Generalized tri-quotient maps and Čech-completeness

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Abstract. For a topological space X let $\mathcal{K}(X)$ be the set of all compact subsets of X. The purpose of this paper is to characterize Lindelöf Čech-complete spaces X by means of the sets $\mathcal{K}(X)$. Similar characterizations also hold for Lindelöf locally compact X, as well as for countably K-determined spaces X. Our results extend a classical result of J. Christensen.

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Classification: 54C10, 54C60

1. Introduction

J.P. Christensen [2] proved the following characterization of Polish spaces which is of great importance in analytic set theory (see [10] for another proof of the same result).

Theorem 1.1. *For a separable metric space Y the following are equivalent:*

(a) *Y is complete.*
(b) *There is a Polish space X and a map $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ such that:*
  (i) *if $A, B \in \mathcal{K}(X)$ with $A \subset B$, then $F(A) \subset F(B)$ (i.e., F is monotone);*
  (ii) *for each $K \in \mathcal{K}(Y)$ there is $H \in \mathcal{K}(X)$ with $K \subset F(H)$ (i.e., $F(\mathcal{K}(X))$ is cofinal in $\mathcal{K}(Y)$).*
(c) *The space $\mathcal{K}(Y)$ endowed with the Vietoris topology is the continuous image of a Polish space.*

Recently A. Bouziad and J. Calbrix [1] generalized the equivalence $(a) \Leftrightarrow (c)$ of Theorem 1.1 as follows.

Theorem 1.2. *A regular q-space Y is Lindelöf and Čech-complete if and only if there exists a continuous map f from a Lindelöf Čech-complete space X into the space $\mathcal{K}(Y)$ endowed with the upper Vietoris topology such that $f(X)$ is cofinal in $\mathcal{K}(Y)$.*

Recall that X is a q-space ([6]) if every $x \in X$ has a sequence $\{U_n\}$ of neighborhoods such that if $x_n \in U_n$ for each $n \in \mathbf{N}$, then $\{x_n\}$ has a cluster point in X. Obviously, every first countable, in particular, every metric space is a q-space.

The importance (and the beauty) of J. Christensen's result is the equivalence $(a) \Leftrightarrow (b)$ because $\mathcal{K}(X)$ is considered without any topology and the map F is required to satisfy only set-theoretical conditions. The main purpose of this note is to generalize the equivalence $(a) \Leftrightarrow (b)$ of Theorem 1.1 in two directions. First, to show that it is true for more general spaces than separable metric ones. Second, to show that local compactness can also be characterized in a similar way.

To state our result, we need some preliminary concepts. Call a set $A \subset X$ *bounded* in X if each locally finite open in X family has a finite restriction on A. For completely regular X boundedness of $A \subset X$ is equivalent to the boundedness of $f(A)$ for each continuous real-valued function f on X. A space X is said to be a *μ-space* or *μ-complete* if every closed and bounded set in X is compact. Every paracompact (in particular, Lindelöf), as well as every realcompact space is μ-complete. Now, we can come to our main result.

Theorem 1.3. *For a regular μ-complete q-space Y the following are equivalent, where $\mathcal{P}$ is either local compactness or Čech-completeness:*

(a) *Y is a Lindelöf space with the property $\mathcal{P}$;*
(b) *there exists a Lindelöf space $X \in \mathcal{P}$ and a monotone map $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ such that $F(\mathcal{K}(X))$ is cofinal in $\mathcal{K}(Y)$;*
(c) *there exists a separable metric space $M \in \mathcal{P}$ and a monotone map $F : \mathcal{K}(M) \to \mathcal{K}(Y)$ such that $F(\mathcal{K}(M))$ is cofinal in $\mathcal{K}(Y)$.*

A similar to Theorem 1.3 result also holds for Lindelöf Σ-spaces, see Theorem 3.3 (recall that Lindelöf Σ-spaces [8], in different terminology, countably K-determined spaces [4], are the images of separable metric spaces under upper semi-continuous compact-valued maps). We introduce in Section 2 generalized tri-quotient maps. The upper semi-continuous compact-valued maps from this class, like Michael's tri-quotient maps [7], preserve sieve-completeness (see Theorem 2.4).

All topological spaces are assumed to be Hausdorff.

2. Generalized tri-quotient maps

Suppose $F : X \to 2^Y$ is a set-valued map. For any $A \subset X$ and $B \subset Y$ we denote $F(A) = \bigcup\{F(x) : x \in A\}$ and $F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$, and the graph of F is the set $G(F) = \{(x, y) : y \in F(x)\} \subset X \times Y$. Note that, in general, $G(F)$ is not closed in $X \times Y$ unless F possesses some type of continuity. We also denote by π_X and π_Y the projections from $G(F)$ into X and Y respectively. The map F is upper semi-continuous (br., usc) if $F^{-1}(H) \subset X$ is closed for any closed $H \subset Y$. Upper semi-continuous compact-valued maps are called usco maps. We also use $\mathcal{F}(X)$ to denote the topology of X.

A map $F : X \to 2^Y$ is said to be *generalized tri-quotient* if one can assign to each open $U \subset X$ an open $t(U) \subset Y$ such that:

(1) $t(U) \subset F(U)$;
(2) $t(X) = Y$;
(3) $U \subset V$ implies $t(U) \subset t(V)$;

(4) if $y \in t(U)$ and if $\mathcal{W}$ is a cover of $F^{-1}(y) \cap U$ by open subsets of X, then $y \in t(\bigcup \mathcal{E})$ for some finite $\mathcal{E} \subset \mathcal{W}$.

We call the function $t : \mathcal{T}(X) \to \mathcal{T}(Y)$ an *assignment* for F. By (1) and (2), every generalized tri-quotient map is surjective, i.e. $Y = F(X)$. When $F : X \to Y$ is single-valued and continuous, the above definition coincides with the definition of a tri-quotient map ([7]).

Proposition 2.1. *$F : X \to 2^Y$ is generalized tri-quotient if and only if $\pi_Y : G(F) \to Y$ is tri-quotient.*

Proof: Suppose F is generalized tri-quotient and t is an assignment for F. For each open $U \subset G(F)$ let $\mathcal{B}_U$ be the family of all pairs $(W, V) \in \mathcal{T}(X) \times \mathcal{T}(Y)$ such that $(W \times V) \cap G(F) \subset U$ and define $t^*(U) = \bigcup \{t(W) \cap V : (W, V) \in \mathcal{B}_U\}$. Obviously, $U_1 \subset U_2$ implies $t^*(U_1) \subset t^*(U_2)$. Let check that t^* satisfies the conditions (1), (2) and (4) with F replaced by π_Y and X by $G(F)$.

(1) If $y \in t^*(U)$, then $y \in t(W) \cap V$ for some $(W, V) \in \mathcal{B}_U$. Since $t(W) \subset F(W)$, there exists $x \in W$ with $y \in F(x)$. Hence $(x, y) \in (W \times V) \cap G(F)$, so $(x, y) \in U$ and $y = \pi_Y((x, y)) \in \pi_Y(U)$.

(2) For a fixed $y \in Y$ and any $x \in F^{-1}(y)$ take arbitrary $W(x) \in \mathcal{T}(X)$ and $V(x) \in \mathcal{T}(Y)$ such that $x \in W(x)$ and $y \in V(x)$. By (4), $y \in t(\bigcup \{W(x_i) : i = 1, 2, \ldots, n\})$ for some finite set $\{x_i : i = 1, 2, \ldots, n\}$. Let $W = \bigcup \{W(x_i) : i = 1, \ldots, n\}$ and $V = \bigcap \{V(x_i) : i = 1, \ldots, n\}$. Then $y \in t(W) \cap V \subset t^*(G(F))$.

(4) Let $y \in t^*(U)$ and $\mathcal{W}$ be a cover of $\pi_Y^{-1}(y) \cap U$ by open subsets of $G(F)$. We can suppose that $\mathcal{W} = \{U_\alpha = (W_\alpha \times V_\alpha) \cap G(F) : \alpha \in \Lambda\}$. There exists $(W, V) \in \mathcal{B}_U$ such that $y \in t(W) \cap V$. It is easily seen that $W \cap F^{-1}(y)$ is covered by $\{W_\alpha : \alpha \in \Lambda\}$. By (4), $y \in t(\bigcup\{W_{\alpha(i)} : i = 1, 2, \ldots, k\})$ for some finite $\{\alpha(i) : i = 1, 2, \ldots, k\} \subset \Lambda$. Denote $W_0 = \bigcup\{W_{\alpha(i)} : i = 1, 2, \ldots, k\}$ and $V_0 = \bigcap\{V_{\alpha(i)} : i = 1, 2, \ldots, k\}$. Then we have $y \in t(W_0) \cap V_0 \subset t^*(\bigcup\{U_{\alpha(i)} : i = 1, 2, \ldots, k\})$.

Now suppose π_Y is tri-quotient with an assignment $t^* : \mathcal{T}(G(F)) \to \mathcal{T}(Y)$. Then $t : \mathcal{T}(X) \to \mathcal{T}(Y)$, $t(U) = t^*(\pi_X^{-1}(U))$, satisfies conditions (1)–(4). Hence F is generalized tri-quotient. □

We say that a map $F : X \to 2^Y$ is contained in a map $\Phi : X \to 2^Y$ if $F(x) \subset \Phi(x)$ for every $x \in X$.

Corollary 2.2. *Let $\mathcal{P}$ be a topological property which is invariant under tri-quotient maps and perfect preimages. If $X \in \mathcal{P}$ and $F : X \to 2^Y$ is a generalized tri-quotient map which is contained in an usco map Φ, then $Y \in \mathcal{P}$.*

Proof: First, let us show that Φ is generalized tri-quotient with the same assignment as F. Conditions (1)–(3) are obviously satisfied. Since $F(x) \subset \Phi(x)$ for all $x \in X$, we have $F^{-1}(y) \subset \Phi^{-1}(y)$ for all $y \in Y$. This implies that condition (4) also holds. Therefore Φ is generalized tri-quotient.

Since Φ is usco, $\pi_X : G(\Phi) \to X$ is perfect, so $G(\Phi) \in \mathcal{P}$. Finally, because $\mathcal{P}$ is preserved by tri-quotient maps, it follows from Proposition 2.1 that $Y \in \mathcal{P}$. □

We need the following analogue of q-spaces: call X an *sq-space* ([11]) if every $x \in X$ has a sequence $\{U_n\}$ of neighborhoods such that if $x_n \in U_n$ for each n, then $\{x_n\}$ has a compact closure in X. The sq-space property is stronger than q-space property and they are equivalent for regular μ-spaces. We express our gratitude to the referee who pointed that, for Hausdorff spaces, the notion of sq-space coincides with Michael's r-space [5].

We say that $F : X \to 2^Y$ is an *sq-map* if every $x \in X$ has a sequence $\{U_n\}$ of neighborhoods such that if $x_n \in U_n$ for each n, then $\bigcup\{F(x_n) : n \in \mathbf{N}\}$ has a compact closure in Y. Most frequently we will have the following particular case of an sq-map: X is an sq-space and $F : X \to 2^Y$ is a map such that $F(K)$ has a compact closure in Y for every (separable) compact set $K \subset X$. Next lemma was proved for that particular case in [12, Lemma 3.5]. The same proof adapted in our situation works. For reader's convenience it is included here.

Lemma 2.3. *Let $F : X \to 2^Y$ be an sq-map with Y a regular μ-space. Then there exists an usco map $\Phi : X \to \mathcal{K}(Y)$ such that $F(x) \subset \Phi(x)$ for every $x \in X$.*

Proof: For each $x \in X$ fix a countable decreasing family $\gamma(x) = \{U_n(x)\}$ of neighborhoods of x in X such that for any sequence $\{x_n\}$ with $x_n \in U_n(x)$ the set $\bigcup\{F(x_n) : n \in \mathbf{N}\}$ has a compact closure in Y. Let

$$\Phi(x) = \bigcap\{\mathrm{cl}_Y(F(U_n(x))) : n \in \mathbf{N}\}, \quad x \in X.$$

Claim 1. *$\Phi(x)$ is compact for each $x \in X$.*

Because each closed and bounded subset of Y is compact, it suffices to show that $\Phi(x)$ is bounded. Assuming the contrary we can find an open locally finite family γ in Y such that $\gamma \cap \Phi(x)$ is infinite. We can suppose that $\gamma = \{W_n\}$ is countable. For each n choose $x_n \in U_n(x)$ and $y_n \in F(x_n) \cap W_n$. Then $K = \mathrm{cl}_Y(\bigcup\{F(x_n) : n \in \mathbf{N}\})$ is compact which contradicts the fact that K intersects infinitely many elements of γ.

Claim 2. *Φ is usc.*

Let $W \subset Y$ be open and $\Phi(x_0) \subset W$ for some $x_0 \in X$. Since $\Phi(x_0)$ is compact and Y is regular, there exists an open $V \subset Y$ such that $\Phi(x_0) \subset \mathrm{cl}_Y(V) \subset W$. It is enough to show that there is n with $F(x) \subset \mathrm{cl}_Y(V)$ for every $x \in U_n(x_0)$. Assuming this is not the case we can find $x_n \in U_n(x_0)$ such that $F(x_n) \cap (Y - \mathrm{cl}_Y(V)) \neq \emptyset$ for each n. Let $H_n = \mathrm{cl}_Y(\bigcup\{F(x_k) : k \geq n\})$. Because all H_n are compact, $\lambda = \{H_n \cap (Y - V)\}$ is a decreasing family of non-empty compact sets. Hence $\bigcap \lambda \neq \emptyset$. But $\bigcap \lambda \subset \bigcap\{H_n\} \subset \Phi(x_0) \subset V$, a contradiction.

Therefore $\Phi : X \to 2^Y$ is an usco map such that $F(x) \subset \Phi(x)$ for every $x \in X$. □

Recall that sieve-completeness (see [3], [8]) is a more general property than Čech-completeness and both they are equivalent for paracompact spaces.

Theorem 2.4. *Let $F : X \to 2^Y$ be a generalized tri-quotient map such that Y is a regular μ-space and $F(K)$ has a compact closure in Y for every separable compact $K \subset X$. If X is regular and sieve-complete, then Y is also sieve-complete.*

Proof: Every regular sieve-complete space is *sq* (even of countable type, see [3, Proposition 4.4]), so F is an *sq*-map. Then, by Lemma 2.3, there exists an usco map $\Phi : X \to \mathcal{K}(Y)$ containing F. Finally, since sieve-completeness is preserved by perfect preimages ([8, Remark (8.7))] and by tri-quotient maps ([8, Theorem 6.3]), Corollary 2.2 yields that Y is sieve-complete. □

Remark 2.5. Actually, the conclusion of Theorem 2.4 remains true if sieve-completeness is replaced by any topological property $\mathcal{P}$ such that $\mathcal{P}$ yields *sq*-space property, and is preserved by tri-quotient maps and perfect preimages. For example, local compactness is such a property (it is preserved by bi-quotient maps [5, Proposition 3.4], and the latter class contains all tri-quotient maps).

It is interesting to compare Theorem 2.4 with Michael's result [8, Theorem 6.3] that sieve-completeness is preserved by tri-quotient maps. Concerning the map F our requirement is weaker, first F is generalized tri-quotient and second it has a weak type of continuity (that the closure of all $F(K)$, $K \in \mathcal{K}(X)$ is separable, are compact). But we require X to be regular and Y to be a regular μ-space.

A map $F : X \to 2^Y$ is called *compact covering* (resp., *countable compact-covering*) if for every compact (resp., countable compact) set $H \subset Y$ there is a compact set $K \subset X$ with $H \subset F(K)$. We also say that F is *open* (resp., *closed*) if $F(A) \subset Y$ is open (resp., closed) for every open (resp., closed) $A \subset X$.

Proposition 2.6. *Let $F : X \to 2^Y$ be a set-valued map. Then each of the following implies that F is generalized tri-quotient.*

(a) *F is open.*
(b) *F is closed and each $F^{-1}(y)$ is compact.*
(c) *F is compact-covering (resp., countable compact-covering) with a closed graph, each $F^{-1}(y)$ is Lindelöf, X is regular and Y is a μ-complete regular q-space (resp., first countable).*

Proof: As in the case of single-valued maps ([8, Theorem 6.5]), we define $t(U) = F(U)$ for (a) and $t(U) = Y - F(X - U)$ for (b). The routine verification that we have assignments for F is omitted.

For part (c), since $\pi_Y^{-1}(y) = F^{-1}(y) \times \{y\}$, each $\pi_Y^{-1}(y)$ is Lindelöf. Because F is compact-covering (resp., countable compact-covering) with a closed graph, π_Y is also compact-covering (resp., countable compact-covering). Then following Michael's arguments from [8, Lemma 5.1 and Theorem 6.5(e)] we can show that π_Y is tri-quotient. Therefore, by Proposition 2.1, F is generalized tri-quotient. □

3. Main results

Let $\mathcal{S}(X) \subset 2^X$. We will use τ_V^+ to denote the *upper Vietoris topology* on $\mathcal{S}(X)$ generated by all collections of the form $\hat{U} = \{H \in \mathcal{S}(X) : H \subset U\}$, where U runs over the open subsets of X. Also, we will use τ_V for the *Vietoris topology* which is generated by the sets

$$\langle \mathcal{U} \rangle = \left\{ H \in \mathcal{S}(X) : H \cap U \neq \emptyset,\ U \in \mathcal{U}, \text{ and } H \subset \bigcup \mathcal{U} \right\},$$

where $\mathcal{U}$ runs over the finite subsets of $\mathcal{T}(X)$.

3.1. Proof of Theorem 1.3.

(a)⇒(b). This implication is obvious.

(b)⇒(c). Suppose that $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ is monotone and $F(\mathcal{K}(X)) \subset \mathcal{K}(Y)$ is cofinal, where X is Lindelöf and $X \in \mathcal{P}$. Since X is paracompact p-space in sense of Arhangel'skii, there exists a separable metric space M and a perfect surjective map $f : X \to M$. Then $M \in \mathcal{P}$ and we define $L : \mathcal{K}(M) \to \mathcal{K}(Y)$ by $L(K) = F(f^{-1}(K))$. It is trivially seen that L is monotone and $L(\mathcal{K}(M))$ is cofinal in $\mathcal{K}(Y)$.

(c)⇒(a). Suppose $F : \mathcal{K}(M) \to \mathcal{K}(Y)$ is monotone and $F(\mathcal{K}(M)) \subset \mathcal{K}(Y)$ is cofinal, where M is a separable metric space with the property $\mathcal{P}$.

First, let show that F, considered as a set-valued map from $(\mathcal{K}(M), \tau_V)$ into Y, is an sq-map. Obviously M is an sq-space. If $\mathcal{H} \subset (\mathcal{K}(M), \tau_V)$ is compact, then $H = \bigcup \mathcal{H} \subset M$ is compact and contains each $K \in \mathcal{H}$. Hence, by monotonicity of F, $F(\mathcal{H}) \subset F(H)$. Consequently, $F(\mathcal{H})$ has a compact closure in Y. Therefore F is an sq-map, and by Lemma 2.3, there exists an usco map $\Phi : (\mathcal{K}(M), \tau_V) \to 2^Y$ such that $F(K) \subset \Phi(K)$ for every $K \in \mathcal{K}(M)$.

Next step is to verify that Φ satisfies the conditions from Proposition 2.6(c). Since Φ is usc, it has a closed graph. Because $(\mathcal{K}(M), \tau_V)$ is a separable metric space each $\Phi^{-1}(y)$, $y \in Y$, is Lindelöf. To show that Φ is compact-covering take compact $K \subset Y$. Then there is $H \in \mathcal{K}(M)$ with $K \subset F(H)$. Hence $K \subset \Phi(H)$ (we actually have that $\{\Phi(A) : A \in \mathcal{K}(M)\}$ is cofinal in $\mathcal{K}(Y)$). So, we can apply Proposition 2.6(c) to conclude that Φ is generalized tri-quotient.

Finally, let show that Y is Lindelöf and $Y \in \mathcal{P}$. Since Φ is compact-covering, $Y = \bigcup\{\Phi(K) : K \in \mathcal{K}(M)\}$. So, Y is an usco image of the Lindelöf space $(\mathcal{K}(M), \tau_V)$. Consequently, Y is Lindelöf. If $\mathcal{P}$ is local compactness, Remark 2.5 and Theorem 2.4 yield that Y is locally compact. If $\mathcal{P}$ is Čech-completeness, we can apply Theorem 2.4 to conclude that Y is sieve-complete, and then [8, Theorem 3.2] (that a paracompact space is sieve-complete iff it is Čech-complete) to conclude that Y is Čech-complete. □

Corollary 3.2. *Let Y be a regular μ-complete q-space and $\mathcal{P}$ be either local compactness or Čech-completeness. Then the following conditions are equivalent:*

(a) *Y is a Lindelöf space with the property $\mathcal{P}$;*

(b) *there exists a Lindelöf space $X \in \mathcal{P}$ and a map $f : X \to \mathcal{K}(Y)$ such that $f(X)$ is cofinal in $\mathcal{K}(Y)$ and each $f(K)$, $K \in \mathcal{K}(X)$, has a compact closure in Y;*
(c) *there exists a Lindelöf space $X \in \mathcal{P}$ and a continuous map $\Phi : X \to (\mathcal{K}(Y), \tau_V^+)$ such that $\Phi(X) \subset \mathcal{K}(Y)$ is cofinal;*
(d) *there exists a Lindelöf space $Z \in \mathcal{P}$ and a compact-covering surjection (not necessarily continuous) $g : Z \to Y$ such that $g(K)$ has a compact closure in Y for each $K \in \mathcal{K}(Z)$.*

Proof: (a)⇒(b). We simply define X to be $(\mathcal{K}(Y), \tau_V)$ and $f(K) = K$, $K \in \mathcal{K}(Y)$.

(b)⇒(c). Let f and X be as in (b). Then f is an sq-map and, by Lemma 2.3, there is an usco map $\Phi : X \to 2^Y$ with $f(x) \subset \Phi(x)$ for all $x \in X$. Therefore Φ considered as a map from X into $(\mathcal{K}(Y), \tau_V^+)$ is continuous and $\Phi(X) \subset \mathcal{K}(Y)$ is cofinal.

(c)⇒(d). If Φ and X satisfy (c), let Z be the graph of Φ and g be the projection π_Y.

(d)⇒(a). Suppose Z and g are as in (d). We define $F : \mathcal{K}(Z) \to \mathcal{K}(Y)$ by $F(K) = cl_Y(g(K))$. Then F is monotone and since g is compact-covering, $F(\mathcal{K}(Z))$ is cofinal in $\mathcal{K}(Y)$. Finally, by Theorem 1.3, Y is Lindelöf with the property $\mathcal{P}$. □

The case of Corollary 3.2 when $\mathcal{P}$ is Čech-completeness is comparable with Theorem 2.1 from [1]. We do not need the maps in (b) and (d) to be continuous (as in [1, Theorem 2.1]), but we need something more from Y — to be μ-complete.

Theorem 3.3. *For a regular μ-complete space Y the following are equivalent:*
(a) *Y is countably K-determined;*
(b) *there is a separable metric space M and a monotone map $F : \mathcal{K}(M) \to \mathcal{K}(Y)$ such that $\{F(K) : K \in \mathcal{K}(M)\}$ covers Y;*
(c) *there is a countably K-determined space X and a map $F : X \to \mathcal{K}(Y)$ such that $F(X) = Y$ and $F(K)$ has a compact closure in Y for every $K \in \mathcal{K}(X)$.*

Proof: (a)⇒(b). Suppose Y is countably K-determined. Take a separable metric space M and an usco map $\phi : M \to 2^Y$ with $\phi(M) = Y$. Then $F(K) = \phi(K)$, $K \in \mathcal{K}(M)$, satisfies the required conditions.

(b)⇒(c). Let M and F be as in (b). The space $(\mathcal{K}(M), \tau_V)$ is separable metric, hence countably K-determined. Then F, considered as a map from $(\mathcal{K}(M), \tau_V)$ into $\mathcal{K}(Y)$, and the space $(\mathcal{K}(M), \tau_V)$ have required properties.

(c)⇒(a). Let M be a separable metric space and $\phi : M \to 2^X$ an usco map with $\phi(M) = X$. Define a map $L : \mathcal{K}(M) \to \mathcal{K}(Y)$ by $L(K) = cl_Y F(\phi(K))$. Then L is monotone and $\{L(K) : K \in \mathcal{K}(M)\}$ covers Y. As in the proof of Theorem 1.3 (implication (c)⇒(a)), one can show that L is an sq-map from $(\mathcal{K}(M), \tau_V)$ into 2^Y.

Then, by Lemma 2.3, there is an usco map $\Phi : (\mathcal{K}(M), \tau_V) \to 2^Y$ such that $L(K) \subset \Phi(K)$ for all $K \in \mathcal{K}(M)$. This implies that Y is an usco image of the space $(\mathcal{K}(M), \tau_V)$. Therefore Y is countably K-determined. □

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