Multiple solutions of a Schrödinger type semilinear equation

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Abstract. Two nontrivial solutions are obtained for nonhomogeneous semilinear Schrödinger equations.

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1. Introduction

The main purpose of this work is to investigate the existence of multiple solutions of the semilinear Schrödinger equation

$$-\triangle u + q(x)u = \lambda u + g(x,u) + f \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $f \in L^2(\mathbb{R}^N)$, $N \geq 3$.

Throughout this paper we assume that

(A1) $q \in L^\infty(\mathbb{R}^N)$ is periodic;

(A2) λ is in the spectral gap of the operator $(-\triangle + q)$.

It is well known that the spectrum $\sigma(T)$ of Schrödinger operator $T = -\triangle + q$ is purely continuous. We denote by E the Sobolev space $H^1(\mathbb{R}^N)$. For $\lambda \in G$, a spectral gap of T, we may decompose E corresponding to the spectral gap G into $E = E^+ \bigoplus E^-$ such that the quadratic form

$$Q(u) = \int_{\mathbb{R}^N} (|\bigtriangledown u|^2 + qu^2 - \lambda u^2)\, dx$$

associated with $T - \lambda I$, $\lambda \in G$, is positive and negative on E^+ and E^- respectively. Both E^+ and E^- are infinite dimensional, so the operator $-\triangle + q - \lambda$ is strongly indefinite. There are many existence results for the case $f \equiv 0$ and we refer to the papers [BJ], [CY], [PP] and references therein. Such a problem is usually resolved by the Linking theorem ([R]), it only yields one solution in general. The nonhomogeneous term f plays a role that the associated functional of (1.1) is no longer even, so the multiple solutions of (1.1) cannot be obtained in a direct way. There are obtained in [CZ] and [J] some multiplicity results for $q = 0$ and $\lambda < 0$.

In this case, the operator $T-\lambda I$ is positive definite. Our problem is different and more involved. We assume further that

(G1) $g(x,t)$ is C^1-function and $g_t'(x,t)\geq 0$ on $\mathbb{R}^N\times\mathbb{R}$,

(G2) there exists $K\in L^1(\mathbb{R}^N)\cap L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ such that $|g(x,t)|\leq K(x)(1+|t|^p)$, where $p\in(1,\dfrac{N+2}{N-2})$, $N\geq 3$,

(G3) $g(x,t)=o(|t|)$ as $t\to 0$ uniformly in $x\in\mathbb{R}^N$,

(G4) there is a constant $\beta>2$ such that

$$0<\beta G(x,t)\leq tg(x,t)$$

for all $t\neq 0$ and $x\in\mathbb{R}^N$, where $G(x,t)=\displaystyle\int_0^t g(x,s)\,ds$.

Therefore, the limits $g_{\pm}=\displaystyle\lim_{t\to\pm\infty}\frac{g(x,t)}{t}=+\infty$ uniformly for $x\in\Omega\subset\subset\mathbb{R}^N$. It reminds one of a type of Ambrosetti-Prodi problem in bounded domains [AP], [F] and [FY]. These Ambrosetti-Prodi type of problems can be viewed as a question of characterizing the range of a perturbation of a linear operator by some nonlinear operator.

In this paper, we obtain two solutions for problem (1.1). The solutions of problem (1.1) will be found as critical points of the functional

$$J(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+qu^2-\lambda u^2)\,dx-\int_{\mathbb{R}^N}G(x,u)\,dx-\int_{\mathbb{R}^N}fu\,dx. \tag{1.2}$$

First we reduce the problem by the Lyapunov-Schmidt reduction to a problem in E^+, and then using variational method, we obtain the following result.

Theorem A. *Assume* (A1)–(A2) *and* (G1)–(G4). *If* $\|f\|_{L^2(\mathbb{R}^N)}$ *is small, problem* (1.1) *possesses at least two solutions.*

Section 2 is dealt with Lyapunov-Schmidt reduction, existence result is proved in Section 3.

2. Lyapunov-Schmidt reduction

Let $E=E^+\bigoplus E^-$ and the quadratic form Q be defined as in Section 1. It is known that Q is positive on E^+ and negative on E^-. We can define a new scalar product $(\cdot,\cdot)_E$ on E with the corresponding norm $\|\cdot\|_E$ such that

$$Q(u)=-\|u\|_E^2\ \text{ for }\ u\in E^-\ \text{ and }\ Q(u)=\|u\|_E^+\ \text{ for }\ u\in E^+.$$

The norm $\|\cdot\|_E$ is equivalent to the original norm on E, see [PP] for details. Let $P^+:E\to E^+$ and $P^-:E\to E^-$ be orthogonal projections of E onto E^+ and E^- respectively. With the aid of these projections, we can write Q in the form $Q(u) = \|P^+u\|_E^2 - \|P^-u\|_E^2$. One may verify that the functional J defined in (1.2) is well defined and C^1 on E. To eliminate the effect of indefinite property, we consider the functional

$$(2.1)\quad I_v(w) = J(v+w) = \frac{1}{2}(\|v\|_E^2 - \|w\|_E^2) - \int_{\mathbb{R}^N} G(x, v+w)\, dx - \int_{\mathbb{R}^N} f(v+w)\, dx$$

defined on E^- for fixed $v \in E^+$. By (A2), (G4) and Hölder's inequality, we have

$$(2.2)\qquad I_v(w) \le \frac{1}{2}(\|v\|_E^2 - \|w\|_E^2) + \varepsilon\|w\|_E^2 + C_\varepsilon\|f\|_{L^2}^2 + \|f\|_{L^2}\|v\|_E.$$

Choose $\varepsilon > 0$ sufficiently small in (2.2), then for any fixed $v \in E^+$, $I_v(w) \to -\infty$ as $\|w\|_E \to \infty$. It implies that $I_v(w)$ is bounded above on E^-. Set

$$(2.3)\qquad M = \sup_{w \in E^-} I_v(w).$$

Lemma 2.1. *Let $K(x)$ be as in* (G2). *If $u_n \overset{n}{\rightharpoonup} u$ weakly in E, then a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, satisfies*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)\big|u_n - u\big|^{p+1} dx = 0.$$

The conclusion follows by the fact that K decays uniformly in "average" sense at infinity. For a proof we refer to [L].

Lemma 2.2. *M is attained by some $w_0 \in E^-$. Furthermore, w_0 satisfies*

$$(2.4)\qquad -\triangle w_0 + q w_0 = \lambda w_0 + g(x, v + w_0) + f \quad \text{in} \quad (E^-)^*.$$

Proof: We follow some ideas from [BJS]. By Ekeland's variational principle [E], we may find a maximizing sequence $\{w_n\} \subset E^-$ of problem (2.3) such that

$$(2.5)\quad \frac{1}{2}(\|v\|_E^2 - \|w_n\|_E^2) - \int_{\mathbb{R}^N} G(x, v + w_n)\, dx - \int_{\mathbb{R}^N} f(v + w_n)\, dx = M + o(1),$$

$$(2.6)\quad \int_{\mathbb{R}^N} (\nabla w_n \nabla \varphi + q w_n \varphi - \lambda w_n \varphi)\, dx - \int_{\mathbb{R}^N} g(x, v + w_n)\varphi\, dx - \int_{\mathbb{R}^N} f\varphi\, dx$$
$$= o(1)\|\varphi\|_E, \qquad \forall\, \varphi \in E^-.$$

Taking $\varphi = -w_n$ in (2.6), we obtain

$$(2.7)\qquad \|w_n\|_E^2 + \int_{\mathbb{R}^N} g(x, v + w_n) w_n\, dx + \int_{\mathbb{R}^N} f w_n\, dx = o(1)\|w_n\|_E.$$

Therefore

$$\|w_n\|_E^2 + \int_{\mathbb{R}^N} g(x, v+w_n)(v+w_n)\,dx \leq \int_{\mathbb{R}^N} g(x, v+w_n)v\,dx + C\|f\|_{L^2}\|w_n\|_E + o(1)\|w_n\|_E.$$

By (G1)–(G4), we have

$$\begin{aligned} |g(x,t)|^2 &\leq Ctg(x,t) \quad \text{if } |t| \leq 1 \text{ and } x \in \mathbb{R}^N, \\ |g(x,t)|^{\frac{p+1}{p}} &\leq Ctg(x,t) \quad \text{if } |t| \geq 1 \text{ and } x \in \mathbb{R}^N \end{aligned}$$

for some constant $C > 0$. It follows

$$\begin{aligned} &|\int_{\mathbb{R}^N} g(x, v+w_n)v\,dx| \\ &\leq C(\int_{\{|v+w_n|\leq 1\}} |g(x, v+w_n)|^2\,dx)^{\frac{1}{2}}\|v\|_{L^2} \\ &\quad + C(\int_{\{|v+w_n|\geq 1\}} |g(x, v+w_n)|^{\frac{p+1}{p}}\,dx)^{\frac{p}{p+1}}\|v\|_{L^{p+1}} \\ &\leq C(\int_{\mathbb{R}^N} (v+w_n)g(x, v+w_n)\,dx)^{\frac{1}{2}}\|v\|_{L^2} \\ &\quad + C(\int_{\mathbb{R}^N} (v+w_n)g(x, v+w_n)\,dx)^{\frac{p}{p+1}}\|v\|_{L^{p+1}} \\ &\leq \varepsilon \int_{\mathbb{R}^N} (v+w_n)g(x, v+w_n)\,dx + C_\varepsilon(\|v\|_E^2 + \|v\|_E^{p+1}). \end{aligned} \tag{2.8}$$

As a result, we obtain

$$\|w_n\|_E \leq C$$

by choosing $\varepsilon > 0$ sufficiently small. Therefore we may assume that $w_n \overset{n}{\rightharpoonup} w_0$ in E and $w_n \xrightarrow{n} w_0$ in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq r < 2^* := \dfrac{2N}{N-2}$ and we have $w_0 \in E^-$ satisfying (2.4). Hence

$$\begin{aligned} &\int_{\mathbb{R}^N} [\nabla(w_n - w_0)\nabla\varphi + q(w_n - w_0)\varphi - \lambda(w_n - w_0)\varphi]\,dx \\ &\quad = \int_{\mathbb{R}^N} [g(x, v+w_n) - g(x, v+w_0)]\varphi\,dx + o(1)\|\varphi\|_E, \quad \forall\,\varphi \in E^-. \end{aligned} \tag{2.9}$$

Let $\varphi = -(w_n - w_0)$ in (2.9). Then

$$\|w_n - w_0\|_E^2 + \int_{\mathbb{R}^N} [g(x, v+w_n)(w_n - w_0) - g(x, v+w_0)(w_n - w_0)]\,dx = o(1)\|w_n - w_0\|_E.$$

By (G2), Hölder's inequality and Lemma 2.1 we obtain

$$\int_{\mathbb{R}^N} g(x, v + w_n)(w_n - w_0)\,dx \xrightarrow{n} 0, \tag{2.10}$$

$$\int_{\mathbb{R}^N} g(x, v + w_0)(w_n - w_0)\,dx \xrightarrow{n} 0. \tag{2.11}$$

Actually, by (G2)

$$\begin{aligned}
&|\int_{\mathbb{R}^N} g(x, v + w_n)(w_n - w_0)\,dx| \\
&\le C \int_{\mathbb{R}^N} K(x)(|v + w_n| + |v + w_n|^p)|w_n - w_0|\,dx \\
&\le C \int_{\mathbb{R}^N} K(x)(|w_n - w_0|^2 + |w_n - w_0|^{p+1})\,dx
\end{aligned} \tag{2.12}$$

since $\{w_n\}$ is bounded in E. (2.12) and Lemma 2.1 imply (2.10). (2.11) can be obtained in the same way. Consequently,

$$w_n \xrightarrow{n} w_0 \ \text{ strongly in } \ E.$$

The assertion follows. □

Lemma 2.3. *There exists $h \in C^1(E^+, E^-)$ such that*

$$J(v + w) < J(v + h(v)), \quad \forall\, w \in \ E^- \quad \textit{and} \quad w \neq h(v).$$

Moreover, $h(v)$ satisfies (2.4).

Proof: Following arguments in [BJS], we let

$$k(v, w) = -\triangle w + qw - \lambda w - P^-(g(x, v + w) + f),$$

where v is fixed, $w \in E^-$. By Lemma 2.2 we have

$$k(v, w_0) = 0.$$

For all $z \in E^-$, $z \neq 0$, we deduce by (G1) that

$$\begin{aligned}
\langle D_w k(v, w_0)z, z\rangle &= \int_{\mathbb{R}^N} (|\bigtriangledown z|^2 + qz^2 - \lambda z^2)\,dx - \int_{\mathbb{R}^N} g_t'(x, v + w_0)z^2\,dx \\
&\le -\|z\|_E^2 < 0.
\end{aligned}$$

Hence $D_w k(v, w_0)$ is bounded in E^*, we conclude that its inverse exists and is bounded. The Implicit Function Theorem yields that there exists $h \in C^1(E^+, E^-)$ such that $w_0 = h(v)$. □

3. Existence results

In this section we prove Theorem A. The first solution is obtained as a local minimum of a functional in a small ball, the second one is found by the Mountain Pass Theorem ([AR]). Let

$$F(v) = J(v + h(v)), \quad \forall\ v \in\ E^+.$$

Then $F \in C^1(E^+, \mathbb{R})$. By (2.4) we know that

$$-\int_{\mathbb{R}^N} fh(0)\,dx = \int_{\mathbb{R}^N} h(0)g(x, h(0))\,dx + \|h(0)\|_E^2.$$

Using (G4) we obtain

$$|\int_{\mathbb{R}^N} fh(0)\,dx| \geq \|h(0)\|_E^2.$$

If $\|P^- f\|_{L^2(\mathbb{R}^N)}$ small, the inequality implies $\|h(0)\|_E$ small. Consequently, $F(0)$ is small provided that $\|P^- f\|_{L^2(\mathbb{R}^N)}$ is small.

Lemma 3.1. *If $\|P^- f\|_{L^2(\mathbb{R}^N)}$ is small, there exist $\alpha, r > 0$ such that*

$$F(v) \geq \alpha > F(0), \quad \forall\ v \in\ E^+,\ \|v\|_E = r. \tag{3.1}$$

Proof: By (G2), (G3), Lemma 2.3 and Hölder's inequality, we have

$$F(v) \geq J(v) \geq (\frac{1}{2} - \varepsilon)\|v\|_E^2 - C_\varepsilon(\|v\|_E^{p+1} + \|f\|_{L^2}^2). \tag{3.2}$$

On the other hand,

$$F(0) \leq C\|f\|_{L^2}\|h(0)\|_E. \tag{3.3}$$

Thus, from (3.2) and (3.3) we obtain (3.1) for $\|v\|_E$ and $\|f\|_{L^2}$ small. □

Lemma 3.2. *For any $v \in E^+$, $\|F'(v)\|_{E^*} = \|J'(v + h(v))\|_{E^*}$.*

Proof: See the proof of Lemma 2.2 in [BJS]. □

A sequence $\{v_n\}$ is said to be the Palais-Smale sequence for F ((PS)-sequence for short) if $|F(v_n)| \leq C$ uniformly in n and $F'(v_n) \xrightarrow{n} 0$ in $(E^+)^*$. We say that F satisfies the Palais-Smale condition ((PS) condition for short) if every (PS)-sequence of F is relatively compact in E^+.

Lemma 3.3. *F satisfies (PS) condition.*

PROOF: Let $v_n \subset E^+$ be a (PS)-sequence of F. We may assume that

$$F(v_n) \xrightarrow{n} c, \quad F'(v_n) \xrightarrow{n} 0.$$

By Lemma 3.2 we have

$$J(v_n + h(v_n)) \xrightarrow{n} c, \quad J'(v_n + h(v_n)) \xrightarrow{n} 0. \tag{3.4}$$

Let $u_n = v_n + h(v_n)$. Then

$$\begin{aligned}
&J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N} g(x,u_n)u_n\,dx - \int_{\mathbb{R}^N} G(x,u_n)\,dx + \frac{1}{2}\int_{\mathbb{R}^N} fu_n\,dx \\
&\le c + o(1)\|u_n\|_E + o(1).
\end{aligned}$$

By (G4)

$$(\frac{1}{2} - \frac{1}{\beta})\int_{\mathbb{R}^N} g(x,u_n)u_n\,dx \le c + o(1)\|u_n\|_E + o(1). \tag{3.5}$$

Since $h(v_n)$ satisfies (2.4),

$$Q(h(v_n)) = \int_{\mathbb{R}^N} g(x,u_n)h(v_n)\,dx + \int_{\mathbb{R}^N} fh(v_n)\,dx.$$

Hence as (2.9) we deduce

$$\begin{aligned}
\|h(v_n)\|_E^2 \le &\Big(\int_{\mathbb{R}^N} |g(x,u_n)|^{\frac{p+1}{p}}\,dx\Big)^{\frac{p}{p+1}}\|h(v_n)\|_{L^{p+1}} \\
&+ C\Big(\int_{\mathbb{R}^N} |g(x,u_n)|^2\,dx\Big)^{\frac{1}{2}}\|h(v_n)\|_{L^2} + C\|f\|_{L^2}\|h(v_n)\|_E.
\end{aligned} \tag{3.6}$$

(3.5) and (3.6) imply $\|h(v_n)\|_E$ is uniformly bounded in n. In the same way, we infer from

$$\langle J'(u_n), v_n\rangle = o(1)\|v_n\|_E$$

that

$$\|v_n\|_E^2 \le C + C\int_{\mathbb{R}^N} g(x,u_n)u_n\,dx + o(1)\|v_n\|_E. \tag{3.7}$$

So $\|v_n\|_E$ is also uniformly bounded. Consequently,

$$\|u_n\|_E \le C.$$

We may assume

$$v_n \stackrel{n}{\rightharpoonup} v_0, \qquad w_n \stackrel{n}{\rightharpoonup} w_0 \ \text{ in } \ E$$

and $v_0 \in E^+$, $w_0 \in E^-$ and

$$u_n \stackrel{n}{\rightharpoonup} u_0 = v_0 + w_0 \ \text{ in } \ E, u_n \stackrel{n}{\longrightarrow} u_0 \quad \text{in} \quad L^r_{\mathrm{loc}}(\mathbb{R}^N), \quad 2 \le r < 2^*.$$

We remark that u_0 is a weak solution of problem (1.1). Therefore

$$\begin{aligned}&\int_{\mathbb{R}^N} [\nabla(u_n - u_0) \nabla \varphi + q(u_n - u_0)\varphi - \lambda(u_n - u_0)\varphi]\, dx \\ &- \int_{\mathbb{R}^N} [g(x, u_n) - g(x, u_0)]\varphi\, dx = o(1)\|\varphi\|_E, \ \forall\, \varphi \in E.\end{aligned}$$

Let $\varphi = v_n - v_0$, then

$$\|v_n - v_0\|_E^2 - \int_{\mathbb{R}^N} g(x, u_n)(v_n - v_0)\, dx - \int_{\mathbb{R}^N} g(x, u_0)(v_n - v_0)\, dx = o(1)\|v_n - v_0\|_E.$$

By Hölder's inequality and Lemma 2.1 again, we infer that

$$\|v_n - v_0\|_E \stackrel{n}{\longrightarrow} 0.$$

The proof is completed. □

Let

$$m = \inf_{v \in B_r} F(v),$$

where $B_r = \{v \in E^+ \ \mid \|v\|_E < r\}$ and r is determined in Lemma 3.1.

Proposition 3.4. *If $\|f\|_{L^2}$ is small, m is attained by some $v_1 \in E^+$, and $v_1 + h(v_1)$ is a solution of* (1.1).

PROOF: Again by the Ekeland's variational principle, we have a minimizing sequence $\{v_n\}$ satisfying

$$F(v_n) \stackrel{n}{\longrightarrow} m, \ F'(v_n) \stackrel{n}{\longrightarrow} 0 \ \text{ and } \ \|v_n\|_E \le r.$$

From Lemma 3.3 we know that there exists a subsequence of $\{v_n\}$ convergent strongly in E. Denote by v_1 the limit function, then $\|v_1\|_E \le r$. Lemma 3.1 implies $\|v_1\| < r$, so v_1 is a critical point of F. By Lemma 3.2, $v_1 + h(v_1)$ is a solution of (1.1). □

Next, we use the Mountain Pass Theorem to obtain the second solution.

Lemma 3.5. *There exists $v \in E^+$, $v \notin B_r(0)$ such that $F(v) < 0$.*

Proof: By assumptions (G1) and (G4), there exists a function $l(x) > 0$, $\forall\, x \in \mathbb{R}^N$ such that

$$G(x,t) \geq l(x)|t|^\beta$$

provided that $|t| \geq \sigma$ for some $\sigma > 0$. Choosing $v \in E^+$ and $\|v\|_E = 1$, we claim that

$$F(tv) < 0 \tag{3.8}$$

for $t > 0$ large.

Let $\{t_n\}$ be a sequence of positive numbers, $t_n \xrightarrow{n} \infty$. Denote $u_n = t_n v + h(t_n v)$, and $w_n = \dfrac{u_n}{\|u_n\|_E}$. We may assume that $w_n \overset{n}{\rightharpoonup} w = w^+ + w^-$ in E, where $w^\pm \in E^\pm$.

We distinguish two cases:

(i) $\dfrac{\|h(t_n v)\|_E}{t_n} \to +\infty$;

(ii) $\dfrac{\|h(t_n v)\|_E}{t_n} \to k \geq 0$, where k is a constant.

In the first case, by (G4) and Hölder's inequality, we deduce

$$\begin{aligned} F(t_n v) &= J(t_n v + h(t_n v)) \\ &\leq \frac{1}{2}\big[t_n^2\|v\|_E^2 - \|h(t_n v)\|_E^2\big] + C\|f\|_{L^2}\|t_n v + h(t_n v)\|_E \\ &\leq \frac{t_n^2}{2}\Big[\|v\|_E^2 - \frac{1}{t_n^2}\|h(t_n v)\|_E^2 + \frac{C}{t_n}\|f\|_{L^2}\|v\|_E + \frac{C}{t_n^2}\|f\|_{L^2}\|h(t_n v)\|_E\Big] \qquad (3.9) \\ &\leq \frac{t_n^2}{2}\Big[\|v\|_E^2 - \frac{1}{t_n^2}(1-\varepsilon)\|h(t_n v)\|_E^2 + C_\varepsilon\|f\|_{L^2}^2 + C\|f\|_{L^2}\|v\|_E\Big]. \end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small, we obtain

$$F(t_n v) \to -\infty$$

as $n \to \infty$.

In the second case, if $\|h(t_n v)\|_E / t_n \to k > 0$, then we may assume $h(t_n v)/t_n \overset{n}{\rightharpoonup} h_1$, it follows that $w = \dfrac{v + h_1}{(1+k^2)^{\frac{1}{2}}} \not\equiv 0$. In fact, were it not the case, we would have $v = -h_1$, it would yield

$$0 = Q(v, h_1) = Q(v, -v) = -\|v\|_E^2$$

a contradiction to the choice of v. By Lemma 2.1

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} l(x)|w_n|^\beta\, dx = \int_{\mathbb{R}^N} l(x)|w|^\beta\, dx.$$

The limit is positive.

For n large we have $\|u_n\|_E \geq t_n > 1$. Let $\omega_n = \{x \in \mathbb{R}^N : |t_n v(x) + h(t_n v(x))| \geq \sigma\}$. We estimate by (G2)

$$\int_{\mathbb{R}^N/\omega_n} G(x, t_n v + h(t_n v))\, dx \leq C$$

and

$$\int_{\mathbb{R}^N/\omega_n} l(x)\big|t_n v + h(t_n v)\big|^\beta\, dx \leq C,$$

where $C > 0$ is independent of n. Hence we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N} G(x, t_n v + h(t_n v))\, dx \\
&= \int_{\omega_n} G(x, t_n v + h(t_n v))\, dx + \int_{\mathbb{R}^N/\omega_n} G(x, t_n v + h(t_n v))\, dx \\
&\geq \int_{\omega_n} l(x)\, |t_n v + h(t_n v)|^\beta\, dx - C \\
&\geq \|u_n\|_E^\beta \int_{\mathbb{R}^N} l(x) \Big|\frac{t_n v + h(t_n v)}{\|u_n\|_E}\Big|^\beta\, dx - C_1 \\
&\geq t_n^\beta \Big(\int_{\mathbb{R}^N} l(x)|w|^\beta\, dx + o(1)\Big) - C_1.
\end{aligned} \tag{3.10}$$

It concludes by (3.10) that

$$\begin{aligned}
F(t_n v) \leq \frac{t_n^2}{2}\Big[\|v\|_E^2 - \frac{1}{t_n^2}(1-\varepsilon)\|h(t_n v)\|_E^2 + C_\varepsilon \|f\|_{L^2}^2 + C\|f\|_{L^2}\|v\|_E\Big] \\
- t_n^\beta\Big(\int_{\mathbb{R}^N} l(x)|w|^\beta\, dx + o(1)\Big) - C \leq 0
\end{aligned} \tag{3.11}$$

for n large.

If $\|h(t_n v)\|_E / t_n \to 0$, then $\|u_n\|_E / t_n \to 1$. By Sobolev embedding, we have $h(t_n v)/t_n \to 0$ a.e. in $\mathbb{R}^N$. It results

$$\int_{\mathbb{R}^N} l(x) \Big|\frac{t_n v + h(t_n v)}{\|u_n\|_E}\Big|^\beta\, dx \to \int_{\mathbb{R}^N} l(x)|v|^\beta\, dx > 0.$$

Then we may argue as before. The conclusion follows. □

Proof of Theorem A: By Lemma 3.5, there exists $e \in E^+$, $e \notin B_r$ such that $F(e) < 0$. Let

$$\Gamma = \{\gamma \in C([0,1], E^+) \mid \gamma(0) = v_1, \gamma(1) = e\},$$

where v_1 is the minimum point of m obtained in Proposition 3.4. Define

$$c = \inf_{\gamma\in\Gamma}\max_{v\in\gamma} F(v).$$

Lemma 3.3 and the Mountain Pass Theorem imply c is a critical value of F, and by Lemma 3.2, corresponding critical point v_2 gives second solution $v_2 + h(v_2)$ of (1.1). □

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References

[AP] Ambrosetti A., Prodi G., *On the inversion of some differentiable mappings with singularities between Banach spaces*, Ann. Mat. Pura Appl. **93** (1972), 231–246.

[AR] Ambrosetti A., Rabinowitz P., *Dual variational methods in critical point theory and application*, J. Funct. Anal. **14** (1973), 231–246.

[BJ] Buffoni B., Jeanjean L., *Minimax characterization of solutions for a semilinear elliptic equation with lack of compactness*, Ann. Inst. H. Poincaré Anal. Nonlineaire **10** (1993), no. 4, 377–404.

[BJS] Buffoni B., Jeanjean L., Stuart C.A., *Existence of a non-trivial solution to a strongly indefinite semilinear equation*, Proc. Amer. Math. Soc. **119** (1993), no. 1, 175–186.

[CY] Chabrowski J., Yang Jianfu, *Existence theorems for the Schrödinger equation involving a critical Sobolev exponent*, Z. Angew. Math. Phys. **49** (1998), 276–293.

[CZ] Cao D.M., Zhou H.S., *Multiple positive solutions of nonhomogeneous semilinear elliptic equations on $\mathbb{R}^N$*, Proc. Royal Soc. Edinburgh **126A** (1996), 443–463.

[E] Ekeland I., *On the variational principle*, J. Math. Anal. Appl. **47** (1974), 324–353.

[F] de Figueiredo D.G., *On superlinear elliptic problems with nonlinearities interacting only with higher eigenvalues*, Rocky Mount. J. Math. **18** (1988), no. 2, 287–303.

[FY] de Figueiredo D.G., Yang Jianfu, *Critical superlinear Ambrosetti-Prodi Problem*, Topol. Methods Nonlinear Anal. **14** (1999), 59–80.

[J] Jeanjean L., *Two positive solutions for a class nonhomogeneous elliptic equations*, Differential Integral Equations **10** (1997), 609–624.

[L] Liu Xiaochun, *Existence theorem of concave and convex effects for nonlinear Schrödinger equations*, J. Nanchang Univ. Nat. Sci. **22** (1998), no. 1, 31–38.

[PP] Pankov A.A., Pflüger K., *On a semilinear Schrödinger equation with periodic potential*, Nonlinear Anal. TMA **33** (1998), 593–609.

[R] Rabinowitz P., *Minimax methods in critical point theory with applications to differential equations*, AMS Conf. Ser. Math. 65 (1986).

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