Hu's Primal Algebra Theorem revisited

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Abstract. It is shown how Lawvere's one-to-one translation between Birkhoff's description of varieties and the categorical one (see [6]) turns Hu's theorem on varieties generated by a primal algebra (see [4], [5]) into a simple reformulation of the classical representation theorem of finite Boolean algebras as powerset algebras.

Keywords: Lawvere theory, equivalence between varieties, Hu's theorem, primal algebra, Post algebras

Classification: Primary 06D25, 06B20; Secondary 18C05

Hu's theorem ([4], [5]) characterizes the varieties (in Birkhoff's sense) equivalent to the variety Bool of Boolean algebras as those varieties which are generated by some primal algebra. The original proof made use of Stone duality; later proofs ([3], [7]) were of a purely algebraic nature, but certainly not straightforward. Since the variety Post_n of Post algebras of order n ($n \in \mathbb{N}$, $n \geq 2$) is generated by a primal n-element algebra (the n-chain) Post_n is equivalent to Bool — a fact which also can be proved directly (see e.g. [1]).

In this note we are going to show how this result — plus an explicit description of all the varieties equivalent to Bool — can be obtained in the most simple manner using instead of Birkhoff's description of varieties the categorical one due to Lawvere ([6]). Our analysis moreover shows that what is needed substantially to obtain these results (besides the appropriate language) is only the elementary fact that finite Boolean algebras are powersets or, more precisely, that the contravariant powerset functor is a dual equivalence between the categories of finite sets and finite Boolean algebras (we refer to this as "restricted Stone duality").

To make this a selfcontained paper we start recalling briefly the fundamentals of Lawvere's approach to varieties as far as they are needed here. As a first application we obtain descriptions of primality of an algebra and of the variety generated by an algebra in this language, which are obvious but seem not to have appeared in print yet.

1 Definitions. A (*Lawvere*) *theory* is a category $\mathbb{T}$ with countably many objects $T_0, T_1, T_2, \ldots$ and, for each $n \in \mathbb{N}$, a distinguished family $(\pi_i^n\colon T_n \to T_1)_{1\leq i\leq n}$ of morphisms making T_n an n-fold power of T_1.

A functor $\Phi\colon \mathbb{S} \to \mathbb{T}$ between theories is *a theory morphism* provided Φ preserves the distinguished product families.

A $\mathbb{T}$-*model* is a product-preserving functor from $\mathbb{T}$ into the category Set of sets and mappings. $\mathsf{Mod}\mathbb{T}$ denotes the category of all $\mathbb{T}$-models considered as a full subcategory of the category of all functors from $\mathbb{T}$ to Set.

2 Definitions. Given concrete categories[1] $(\mathcal{K}, \mathsf{U})$ and $(\mathcal{L}, \mathsf{V})$, a functor $\mathsf{F}\colon \mathcal{K} \to \mathcal{L}$ is called *concrete functor* provided $\mathsf{VF} = \mathsf{U}$[2].

Concrete categories $(\mathcal{K}, \mathsf{U})$ and $(\mathcal{L}, \mathsf{V})$ are called *concretely equivalent* (or *concretely isomorphic*) provided there exists a concrete functor $\mathsf{F}\colon (\mathcal{K}, \mathsf{U}) \to (\mathcal{L}, \mathsf{V})$ which is an equivalence (an isomorphism, respectively) as a functor.

3 Facts ([6], [2, Chapter 3])**.** The category $\mathsf{Mod}\mathbb{T}$ is concrete by means of the underlying functor "evaluation at T_1", i.e., the functor $\mathsf{ev}_{T_1}\colon \mathsf{Mod}\mathbb{T} \to \mathsf{Set}$ mapping a model G to $\mathsf{G}(T_1)$ and a natural transformation λ to its T_1-component λ_{T_1}.

The fundamental observation leading to the concepts introduced above is the following. For any variety $\mathcal{V}$ one can construct a theory $\mathsf{Th}\mathcal{V}$, called the *theory of* $\mathcal{V}$, as follows: objects of $\mathsf{Th}\mathcal{V}$ are the finite powers $(F\omega)^n$, $n \geq 0$, of the free $\mathcal{V}$-algebra $F\omega$ on a countable set of generators[3]; morphisms $(F\omega)^m \to F\omega$ are all m-ary term operations of $F\omega$ and morphisms $(F\omega)^m \to (F\omega)^n$ are all maps $t := \langle t_1, \ldots, t_n \rangle$ where the i-th component t_i of t is an m-ary term operation. Note that the theory just described is nothing but the clone of $F\omega$ extended by the maps $\langle t_1, \ldots, t_n \rangle$ and that, what often is called *clone composition*, becomes ordinary composition in this category.

Now any $\mathcal{V}$-algebra A determines a $\mathsf{Th}\mathcal{V}$–model, denoted by A, mapping $(F\omega)^n$ to A^n and an m-ary term (operation of $F\omega$) to its interpretation on A. Then a homomorphism $f\colon A \to B$ determines a natural transformation $\mathsf{f}\colon \mathsf{A} \to \mathsf{B}$ with $\mathsf{f}_{(F\omega)^n} = f^n = f \times \ldots \times f$. This construction yields a concrete equivalence between $\mathcal{V}$ and the category of models of $\mathsf{Th}\mathcal{V}$.

Conversely, for every Lawvere theory $\mathbb{T}$ there exists a variety $\mathcal{V}_{\mathbb{T}}$ — called the *variety determined by* $\mathbb{T}$ — such that $\mathsf{Mod}\mathbb{T}$ and $\mathcal{V}_{\mathbb{T}}$ are concretely equivalent. Moreover, the correspondence between varieties and Lawvere theories established above is essentially bijective (and functorial), i.e.,

1. every variety $\mathcal{V}$ is concretely isomorphic to the variety $\mathcal{V}_{\mathsf{Th}\mathcal{V}}$ determined by $\mathsf{Th}\mathcal{V}$;
2. every Lawvere theory $\mathbb{T}$ is isomorphic to the theory $\mathsf{Th}\mathcal{V}_{\mathbb{T}}$ of the variety $\mathcal{V}_{\mathbb{T}}$ determined by $\mathbb{T}$;
3. every theory morphism $\Phi\colon \mathbb{S} \to \mathbb{T}$ determines a functor $\Phi^*\colon \mathsf{Mod}\mathbb{T} \to \mathsf{Mod}\mathbb{S}$ given by $\Phi^*(\mathsf{H} \xrightarrow{\lambda} \mathsf{K}) = \mathsf{H}\Phi \xrightarrow{\lambda\Phi} \mathsf{K}\Phi$ which is concrete;
4. every concrete functor $\mathsf{F}\colon \mathsf{Mod}\mathbb{T} \to \mathsf{Mod}\mathbb{S}$ is $\mathsf{F} = \Phi^*$ for a unique theory morphism $\Phi\colon \mathbb{S} \to \mathbb{T}$.

[1] By a *concrete category* $(\mathcal{K}, \mathsf{U})$ is meant a category $\mathcal{K}$ together with a faithful functor $\mathsf{U}\colon \mathcal{K} \to \mathsf{Set}$; a variety can always be considered as a concrete category by means of its canonical forgetful functor.

[2] Concrete functors between varieties are often called *interpretations*.

[3] We do not distinguish notationally between an algebra and its underlying set.

4 Examples. Each algebra A in a variety $\mathcal{V}$ gives rise to two different theories as follows:

a) The *theory of* A, denoted by $\mathbb{A}$, is simply the image of $\mathsf{A}\colon \mathsf{Th}\mathcal{V} \to \mathsf{Set}$. Thus $\mathbb{A}$ is the clone of the algebra A together with the additional maps $A^n \xrightarrow{\langle t_i^A \rangle} A^m$ induced by families of $(t_1^A, \ldots, t_m^A)$ of n-ary derived operations. Note that for $A = F\omega$, $\mathbb{A} = \mathsf{Th}\mathcal{V}$.

b) The *theory generated by* A, denoted by $\mathsf{Th}_{\mathcal{V}}(A)$, is the dual of the full subcategory of $\mathcal{V}$ spanned by all finite copowers $m \cdot A$, $m \in \mathbb{N}$. Explicitly: $\mathsf{Th}_{\mathcal{V}}(A)$ has objects $T_i = i \cdot A$ (the i-fold copower of A in $\mathcal{V}$) and morphism sets $\hom_{\mathsf{Th}_{\mathcal{V}}(A)}(T_i, T_j) = \hom_{\mathcal{V}}(j \cdot A, i \cdot A)$ and chosen coproduct injections $\mu_i^n\colon A \longrightarrow n \cdot A$ $(i = 1, \ldots, n)$. Note that $\mathsf{Th}_{\mathcal{V}}(F1)$ is isomorphic to $\mathsf{Th}\mathcal{V}$.

For each algebra A the functor A factors, in the notation above, as

$$\mathsf{Th}\mathcal{V} \xrightarrow{\mathsf{A}} \mathsf{Set} = \mathsf{Th}\mathcal{V} \xrightarrow{\Phi_A} \mathbb{A} \xrightarrow{\mathsf{E}_A} \mathsf{Set}$$

where E_A is the embedding of the subcategory $\mathbb{A}$ into Set. Φ_A clearly is a theory-morphism which is surjective (on objects and morphisms). Note that in general $\mathbb{A}$ is not a full subcategory of Set. This being the case means that each map $A^n \to A$ is a derived operation of the algebra A. A is then called a *primal algebra* (provided A is finite with more than one element). Thus we have shown the first of the following propositions.

5 Proposition. *Let A be a finite algebra in $\mathcal{V}$ having more than one element. Then A is primal if and only if A is full.*

6 Proposition. *For any algebra A in a variety $\mathcal{V}$ the following hold:*

1. *$\mathbb{A}$ is the theory of $\mathsf{HSP}(A)$, the variety generated by A.*
2. *$\mathcal{V} = \mathsf{HSP}(A)$ if and only if A is faithful.*

PROOF: Statement 1 is equivalent to the following lemma which — in the language of universal algebra — states that the variety determined by $\mathbb{A}$ is the smallest subvariety of $\mathcal{V}$ containing A[4]. The second statement is an immediate consequence of the first one: $\Phi_A\colon \mathsf{Th}\mathcal{V} \to \mathbb{A}$ is an isomorphism iff A is a faithful model; thus, in this case $\mathcal{V}$ and $HSP(A)$ coincide by 1. and bijectivity of the correspondence between theories and varieties. □

7 Lemma. *Let $\mathbb{S}$ be a theory and $\mathsf{E}\colon \mathsf{Mod}\mathbb{S} \to \mathsf{Mod}\mathbb{T}$ be a full concrete embedding. Assume that the $\mathbb{T}$-model A is $\mathsf{E}\,(\mathsf{A}')$ for some $\mathbb{S}$-model A'. Then there exists a full concrete embedding $\mathsf{Mod}\mathbb{A} \hookrightarrow \mathsf{Mod}\mathbb{S}$.*

PROOF: Let E be Φ^* for $\Phi\colon \mathbb{T} \longrightarrow \mathbb{S}$. Then $\mathsf{A} = \mathsf{A}' \circ \Phi$ and therefore $\Phi_A = \Phi_{A'} \cdot \Phi\colon \mathbb{T} \longrightarrow \mathbb{A}$. Then $\Phi_{A'}$ is surjective since Φ_A is. Now apply the following lemma, the proof of which is an easy exercise. □

[4] A somewhat informal way of proving this is the following: an algebra B is in $HSP(A)$ iff B satisfies all equations which are valid in A. The $\mathbb{T}$-model B "is" an $\mathbb{A}$-model iff B factors over Φ_A iff the algebra B satisfies all equations of A.

8 Lemma. *The functor* $\Phi^*\colon \mathsf{Mod}\mathbb{T} \to \mathsf{Mod}\mathbb{S}$ *is a full embedding for every surjective theory morphism* $\Phi\colon \mathbb{S} \to \mathbb{T}$.

It is well known that varieties might be equivalent as categories without being concretely equivalent (in the latter case they would in fact be concretely isomorphic) as, e.g., the varieties of left modules over Morita-equivalent rings or the varieties of Post algebras of the various fixed (finite) orders. In order to describe the varieties $\mathcal{W}$ equivalent to a given variety $\mathcal{V}$ the following notion is crucial:

9 Definition. An algebra G in a variety $\mathcal{V}$ is called a *varietal generator*[5], provided G is a retract of some finitely generated free $\mathcal{V}$–algebra Fn and the free algebra $F1$ on one generator is a retract of some finite copower of G.

10 Fact ([6], [8])**.** Let $\mathcal{V}$ be a variety with theory $\mathbb{T}$. A variety $\mathcal{W}$ with theory $\mathbb{S}$ is equivalent to $\mathcal{V}$ iff $\mathbb{S}$ is (isomorphic to) some theory $\mathsf{Th}_{\mathcal{V}}(G)$ where G is a varietal generator in $\mathcal{V}$.

11 Example. Let $\mathcal{V}$ be the variety Bool of Boolean algebras. The varietal generators of Bool are (up to isomorphism) precisely the powerset algebras $\mathbf{2}^n$, $n \in \mathbb{N}$, $n \geq 2$. This is clear since each retract of a finitely generated free Boolean algebra (which is finite) is finite, thus $\mathbf{2}^n$ for some $n \in \mathbb{N}$ due to restricted Stone duality. Moreover $\mathbf{2}^2$ is a retract of $\mathbf{2}^n$ iff $n \geq 2$ (again by restricted Stone duality).

It is easy to describe the theory generated by $\mathbf{2}^n$: the contravariant powerset functor, restricted to the full subcategory $\mathbb{T}_n$ of Set spanned by the finite powers of the n-element set $\{0, 1, \dots, n-1\}$ provides a theory-isomorphism $\mathbb{T}_n \simeq \mathsf{Th}_{\mathsf{Bool}}(\mathbf{2}^n)$.

As a consequence of the above a proof of the following sharpening of Hu's theorem ([4] [5]) becomes nearly trivial, making thereby clear in addition that this theorem is essentially — up to the well known categorical result stated as Fact 10 above — a reformulation of *restricted* Stone duality.

12 Theorem. *The following are equivalent for a variety* $\mathcal{V}$*:*

(i) $\mathcal{V}$ *is equivalent to* Bool, *the variety of Boolean algebras;*

(ii) $\mathcal{V}$ *is generated by a primal algebra;*

(iii) $\mathsf{Th}\mathcal{V} \simeq \mathbb{T}_n$ *for some* $n \in \mathbb{N}$, $n \geq 2$.

PROOF: By Fact 10 and Example 11 $\mathcal{V}$ is equivalent to Bool iff $\mathsf{Th}\mathcal{V} \simeq \mathbb{T}_n$ for some $n \geq 2$.

By Propositions 5 and 6 one gets: $\mathcal{V} = \mathsf{HSP}(A)$ with an n-element primal algebra A iff A is a full and faithful $\mathsf{Th}\mathcal{V}$-model, iff $\mathsf{Th}\mathcal{V} \simeq \mathbb{T}_n$. (Note that $\mathbb{T}_n$ has a full and faithful n-element model trivially: its embedding into Set). □

[5]In categorical terms, this is equivalent to saying that G is an extremally projective, finitely presentable extremal generator (see e.g. [8]).

13 Remark. Since the correspondence between varieties and Lawvere theories is essentially bijective the above theorem shows in particular that, for each $n \in \mathbb{N}$, $n \geq 2$, there is (up to concrete isomorphism) precisely one variety which is generated by an n-element primal algebra, namely the variety $\mathcal{V}_{\mathbb{T}_n}$ determined by the theory $\mathbb{T}_n$, and that any variety equivalent to Bool has to be one of those (again up to concrete isomorphism). Now the variety $\mathcal{V}_{\mathbb{T}_n}$ is easily identified as the variety Post_n of Post algebras of order n since it is well known (see e.g. [1]) that the latter is equivalent to Bool and generated by an n-element primal algebra. For a more categorical argument see [8].

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(*Received* February 14, 2000)