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Extensions of topological and semitopological groups and the product operation

A.V. Arhangel'skii, M. Hušek

Abstract. The main results concern commutativity of Hewitt-Nachbin realcompactification or Dieudonné completion with products of topological groups. It is shown that for every topological group G that is not Dieudonné complete one can find a Dieudonné complete group H such that the Dieudonné completion of $G \times H$ is not a topological group containing $G \times H$ as a subgroup. Using Korovin's construction of G_δ-dense orbits, we present some examples showing that some results on topological groups are not valid for semitopological groups.

Keywords: topological group, Dieudonné completion, PT-group, realcompactness, Moscow space, C-embedding, product

Classification: 22A05, 54H11, 54D35, 54D60

§0. Introduction

Although many of our general results are valid for more general spaces, we shall assume that all the spaces under consideration are Tychonoff.

The following notion was introduced in [Ar1]. A space X is called *Moscow*, if, for each open subset U of X, the closure of U in X is the union of a family of G_δ-subsets of X, that is, for each $x \in \bar{U}$ there exists a G_δ-subset P of X such that $x \in P \subset \bar{U}$. A topological group is called Moscow if it is a Moscow space. The techniques based on the notion of Moscow space played a vital role in the recent solution of the next problem posed by V.G. Pestov and M.G. Tkačenko in [PT]. They asked the following question, which we call below *the PT-problem*. Let G be a topological group, and μG the Dieudonné completion of the space G. Can the operations on G be extended to μG in such a way that μG becomes a topological group, containing G as a topological subgroup? That means, is μG a topological group such that the reflection map $G \to \mu G$ is a homomorphism?

Recall that the Dieudonné completion μG of G is the completion of G with respect to the maximal uniformity on G compatible with the topology of G. In other words, it is a reflection of G in the smallest productive and closed hereditary class containing all metrizable spaces. It is well known that the Dieudonné completion of a topological space X is always contained in the Hewitt-Nachbin

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completion υX of X (the smallest productive and closed hereditary class containing the space of reals). Clearly, μX is the smallest Dieudonné complete subspace of υX containing X.

Moreover, if there are no Ulam-measurable cardinals, then υX and μX coincide by Shirota theorem ([Sh]). Therefore, the next question, also belonging to Pestov and Tkačenko (see [Tk]), is almost equivalent to the above question. Let G be a topological group, and let υG be the Hewitt-Nachbin completion of the space G. Can the operations on G be extended to υG in such a way that υG becomes a topological group, containing G as a topological subgroup?

A counterexample to the PT-problem was described in [Ar2]. We call a topological group G *a PT-group* if μG is a topological group, containing G as a topological subgroup. It was shown in [Ar2] that every Moscow group is a PT-group.

One of the main questions we consider in this note is as follows: when the product of PT-groups is a PT-group? We also consider some open questions concerning various natural extensions of topological groups G.

In what follows ρG is the Rajkov completion of G and $\rho_\omega G$ is the G_δ-closure of G in ρG (see [RD]), that is, the set of all points in ρG which can not be separated from G by a G_δ subset of ρG. Clearly, $\rho_\omega G$ is also a topological subgroup of ρG. The problems we deal with deeply involve the notion of C-embedding and are all closely related to the PT-problem.

Some results we formulate for more general situations. Recall that a group endowed with a topology is called

a *right topological group* if the binary group operation is continuous in the left variable, when the right one is fixed;

a *semitopological* group if the binary group operation is separately continuous;

a *paratopological* group if the binary group operation is jointly continuous;

a *quasitopological* group if the binary group operation is separately continuous and the inversion is continuous;

a *topological* group if the binary group operation is jointly continuous and the inversion is continuous.

The first two concepts and terms are used also for the case when G is a semigroup. As for the third concept, a semigroup having its binary operation continuous is called a topological semigroup. Although it is not always needed, we shall assume that semigroups have units and homomorphisms preserve units. Note that, unlike in right topological semigroups, the translations in right topological groups are homeomorphisms (and the spaces are homogeneous).

Nature of some of the next results for μX is categorical and we shall formulate them for a general reflection cX of a topological space X (in fact, c need not be defined on all the Tychonoff spaces, it suffices it is defined on the spaces under consideration). In most situations, X is densely embedded into cX (like βX or υX or μX for Tychonoff spaces X). There are also some natural situations where the reflection map sends X onto a dense subspace of cX but the map is not one-to-one (like zerodimensional compactifications). Other situations do not happen too often (like Herrlich's example of reflections into powers of a strongly rigid space).

Take a family $\{X_i\}_{i \in I}$ of nonvoid topological spaces. Denote by p the canonical continuous map $c(\Pi X_i) \to \Pi cX_i$ that is a reflective extension of the product of reflection maps $\Pi X_i \to \Pi cX_i$. It is also generated (as a mapping into a product) by the extensions $c(\mathrm{pr}_j) : c(\Pi X_i) \to cX_j$ of $\mathrm{pr}_j : \Pi X_i \to X_j$ for $j \in I$. Since X_j is a retract of ΠX_i, the reflection cX_j is a retract of $c(\Pi X_i)$, where the retraction is the composition of p and of the projection onto cX_j. In case when spaces under consideration are densely embedded in their c-reflections, the restriction of p to the closure in $c(\Pi X_i)$ of $X_j \times \{a\}$ (where $a \in \Pi_{i \in I \setminus \{j\}} X_i$) is a homeomorphism onto $cX_j \times \{a\}$.

§1. Products of extensions

If we say for a group (or semigroup) X endowed with a topology that cX is a group (or semigroup, resp.), we also assume that the corresponding reflection map $X \to cX$ is a homomorphism. Clearly, if $c(\Pi X_i)$ is a semitopological group (or semigroup), then cX_i are groups (or semigroups, resp.), too (using extensions of the embeddings $X_j \to X_j \times \{e\} \subset \Pi X_i$ and then the above retractions). Moreover, then the map p is a homomorphism; we can show it for dense embedding reflections (the general case has the same idea but is formally more complicated): the continuous maps $p(x \cdot y)$ and $p(x) \cdot p(y)$ for a fixed $x \in \Pi X_i$ coincide on the dense subset ΠX_i, so they are equal, and now repeat the same procedure for the same maps for fixed $y \in c(\Pi X_i)$.

We conjecture that the situation above does not hold for right topological groups although the next result would suggest its more general application than for semitopological groups.

Lemma 1.1. *Let G_1 and G_2 be right topological groups, and let H be a dense subspace of G_1. Suppose further that $f : G_1 \to G_2$ is a continuous homomorphism of G_1 into G_2 such that the restriction of f to H is a topological embedding. Then f is a topological embedding of G_1 into G_2.*

PROOF: Assume the contrary. Then there exist $x \in G_1$ and $A \subset G_1$ such that x is not in the closure of A, while $y = f(x)$ is in the closure of $B = f(A)$. Since the natural translations in G_1 and G_2 commute with the homomorphism f, we may assume that x belongs to H. Since the space G_1 is regular, we may also assume that $A \subset H$.

However, now the contradiction is obvious: since $f|H$ is a homeomorphism of H onto $f(H)$, and $x \in H$, $A \subset H$, it follows that $f(x)$ is not in the closure of $f(A)$. $\square$

As a direct consequence of the above lemma, we get the following easy but important result. Since we do not know whether p is a homomorphism for right topological groups, we have to formulate it for semitopological groups. See Theorem 2.8, where also the converse statement is proved for $c = \mu$.

Theorem 1.2. *Suppose that c is a dense quotient reflection. Let X_i, where $i \in I$, be semitopological groups. If $c(\Pi\{X_i : i \in I\})$ is a semitopological group, then $p : c(\Pi\{X_i : i \in I\}) \to \Pi\{c(X_i) : i \in I\}$ is a topological and algebraic embedding.*

PROOF: In case c is a dense embedding reflection, it suffices to apply Lemma 1.1 with $c(\Pi\{X_i : i \in I\})$, $\Pi\{X_i : i \in I\}$, $\Pi\{cX_i : i \in I\}$ and p in the roles of G_1, H, G_2 and f, respectively.

In case c is a dense quotient reflection, we may use Corollary 2 of Theorem 3 from [HdV] to see that the restriction of p to the image of ΠX_i under c is a homeomorphism onto the image of ΠX_i into $\Pi c X_i$. Now, we can use Lemma 1.1. □

It is clear that for $c = \beta$ the map p from Theorem 1.2 is always surjective (also for other compact reflections, like zerodimensional ones). We do not know whether p is surjective for other reflections, like μ. We remind that [Ar2, Theorem 2.9] gives the affirmative answer in case ΠX_i is a Moscow group (and thus, e.g., if every group X_i is totally bounded, or has precaliber ω_1, or is κ-metrizable, or if the product ΠX_i has countable Souslin number, or countable tightness) — it is not clear whether total boundedness (i.e., ω-boundedness) can be replaced by ω_1-boundedness.

Question 1. Can it happen that p from Theorem 1.2 is not onto $\Pi c X_i$ (especially for $c = \mu$ or $c = \upsilon$)?

We know the answer (in the negative) to Question 1 for finite products. In fact, we can prove a little more if we use methods from [HdV], where one can find general theorems on commutation of reflections and products, applicable mainly for topological spaces bearing some algebraic structure. Suppose c is an epireflection in some class $\mathcal{C}$ of Tychonoff spaces. Denote by $\mathcal{C}'$, say, all the topological groups the underlying topology of which belong to $\mathcal{C}$. If one takes now a topological group X and makes the reflections cX in $\mathcal{C}$ and gX in $\mathcal{C}'$, then our expression that cX is a topological group means exactly that $cX = gX$. So, if $c(\Pi X_i)$ is a topological group, then $c(\Pi X_i) = g(\Pi X_i)$. If we know that $g(\Pi X_i) = \Pi g X_i$, then we know that $c(\Pi X_i) = \Pi c X_i$. Since the procedures in [HdV] are more general than needed here (and more complicated) we shall repeat simplified methods from [HdV] to get what we need.

The following lemma is basic for our consideration in case of finite products.

Lemma 1.3. *If $c(\Pi X_i)$ is a semitopological semigroup and I is finite, then the map p is a bijection.*

PROOF: Suppose $|I| < \omega$. There are canonical continuous homomorphisms $h_i : cX_i \to c(\Pi X_i)$ (coretractions of p). The map $h : \Pi c X_i \to c(\Pi X_i)$ that assigns to $\{x_i\}$ the semigroup-product of $h_i(x_i)$'s, is a coretraction of p (noncontinuous, in general). Thus p must be an onto map.

Moreover, hp is the identity map on $c(\Pi X_i)$. To prove that, it suffices to show the equality $hpc = c$ for the reflection map $c : \Pi X_i \to c(\Pi X_i)$. We shall prove it for two factors X, Y. If we denote by $c_X : X \to cX$ and $c_Y : Y \to cY$ the corresponding reflection maps, then $hpc(x, y) = h(c_X(x), c_Y(y)) = hc_X(x) \cdot hc_Y(y) = c(x, e) \cdot c(e, y) = c(x, y)$, and we are ready. It means that p is one-to-one. $\square$

The map h need not be continuous; it is continuous, when $c\Pi X_i$ is a topological semigroup. Then p is a homeomorphism onto ΠcX_i, which we express by the equality $c(\Pi X_i) = \Pi cX_i$. In fact, in case c is a quotient-dense reflection, Theorem 1.2 and Lemma 1.3 (the part that p is onto) imply the equality $c(\Pi X_i) = \Pi cX_i$ provided $c(\Pi X_i)$ is a semitopological group.

Clearly, the map p is one-to-one in case when c is a monoreflection and I is arbitrary.

So, we have got the following result.

Theorem 1.4. *Let I be a finite set. The equality $c(\Pi X_i) = \Pi cX_i$ holds in either of the next two cases:*
1. *X_i and $c(\Pi X_i)$ are semitopological groups and c is a quotient-dense reflection;*
2. *X_i and $c(\Pi X_i)$ are topological semigroups.*

Question 2. Is $c(\Pi X_i) = \Pi cX_i$ when X_i and $c(\Pi X_i)$ are semitopological semigroups or semitopological groups (without any condition on c)?

Corollary 1.5. *Let X be a topological group (or semigroup). Then $c(X \times X)$ is a topological group (or semigroup, resp.) iff $c(X \times X) = cX \times cX$.*

When we take for c the Čech-Stone compactification β, Corollary 1.5 gives the known result that (for a topological group X) $\beta(X \times X)$ is a topological group iff X is pseudocompact.

When we take for c the Hewitt-Nachbin realcompactification υ, we get for a topological group X that if $\upsilon(X \times X) \neq \upsilon X \times \upsilon X$ then either υX is not a topological group or $\upsilon(X \times X)$ is not a topological group.

It is well known that for $c = \beta$ and G a topological group it cannot happen that βG is a topological group but $\beta(G \times G)$ is not. We do not know whether the same is valid for other c, especially for $c = \mu$ or $c = \upsilon$:

Question 3. Can it happen that G is a PT-group but $G \times G$ is not a PT-group?

Note that in [Ar2] two different Moscow groups (thus PT-groups) were presented such that their product is not a PT-group. The next question is a special version of Question 3.

Question 4. Is the square of a Moscow group a PT-group? A Moscow group?

Using Theorem 1.4, one can now construct many examples of topological groups reflections of which are not topological groups. For example, it follows easily from Theorem 4 in [Hu] that

Theorem 1.6. *For every topological group G that is not Dieudonné complete there exists a Dieudonné complete topological group H such that $G \times H$ is not a PT-group.*

PROOF: Take a topological group G that is not Dieudonné complete and take some point $\eta \in \mu G \setminus G$. Find an open cover $\mathcal{A}$ of G such that η does not belong to the closures (in μG) of members of $\mathcal{A}$. Then the space $C_{\mathcal{A}}(G)$ of (bounded) continuous real-valued functions on G, endowed with the topology of uniform convergence on members of $\mathcal{A}$, is Dieudonné complete; denote that space by H. The equality $\mu(G \times H) = \mu G \times \mu H$ does not hold, because the evaluation *eval* $: G \times H \to R$ (which is continuous) does not extend continuously to $\{\eta\} \times H$. $\square$

One can formulate a similar theorem for realcompactness instead of Dieudonné completeness: one must take G of nonmeasurable cardinality or omit the assumption that H is realcompact.

In the next section, we explain a topological approach how to get Theorem 1.4 and its improvement in case $c = \mu$.

S 2. Products and minimal extensions

A topological property $\mathcal{P}$ will be called *invariant under intersections* if for every family γ of subspaces of a topological space X such that every $Y \in \gamma$ has $\mathcal{P}$, the subspace $Z = \cap\gamma$ also has the property $\mathcal{P}$.

The next statement is a generalization of Proposition 1.4 in [Ar2]. For the sake of completeness, we present the proof of it, though this proof is an easy adaptation of the proof of Proposition 1.4 in [Ar2].

Proposition 2.1. *Let H be a subgroup of a quasitopological group G, and $\mathcal{P}$ a topological property invariant under intersections. Let $\gamma_{\mathcal{P}}$ be the family of all subspaces X of G such that X has the property $\mathcal{P}$ and $H \subset X$. Then either $\gamma_{\mathcal{P}}$ is empty, or there exists the smallest element M in $\gamma_{\mathcal{P}}$, and M is a subgroup of G (containing H).*

PROOF: Assume that $\gamma_{\mathcal{P}}$ is not empty, and let M be the intersection of the family $\gamma_{\mathcal{P}}$. Clearly, $H \subset M$. Since $\mathcal{P}$ is invariant under intersections, M also has the property $\mathcal{P}$. Therefore, $M \in \gamma_{\mathcal{P}}$, and M is the smallest element of $\gamma_{\mathcal{P}}$.

It remains to show that M is a subgroup of G. Note that M^{-1} is homeomorphic to M; therefore, M^{-1} also has the property $\mathcal{P}$. Since $H \subset M^{-1} \subset G$, it follows that $M^{-1} \in \gamma_{\mathcal{P}}$ and, therefore, $M \subset M^{-1}$. Hence, $M^{-1} \subset (M^{-1})^{-1} = M$ and, finally, $M = M^{-1}$.

For every $a \in H$ we have: $H \subset aH \subset aM \subset aG = G$, which implies that $M \subset aM$, since aM is homeomorphic to M, and therefore, has the property $\mathcal{P}$. It follows that $a^{-1}M \subset M$. Since $H = H^{-1}$, this implies that $aM \subset M$, for each $a \in H$. Since $M = M^{-1}$, we also have $aM^{-1} \subset M$, for each $a \in H$. Now take any $b \in M$. Let us show that $H \subset Mb$. Indeed, take any $a \in H$. Then $ab^{-1} \in M$, that is, $ab^{-1} = c$, for some $c \in M$. It follows that $a = cb \in Mb$.

Hence, $H \subset Mb$. Since Mb is homeomorphic to M, it follows that Mb is in $\gamma_{\mathcal{P}}$ and $M \subset Mb$. Hence, $Mb^{-1} \subset M$. Since $M = M^{-1}$, it follows that $Mb \subset M$, for each $b \in M$. Now it is clear that M is closed under multiplication. Hence, M is a subgroup of G. $\square$

A space X is called *a minimal Dieudonné extension* of Y if Y is dense in X, X is Dieudonné complete, and every Dieudonné complete subspace of X containing Y coincides with X ([Ar2]).

Hewitt-Nachbin completeness and Dieudonné completeness (as any reflection) are both invariant under intersections. They are also closed hereditary. Therefore, Proposition 2.1 implies the next result:

Proposition 2.2. *If H is a subgroup of a quasitopological group G, and there exists a Dieudonné complete subspace X of G such that $H \subset X$, then there exists a subgroup M of G such that the space M is a minimal Dieudonné extension of H and $M \subset X$.*

A similar statement holds for Hewitt-Nachbin completeness.

It is obvious that the Dieudonné completion of X is a minimal Dieudonné extension of X (in which X is C-embedded). The next statement is a part of the folklore (see [Ar2] and [En]):

Proposition 2.3. *If a space X is a minimal Dieudonné extension of a subspace Y, and Y is C-embedded in X, then $X = \mu Y$, that is, X is the Dieudonné completion of Y.*

Theorem 2.4. *Let H be a subgroup of a quasitopological group G, and X a subspace of G such that X is the Dieudonné completion of the space H. Then X is a subgroup of G.*

Proof: By Proposition 2.2, there is a subgroup M of G contained in X such that $H \subset M$ and the space M is Dieudonné complete. However, X is a minimal Dieudonné extension of the space H. Therefore, $M = X$, and X is a subgroup of G. $\square$

Corollary 2.5. *Suppose G is a topological group. Then μG is a topological group if and only if there exists a subspace X of the Rajkov completion ρG of G such that $G \subset X$ and X is the Dieudonné completion of G. That is, G is a PT-group if and only if ρG naturally contains the Dieudonné completion of G.*

The next result is a version of Corollary 2.5 for the Hewitt-Nachbin completion of a topological group. The proof of it is similar to the proof of Corollary 2.5.

Corollary 2.6. *Suppose G is a topological group and X a subspace of the Rajkov completion ρG of G such that $G \subset X$ and X is the Hewitt-Nachbin completion of G. Then X is a subgroup of $\rho_\omega G$.*

We now give a criterion for the product of two topological groups to be a PT-group.

Theorem 2.7. *The product $G \times H$ of topological groups G and H is a PT-group if and only if G and H are PT-groups and the formula $\mu(G \times H) = \mu G \times \mu H$ holds.*

PROOF: *Sufficiency.* If the formula $\mu(G \times H) = \mu G \times \mu H$ holds, then $\mu(G \times H)$ is a topological group, since μG and μH are topological groups. Hence, $G \times H$ is a PT-group.

Necessity. Suppose $G \times H$ is a PT-group. Then G and H are PT-groups. It remains to show that the formula $\mu(G \times H) = \mu G \times \mu H$ holds. Since $\mu(G \times H)$ is a topological group, containing $G \times H$ as a dense subgroup, $\mu(G \times H)$ can be represented as a topological subgroup of the Rajkov completion $\rho(G \times H)$ of the group $G \times H$. Similarly, since $G \times H$ is a dense subgroup of the topological group $\mu G \times \mu H$, the group $\mu G \times \mu H$ can be also represented as a topological subgroup of $\rho(G \times H)$. Both spaces $\mu(G \times H)$ and $\mu G \times \mu H$ are minimal Dieudonné extensions of the space $G \times H$, by Propositions 2.1 and 2.2 in [Ar2]. Since the intersection of $\mu(G \times H)$ and $\mu G \times \mu H$ (as subsets of $\rho(G \times H)$) is again a Dieudonné extension of $(G \times H)$, it follows from the minimality of $\mu(G \times H)$ and $\mu G \times \mu H$ that they coincide. Thus, $\mu(G \times H) = \mu G \times \mu H$. □

Theorem 2.7 obviously generalizes to finite products of PT-groups. However, for the product of arbitrary family of topological groups the corresponding criterion takes a slightly different form.

Theorem 2.8. *Let $\mathcal{F} = \{G_i : i \in I\}$ be a family of topological groups and $G = \Pi\{G_i : i \in I\}$ be their topological product. Then G is a PT-group if and only if G_i is a PT-group, for each $i \in I$, and the formula*

(1) $\mu(\Pi\{G_i : i \in I\}) \subset \Pi\{\mu(G_i) : i \in I\}$

holds.

PROOF: *Sufficiency.* If the formula (1) holds, and G_i is a PT-group, for each $i \in I$, then $\Pi\{\mu(G_i) : i \in I\}$ is a topological group, containing the Dieudonné completion μG of the topological group G. Hence, by Theorem 2.4, G is a PT-group.

Necessity. Suppose G is a PT-group. Then, clearly, G_i is a PT-group, for each $i \in I$. It remains to show that the formula (1) holds.

Since G is a PT-group, μG can be represented as a topological subgroup of the Rajkov completion ρG of G. Similarly, the group $\Pi\{\mu(G_i) : i \in I\}$ can be also represented as a topological subgroup of ρG. The space μG is a minimal Dieudonné extension of the space G. Since the intersection of μG and $\Pi\{\mu(G_i) : i \in I\}$ (as subsets of ρG) is again a Dieudonné extension of G, it follows from minimality of μG that μG is contained in $\Pi\{\mu(G_i) : i \in I\}$. Thus, formula (1) holds. □

§3. Some examples

In [Ko], Korovin constructed for a given topological space X and a group G a topology on G such that G became a semitopological group embedded in the power X^G in such a way that its projections onto countable subpowers were surjections. The only conditions are $|G| = |G|^\omega \geq |X|^\omega$.

We shall now use Korovin's construction for obtaining some interesting examples. Let $(G, +)$ be an Abelian group and Z be a topological space. If $\varphi : G \times Z \to Z$ is an action (i.e., $\varphi(e, z) = z, \varphi(a, \varphi(b, z)) = \varphi(a + b, z)$ for each $a, b \in G$ and $z \in Z$, and $\varphi(a, -) : Z \to Z$ is a continuous map for every $a \in G$), then every orbit of φ forms a semitopological group (with the topology inherited from Z). Recall that the orbit of some $z_0 \in Z$ is the subset $O(z_0) = \{\varphi(a, z_0) : a \in G\}$ of Z. This set has a group structure as a factor group of G along the subgroup $\{a \in G : \varphi(a, z_0) = z_0\}$ (defining $\varphi(a, z_0) + \varphi(b, z_0) = \varphi(a + b, z_0)$). The group $O(z_0)$ is commutative and if $\{\varphi(a_i, z_0)\}$ converges to $\varphi(a, z_0)$ in Z, then for any $b \in G$ the net $\{\varphi(b, \varphi(a_i, z_0))\}$ converges to $\{\varphi(b, \varphi(a, z_0))\}$ (i.e., $\{\varphi(b+a_i, z_0)\}$ converges to $\{\varphi(b+a, z_0)\}$) which means that the binary operation on $O(z_0)$ is separately continuous.

Take now for Z the power X^G of some topological space X. For the action $\varphi : G \times X^G \to X^G$ we take the shift $\varphi(a, f)(b) = f(a + b)$. The orbit of f_0 is isomorphic to the factor group of G along its subgroup of periods of f_0 (thus in case f_0 has no nontrivial period, its orbit is isomorphic to G).

Korovin in [Ko] showed that if $|G| = |G|^\omega \geq |X|^\omega$ then one can find such an $f_0 : G \to X$ that the projections of its orbit into all countable subpowers are surjections. We call such orbits *Korovin's orbits*. It is possible to repeat his procedure for an infinite regular cardinal κ and for groups G and topological spaces X satisfying $|G| = |G|^{<\kappa} \geq |X|^{<\kappa}$. For the resulting orbit, all its projections into X^λ, where $\lambda < \kappa$ are surjections. If, for instance, we have $\kappa = \omega$, then we assume that $|G|$ is infinite and not smaller than $|X|$. So, if X is at most countable, we may take G countable. But for the case $\kappa = \omega_1$ that is used mostly, the group G must have cardinality at least 2^ω.

The constructed semitopological group will be denoted by $K(X^G, \kappa)$ (or $K(X^G)$ if $\kappa = \omega_1$). We shall always assume that $|G| \geq \omega, |X| > 1$ and κ is a regular infinite cardinal. In this case, the maps in Korovin's orbit have no nontrivial period and, consequently, the orbit is algebraically isomorphic to G (indeed, for every $a \in G$ there exists some $b \in G$ such that $f_0(a + b)$ and $f_0(b)$ have different values). In case the group G is of order 2, its inversion is identity and the constructed orbit is a quasitopological group.

If every countable power of X is pseudo-ω_1-compact (i.e., every discrete collection of open sets is at most countable) and $\kappa > \omega$, then every continuous mapping on $K(X^G, \kappa)$ into a metrizable space depends on countably many coordinates and, thus, can be continuously extended onto the whole power X^G (see [CN]). Since $K(X^G, \kappa)$ is G_δ-dense in X^G, every continuous map from $K(X^G, \kappa)$ into a metrizable space can be continuously extended to X^G provided X is metrizable, [HP] (in fact, a paracompact p-space suffices). Thus we have the following result, the first assertion of it was used by Korovin in [Ko]. We shall formulate it for $\kappa = \omega_1$ but it is true for any regular uncountable κ.

Proposition 3.1. 1. *If X is compact, then $\beta(K(X^G)) = X^G$.*

2. *If X is Dieudonné complete and X^ω is pseudo-ω_1-compact, then $\mu K(X^G) = X^G$.*
3. *If X is metrizable, then $\mu(K(X^G)) = X^G$.*

The part 1 can be generalized for κ-compact spaces in the sense of Herrlich.

Taking for X various spaces, we can get interesting examples. Korovin used a nondyadic compact space X and obtained a pseudocompact semitopological group such that its Čech-Stone compactification is not a topological group (the main purpose was to find Y such that Y^ω is pseudocompact and some continuous image of Y in $C_p(Y)$ is not relatively compact in $C_p(Y)$ — Y equals to $K(X^G)$). Reznichenko noticed in [Re] that the construction (when using a group of order 2) gave an example of a pseudocompact quasitopological group that is not a topological group.

Example 1. *There exists a pseudocompact quasitopological group the Dieudonné completion of which is not homogeneous.*

Take for X a metrizable space having a point with a clopen base (e.g., an isolated point) and a point not having a clopen base. Then X^G has also such two points and no homeomorphism can assign one of those points to the other. We may take for G a countable group of order 2 and then $K(X^G)$ is the requested quasitopological group. □

In Example 1 it may happen that characters of all points of X^G are the same. We shall now show that there is an X for which the characters of points in X^G do not coincide.

Example 2. *There exists a pseudocompact quasitopological group the Dieudonné completion of which has points of different characters.*

Take for X a space $C_p(Y) \cup \{w\}$ for a topological space Y having the following properties: Y is separable and has the cardinality 2^{2^ω} (e.g., $Y = \mathbb{R}^{2^\omega}$). Then $C_p(Y)$ is realcompact, has cardinality 2^ω and character bigger than 2^ω. The point w does not belong to C_pY and is isolated in X. The space X^ω has ccc.

Take for G again a group of order 2 and cardinality 2^ω. Then the quasitopological group $G(X)$ has ccc and, thus, it is pseudo-ω_1-compact. Since X is realcompact, the power X^G is a Hewitt-Nachbin realcompactification (and a Dieudonné completion) of $K(X^G)$. The space X^G contains points of character 2^ω and points of character bigger than 2^ω.

The corresponding situation for nonpseudocompact topological groups is not yet clear and we repeat here the following question.

Question 5. Does there exist a topological group G such that μG is not homogeneous (or even such that the characters of points in μG do not coincide)?

In Example 2, the character of $K(X^G)$ coincides with that of $\mu K(X^G)$. Using a little more complicated procedure, we can construct spaces, where $\chi\mu(K(X^G)) > \chi K(X^G)$.

Example 3. *There exists a pseudocompact quasitopological group the Dieudonné completion of which has bigger character than the group has.*

Take a pseudocompact space X such that $|X| = |X|^\omega$, X^ω is pseudocompact, $\chi(X) \leq |X|$ and $\chi(\beta X) \geq 2^{|X|}$. We take for G again a group of order 2. Then the quasitopological group $K(X^G)$ is pseudocompact and its Dieudonné completion is the space $(\beta X)^G$ (since $K(X^G)$ is C-embedded in X^G and X^G is C-embedded in $\beta(X^G) = (\beta X)^G$). The character of $K(X^G)$ is at most that of X^G that equals to $\chi(X) \cdot |G| = |X|$. But $\chi(\beta X)^G \geq \chi(\beta X) \geq 2^{|X|}$.

It remains to show that a space X with the requested properties exists. It suffices to take a discrete space P of a cardinality 2^{2^ω} and for X the subspace of βP equal to the union of closures of countable subsets of P. Then $|X| = |P|$, all the powers of X are pseudocompact (because closures of countable subsets of X are compact), $\chi(X) = |P|$ and $\chi(\beta X) = \chi(\beta P) > |P|$. □

Again we do not know what is the similar situation with topological groups:

Question 6. Is $\chi(\mu G) = \chi G$ for every topological group G?

Proposition 3.2. *If G is a PT-group, then $\chi(\mu G) = \chi G$.*

Proof: The space μG is homogeneous, since μG is a topological group. Since G is dense in μG, the character of μG coincides with the character of G at each point of G. Since G is nonempty and μG is homogeneous, it follows that the character of μG is equal to the character of G. □

Corollary 3.3. *If G is a Moscow group, then $\chi(\mu G) = \chi G$. In particular, $\chi(\mu G) = \chi G$ in each of the following cases (where G is a topological group):*

(1) *the tightness of G is countable;*
(2) *G is separable;*
(3) *the Souslin number of G is countable;*
(4) *G is a k-space;*
(5) *G is totally bounded.*

The next example shows a situation that cannot occur in topological groups (by Comfort and Ross [CR], every pseudocompact topological group is totally bounded). We say, that a semitopological group H is *κ-bounded* if for every neighborhood U of the neutral element there is a subset F of H with $|F| < \kappa$ such that both the left and right shifts of U by F cover H; ω-bounded groups are called totally bounded.

Example 4. *For every infinite cardinal κ there exists a pseudocompact quasitopological group, that is not κ-bounded.*

Let $K(X^G, \kappa)$ be the orbit of a map $h : G \to X$. For this proof, we shall identify G with $K(X^G, \kappa)$. Take an open neighborhood U of $h(0), U \neq X$; U forms a canonical neighborhood $\tilde{U}$ of 0 in G, $\tilde{U} = \{b \in G : h(b) \in U\}$. For any $a \in G$, the shift of $\tilde{U}$ by a is the set $\{b \in G : h(b - a) \in U\}$. Take now any subset F of G with $|F| < \kappa$. There is a point $c \in G$ such that h_c, the shift of h by c, restricted to $-F$ is constant with a value not lying in U; then h_c does not belong to the shift of $\tilde{U}$ by F. □

Question 7. Can every (semi)topological group be embedded into a totally bounded semitopological group?

Question 8. Are products of pseudocompact semitopological groups pseudocompact?

Example 5. *There exists a zero-dimensional pseudocompact quasitopological group H such that μH is homogeneous and Moscow, and is not homeomorphic to any semitopological group. Moreover, μH is the unique homogeneous Dieudonné complete extension of H.*

Take for X the "double arrow" space and for G the Cantor group. The space X^G is zero-dimensional, compact and homogeneous, since X is zero-dimensional, compact and homogeneous. It is Moscow, since the product of any family of first countable spaces is Moscow ([Ar3]). Define $H = K(X^G)$.

Since X^G is Moscow and H is G_δ-dense in X^G, the space H is C-embedded in X^G (see [Ar2]). Since H is pseudocompact, it follows that $\mu(H) = \beta(H) = X^G$. Finally, H is Moscow, since it is a dense subspace of the Moscow space X^G.

Assume now that X^G is homeomorphic to a semitopological group Z. Then Z is compact, and therefore, by R. Ellis's theorem [El], Z is a topological group. It follows that the spaces Z and X^G are dyadic (see [Us]). Hence, the space X is dyadic, which it is not (since every first countable dyadic compactum is metrizable). Thus X^G is not homeomorphic to any semitopological group.

Assume that H is a dense subspace of a Dieudonné complete homogeneous space Z. Then Z is pseudocompact, since H is pseudocompact. However, every Dieudonné complete pseudocompact space is compact. Hence, Z is compact. The subspace H is G_δ-dense in Z, since H is pseudocompact. Since H is Moscow, and Z is homogeneous, it follows, by a theorem in [Ar2], that H is C-embedded in Z. Hence, $Z = \beta(H) = X^G$. □

We can see that the pseudocompact quasitopological group G from Example 5 is not a dense subspace of any Dieudonné complete semitopological group.

Example 6. *There exists a pseudocompact quasitopological group that is not Moscow.*

Take a compact homogeneous space that is not Moscow for X (see [Pa]) and a group of order 2 for G. The space X^G is homogeneous, since X is homogeneous. Assume that $K(X^G)$ is Moscow. Then, by a key lemma in [Ar2], X^G is Moscow. However, the space X^G is not Moscow, since otherwise the space X would be Moscow, which is not the case. It follows that $K(X^G)$ is not Moscow. □

Note, that every pseudocompact paratopological group G is Moscow. Indeed, by a result of E.A. Reznichenko [Re], G is a topological group, and every pseudocompact topological group is Moscow (see [Ar2]).

Question 9. Suppose G is a topological group such that μG is homogeneous. Is then μG a topological group?

Question 10. Is μG homogeneous for *every* topological group G?

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Department of Mathematics, Ohio University, 321 Morton Hall, Athens, Ohio 45701, USA

E-mail: arhangel@bing.math.ohiou.edu (from January 1 to June 15, 2000)
arhala@arhala.mccme.ru (from July 1 to December 31, 2000)

E-mail: mhusek@karlin.mff.cuni.cz

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