. Y) in the above Proposition 5 may be substituted by more general assumption that the Banach space X (resp. Y) is weakly compactly generated and Asplund. Namely we may show that the following generalization of Corollaries 1, 2 and Propositions 3 holds:

Let λ be a scalar with $1 < \lambda < 2$ and suppose that $L(X,Y) \neq K(X,Y)$. Suppose that one of the assumption (i) *or* (ii) *is valid.*

(i) *X is a weakly compactly generated Banach space, X is an Asplund space and X^* has the approximation property.*
(ii) *Y is a weakly compactly generated Banach space, Y is an Asplund space and Y^* has the approximation property.*

Then X (resp. Y) can be equivalently renormed so that any projection P of $L(X,Y)$ onto $K(X,Y)$ has the norm greater or equal to λ.

The proof is formally the same as that of the Proposition 3. The separability of $(TX)^*$ (resp. $(TY)^*$) is a consequence of the Asplundness assumption. To get a projection Q with a separable range in X such that $T^*Y^* \subset Q^*X^*$ we use [AmLin, Lemma 4] and the w^*-separability of T^*Y^*.
For the last remark we will repeat the extended definition of the Godun set $G(E,M)$ from [Sap]:

Definition 2. Let E be a Banach space and a subspace $M \subset E^{**}$ with $E \subset M$. We define the set $G(E,M)$ of positive scalars λ such that for any $x^{**} \in M$ there exists a net $\{x_\alpha\} \subset E$ which verifies the following properties:

1) $x_\alpha \longrightarrow x^{**}$ in the w*-topology $\sigma(E^{**}, E^*)$,
2) $\limsup_\alpha \|x^{**} - \lambda x_\alpha\| \leq \|x^{**}\|$.

Remark 2. As it was mentioned at the beginning of the paper Saphar [Sap] deduces the lower estimates of the possible projections P of $L(X,Y)$ onto $K(X,Y)$ from statements on the Godun set $G(K(X,Y), L(X,Y))$. We have preferred to use the simple direct proofs. Nevertheless the statements on the Godun set $G(K(X,Y), L(X,Y))$ are also possible in our cases. For example we have

Proposition 1′. *Let the Banach space X or the Banach space Y belong to the class K^1_λ. Then there is an isometric imbedding $J : L(X,Y) \longrightarrow K(X,Y)^{**}$ and we have $\lambda \in G(K(X,Y), L(X,Y))$.*

Proof: We proceed as in [Jo, Lemma 2]. Denote by K the closed unit ball in the space $K(X,Y)^{**}$ and consider in K the w^* topology. Let $T_\alpha \in K(X,Y)$ be the approximations of T defined in the proof of Proposition 1. Let $B_{L(X,Y)}$ be a closed unit ball in $L(X,Y)$ and let $\{J_\alpha\}$ be a net in $K^{B_{L(X,Y)}}$ defined by

$$J_\alpha(T) = T_\alpha.$$

The space $K^{B_{L(X,Y)}}$ being compact we may choose a subnet $\{J_{\alpha_\beta}\}$ converging w^* to $J \in K^{B_{L(X,Y)}}$. Let us extend J by homogeneity to the whole of $L(X,Y)$. Evidently J is linear map of $L(X,Y)$ into $K(X,Y)^{**}$ and

$$J(T)(\phi) = \lim_\beta \phi(T_{\alpha_\beta}) \tag{3}$$

for all $\phi \in K(X,Y)^*$.
Now let $\limsup_\alpha \|k_\alpha\| \leq 1$ for all α, where k_α satisfy (i) and (ii) from the Definition 1. Considering $\phi = x \otimes y^* \in K(X,Y)^*$ we get from (3) and (i)

$$\begin{aligned}\|T\| &= \sup\{\lim_\beta y^*(T_{\alpha_\beta}(x)); \|x \otimes y^*\| = 1\} = \sup\{|JT(x \otimes y^*)|; \|x \otimes y^*\| = 1\}\\ &\leq \|jT\|^{**} = \sup\{|JT(\phi)|; \|\phi\|^* \leq 1\} = \sup\{\lim_\beta \phi(T_{\alpha_\beta})\}\\ &\leq \limsup_\beta \|T_{\alpha_\beta}\| \leq \|T\| \limsup \|k_\alpha\| \leq \|T\|.\end{aligned}$$

Thus J is an isometry of $L(X,Y)$ into $K(X,Y)^{**}$. If T is any element from $L(X,Y)$ we have

$$\limsup_\alpha \|JT - \lambda JT_\alpha\|^{**} = \limsup_\alpha \|T - \lambda T_\alpha\| \leq \|T\| \limsup_\alpha \|Id - \lambda k_\alpha\| \leq \|T\|.$$

Proposition $1'$ together with Proposition 2 combine to give statements similar to the Corollaries 1 and 2, Proposition 3 and the proposition stated in the Remark 1. For example the last one reads:

Let λ be a scalar with $1 < \lambda < 2$ and suppose that one of the assumption (1) *or* (2) *is valid.*

(1) *X is a weakly compactly generated Banach space, X is an Asplund space and X^* has the approximation property.*
(2) *Y is a weakly compactly generated Banach space, Y is an Asplund space and Y^* has the approximation property.*

Then X in the case (1) (*resp. Y in the case* (2)) *can be equivalently renormed so that $\lambda \in G(K(X,Y), L(X,Y))$.* □

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