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On Kelvin type transformation for Weinstein operator

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Abstract. The note develops results from [5] where an invariance under the Möbius transform mapping the upper halfplane onto itself of the Weinstein operator $W_k := \Delta + \frac{k}{x_n}\frac{\partial}{\partial x_n}$ on $\mathbb{R}^n$ is proved. In this note there is shown that in the cases $k \neq 0$, $k \neq 2$ no other transforms of this kind exist and for case $k = 2$, all such transforms are described.

Keywords: harmonic morphisms, Kelvin transform, Weinstein operator

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1. Introduction

It is well known from last century that the Kelvin transformation for functions of n real variables

$$(Ku)(x) := \frac{1}{\|x\|^{n-2}} u\left(\frac{x}{\|x\|^2}\right)$$

transforms a harmonic function u to a harmonic function Ku. Another transformation of this property is changing variables of a function u by a similarity, i.e.

$$(Su)(x) := u(rRx + a)$$

where r is a positive real number, R is an orthonormal matrix and $a \in \mathbb{R}^n$ is a vector. Transformations K and S generate the group of transformations

$$(1) \qquad (Tu)(x) := \varkappa(x)^{(n-2)/2} u(M(x)),$$

where M is a Möbius transformation with dilatation factor $\varkappa$, i.e. $\varkappa(x)$ is such a positive real number that the Jacobian matrix $M'(x)$ divided by $\varkappa(x)$ is an orthonormal matrix. It can be computed from the coordinates of the mapping $M = (M_1, \dots, M_n)$ as

$$(2) \qquad \varkappa = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial M_l}{\partial x_i}\right)^2}$$

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for an arbitrary $l = 1, \ldots, n$.

O.D. Kellogg shows in his monograph [1] that no further transformation of the type

$$(Tu)(x) := \varphi(x) u(\Psi(x)),$$

where φ is a positive function and Ψ a bijection, which preserves harmonicity exists. Kellogg considers the case $n = 3$ only, but it is also valid in the case $n > 3$.

A similar problem for caloric functions is solved by H. Leutwiler in [2]. The role of the Kelvin transformation is played by the Appell transformation. The problem for Kolmogorov and Kolmogorov type operator is solved in [3], [4].

This paper deals with the Weinstein operator

$$W(u) := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{k}{x_n} \frac{\partial u}{\partial x_n}$$

where k is an arbitrary real number. In [5] is shown that for this operator the transformations (1) have the form

$$(Tu)(x) := \varkappa(x)^{(n+k-2)/2} u(M(x))$$

and they preserve the solution of the Weinstein operator provided that M is a Möbius transformation which maps the halfspace $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n, x_n > 0\}$ onto $\mathbb{R}^n_+$. We show in this paper that in cases $k \neq 0, 2$, $n \geq 3$ no else transformation of this type exists. Further we prove that in the case $k = 2$ the bijection M can be an arbitrary Möbius transformation and (1) has the form

$$(Tu)(x) := \frac{M_n(x)}{x_n} \varkappa(x)^{(n-2)/2} u(M(x)).$$

Note that the condition $M_n = x_n \varkappa$ is equivalent to the invariance of the halfspace $\mathbb{R}^n_+$ under transformations M and M^{-1}.

2. Equations describing W-morphisms

In this section we give the definition of a W-morphism and equivalent conditions which describe it. First we introduce some notation. Let $\mathbb{R}^n_+$ be as above, $U, V \subset \mathbb{R}^n_+$ domains, $\Psi : U \to V$ a bijection, $\varphi : U \to (0, \infty)$ a positive function and W the Weinstein operator with $k \in \mathbb{R}$. The transformation $T : \mathcal{C}^2(V) \to \mathcal{C}^2(U)$ defined by $Tu := \varphi \cdot (u \circ \Psi)$ is called a *W-morphism* provided that $W(Tu) = 0$ on U whenever $W(u) = 0$ on V.

Theorem. *Let $n \geq 2$, $W, T, \varphi \in \mathcal{C}^2(U)$, $\Psi = (\psi_1, \ldots, \psi_n) \in \left(\mathcal{C}^2(U)\right)^n$, U, V be as above. Then the following conditions are equivalent:*

1. *T is a W-morphism;*
2. *T maps every polynomial*

$$\left.\begin{array}{c} 1, x_l, x_l x_p, x_n^2-(k+1)x_l^2, 3x_n^2 x_l-(k+1)x_l^3, \\ 3x_n^4-6(k+3)x_n^2 x_l^2+(k+1)(k+3)x_l^4 \end{array}\right\} \text{ for } l,p=1,\dots,n-1,\ l\neq p$$

to a solution of the equation $W(\cdot)=0$*;*

3. *The functions* $\varphi, \psi_1, \dots, \psi_n$ *satisfy on* U *the equations:*

$$W(\varphi)=0, \tag{3}$$

$$\nabla\psi_i\cdot\nabla\psi_j=0, \quad \text{for } i,j=1,\dots,n,\ i\neq j, \tag{4}$$

$$\|\nabla\psi_i\|=\|\nabla\psi_j\| \quad \text{for } i,j=1,\dots,n, \tag{5}$$

$$\varphi W(\psi_i)+2\nabla\varphi\cdot\nabla\psi_i=\frac{k\varphi\|\nabla\psi_n\|^2}{\psi_n}\delta_{in}, \tag{6}$$

here δ_{in} *is the Kronecker delta equal to 1 for* $i=n$ *and to 0 for* $i\neq n$*. Note that* $V\subset\mathbb{R}^n_+$ *and hence* $\psi_n>0$ *on* U*;*

4. *There exists a function* $\varkappa$ *on* U *such that*

$$W(Tu)=\varkappa^2\varphi\cdot(W(u)\circ\Psi) \tag{7}$$

for every $u\in\mathcal{C}^2(V)$*.*

Proof: We show that $1\Rightarrow 2\Rightarrow 3\Rightarrow 4\Rightarrow 1$.

To prove $1\Rightarrow 2$ it is enough to check that all considered polynomials are solutions of the Weinstein operator. We omit it.

Now we prove $2\Rightarrow 3$: the equation (3) is a straightforward consequence of $W(1)=0$. To prove the other equations we will use the following identities which are valid for all functions $u, v, v_1, v_2\in\mathcal{C}^2(U)$

$$W(uv)=u\,W(v)+vW(u)+2\nabla u\cdot\nabla v, \tag{8}$$

$$W(uv^m)=mv^{m-1}W(uv)-(m-1)v^mW(u)+m(m-1)uv^{m-2}\|\nabla v\|^2, \tag{9}$$

$$W(uv_1v_2)=v_1W(uv_2)+v_2W(uv_1)-v_1v_2W(u)+2u\nabla v_1\cdot\nabla v_2. \tag{10}$$

In the sequel we consider indexes $l,p=1,\dots,n-1$, $l\neq p$ and $i=1,\dots,n$. Substituting $u=\varphi$, $v_1=\psi_l$, $v_2=\psi_p$ to (10) and using $W(\varphi)=W(\varphi\psi_l)=W(\varphi\psi_p)=W(\varphi\psi_l\psi_p)=0$, we obtain $\varphi\nabla\psi_l\cdot\nabla\psi_p=0$ which, due to the positivity of φ, gives (4) for $l,p\neq n$. Now we claim to prove $\nabla\psi_l\cdot\nabla\psi_n=0$ for $l\neq n$. Condition 2 in the theorem gives

$$W(\varphi\psi_n^2)=(k+1)W(\varphi\psi_l^2), \tag{11}$$

$$3W(\varphi\psi_n^2\psi_l)=(k+1)W(\varphi\psi_l^3). \tag{12}$$

On the other hand, substituting $u=\varphi$, $v=\psi_l$ to (9) and using $W(\varphi)=W(\varphi\psi_l)=0$ we obtain

$$W(\varphi\psi_l^m)=m(m-1)\varphi\psi_l^{m-2}\|\nabla\psi_l\|^2 \tag{13}$$

and putting $u = \varphi$, $v_1 = \psi_l$, $v_2 = \psi_n^2$ to (10) we get

$$W(\varphi\psi_l\psi_n^2) = \psi_l W(\varphi\psi_n^2) + 2\varphi\nabla\psi_l \cdot \nabla(\psi_n^2). \tag{14}$$

Now using (11), (13), (14) to simplify (12) we get

$$\varphi\nabla\psi_l \cdot \nabla(\psi_n^2) = 0$$

which can be simplified to

$$\varphi\psi_n\nabla\psi_l \cdot \nabla\psi_n = 0.$$

This implies, due to the positivity of φ and ψ_n (Ψ is a mapping to $\mathbb{R}_+^n$ and hence $\psi_n > 0$),

$$\nabla\psi_l \cdot \nabla\psi_n = 0.$$

To prove (5) we use

$$3W(\varphi\psi_n^4) - 6(k+3)W(\varphi\psi_n^2\psi_l^2) + (k+1)(k+3)W(\varphi\psi_l^4) = 0. \tag{15}$$

Substituting $u = \varphi$ and $v_1 = v_2 = \psi_n^2$ or $v_1 = \psi_n^2, v_2 = \psi_l^2$, respectively, to (10) and using (4) we get

$$W(\varphi\psi_n^4) = 2\psi_n^2 W(\varphi\psi_n^2) + 8\varphi\psi_n^2\|\nabla\psi_n\|^2, \tag{16}$$

$$W(\varphi\psi_n^2\psi_l^2) = \psi_n^2 W(\varphi\psi_l^2) + \psi_l^2 W(\varphi\psi_n^2). \tag{17}$$

Using (11), (13) for $m = 2$ we can rewrite (16) and (17) as

$$W(\varphi\psi_n^4) = 4(k+1)\varphi\psi_n^2\|\nabla\psi_l\|^2 + 8\varphi\psi_n^2\|\nabla\psi_n\|^2, \tag{18}$$

$$W(\varphi\psi_n^2\psi_l^2) = 2\varphi\psi_n^2\|\nabla\psi_l\|^2 + 2(k+1)\varphi\psi_l^2\|\nabla\psi_l\|^2. \tag{19}$$

Now substituting (18), (19) and (13) with $m = 4$ to (15) we get

$$\begin{aligned} 12(k+1)\varphi\psi_n^2\|\nabla\psi_l\|^2 + 24\varphi\psi_n^2\|\nabla\psi_n\|^2 - 12(k+3)\varphi\psi_n^2\|\nabla\psi_l\|^2 \\ - 12(k+1)(k+3)\varphi\psi_l^2\|\nabla\psi_l\|^2 + 12(k+1)(k+3)\varphi\psi_l^2\|\nabla\psi_l\|^2 = 0 \end{aligned}$$

which can be simplified to

$$\varphi\psi_n^2\|\nabla\psi_n\|^2 = \varphi\psi_n^2\|\nabla\psi_l\|^2. \tag{20}$$

Relation (20) is valid for all $l = 1, \ldots, n-1$ and gives (5).
Equations (6) for $i = 1, \ldots, n-1$ are direct consequences of $W(\varphi\psi_i) = 0$, $W(\varphi) = 0$ and (8).
Equation (6) for $i = n$ can be derived from (11) using (9), (13), (5), and (3).

To prove $3 \Rightarrow 4$ we use the chain rule for the relation $Tu = \varphi \cdot (u \circ \Psi)$

$$\frac{\partial (Tu)}{\partial x_i} = \frac{\partial \varphi}{\partial x_i}(u \circ \Psi) + \varphi \sum_{j=1}^{n} \left(\frac{\partial u}{\partial x_j} \circ \Psi \right) \frac{\partial \psi_j}{\partial x_i} \tag{21}$$

$$\begin{aligned} \frac{\partial^2 (Tu)}{\partial x_i^2} &= \frac{\partial^2 \varphi}{\partial x_i^2}(u \circ \Psi) + 2\frac{\partial \varphi}{\partial x_i} \sum_{j=1}^{n} \left(\frac{\partial u}{\partial x_j} \circ \Psi \right) \frac{\partial \psi_j}{\partial x_i} \\ &+ \varphi \sum_{j,l=1}^{n} \left(\frac{\partial^2 u}{\partial x_j \partial x_l} \circ \Psi \right) \frac{\partial \psi_j}{\partial x_i} \frac{\partial \psi_l}{\partial x_i} + \varphi \sum_{j=1}^{n} \left(\frac{\partial u}{\partial x_j} \circ \Psi \right) \frac{\partial^2 \psi_j}{\partial x_i^2} . \end{aligned} \tag{22}$$

Now we are able to compute (we use the summation convention over indexes i, j, l)

$$\begin{aligned} W(Tu) = W(\varphi)\,(u \circ \Psi) + \varphi \left(\frac{\partial^2 u}{\partial x_j \partial x_l} \circ \Psi \right) \frac{\partial \psi_j}{\partial x_i} \frac{\partial \psi_l}{\partial x_i} \\ + \left(\frac{\partial u}{\partial x_j} \circ \Psi \right) \left(\frac{\varphi k}{x_n} \frac{\partial \psi_j}{\partial x_n} + 2\frac{\partial \varphi}{\partial x_i} \frac{\partial \psi_j}{\partial x_i} + \varphi \frac{\partial^2 \psi_j}{\partial x_i^2} \right) \end{aligned}$$

which together with (3)–(6) gives (7) with $\varkappa = \|\nabla \psi_n\|$.

The last implication, $4 \Rightarrow 1$, is evident. □

3. Some facts about Möbius transformation

Consider non-empty domains $U, V \subset \mathbb{R}^n_+$ with the bijection $\Psi : U \to V$ now. Let components $\psi_1, \dots, \psi_n \in \mathcal{C}^2(U)$ of the bijection Ψ fulfill the conditions (4), (5). Put $\varkappa = \|\nabla \psi_1\|$ and let $U_1 := \{x \in U, \varkappa(x) \neq 0\}$. As $\varkappa$ is a continuous function, the set U_1 should be open and as the mapping Ψ is a bijection, the complement $U - U_1$ has no inner points and $U_1 \neq \emptyset$. From the equations (4), (5) is clear that the bijection Ψ preserves angles of curves i.e. it is conformal on U_1. We will use

Liouville theorem. *Let $n \geq 3$, $U_2, V_2 \subset \mathbb{R}^n$ be domains and let $\Psi : U_2 \to V_2$ be a conformal mapping. Then Ψ is a Möbius transformation, i.e. its components $\psi_1, \dots, \psi_n$ can be written either in the form*

$$\psi_i(x) = b_i + r \sum_{j=1}^{n} R_{ij} \frac{x_j - a_j}{\|x - a\|^2} \tag{23}$$

or in the form

$$\psi_i(x) = b_i + r \sum_{j=1}^{n} R_{ij} x_j , \tag{24}$$

where $r > 0$, a_i, b_i for $i = 1, \dots, n$ are real numbers with $(a_1, \dots, a_n) \notin U_2$ and R_{ij} are components of an orthonormal matrix.

We cannot use the theorem for U_1 as it may not be connected. But we can take as U_2 one of the components of U_1 and if we compute $\varkappa = \|\nabla \psi_1\|$, we obtain

$$\varkappa(x) = \frac{r}{\|x-a\|^2} \tag{25}$$

in the case (23) and

$$\varkappa(x) = r \tag{26}$$

in the case (24). We see that $\varkappa$ is a positive function and therefore $U_2 = U_1 = U$.

In the sequel we will need the following

Lemma 1. *Let $\Psi = (\psi_1, \dots, \psi_n)$ be a Möbius transformation given by* (23) *or* (24), *respectively, and let $\varkappa$ be given by* (25) *or* (26), *respectively. Then Ψ is a bijection on $\mathbb{R}^n_+$ iff $\psi_n = x_n \varkappa$ on $\mathbb{R}^n_+$.*

Proof: To prove the implication $\Leftarrow$ is enough to realize that from $\psi_n = x_n \varkappa$ it follows that $\psi_n > 0$ iff $x_n > 0$.

To prove the implication $\Rightarrow$ we consider the case (23) first. As Ψ is a continuous bijection, the boundary $\{(x_1, \dots, x_{n-1}, 0); x_1, \dots, x_{n-1} \in \mathbb{R}\}$ of $\mathbb{R}^n_+$ should be mapped bijectively by Ψ onto itself with exception of at most one point a. Then

$$b_n + r \sum_{j=1}^{n-1} \left(\frac{R_{nj}(x_j - a_j)}{\|x-a\|^2} \right) - r \frac{R_{nn} a_n}{\|x-a\|^2} = 0 \tag{27}$$

holds for all $(x_1, \dots, x_{n-1}, 0) \neq a$. Let $\bar{a} = (a_1, \dots, a_{n-1}, 0)$ be an orthogonal projection of the vector a to the plane $x_n = 0$, let e_i, $i = 1, \dots, n-1$ be a canonical vector in the same plane and let $x = \bar{a} + e_i$. Then $\|x-a\|^2 = 1 + a_n^2$ and (27) gives

$$0 = b_n + r \frac{R_{ni}}{1 + a_n^2} - r \frac{R_{nn} a_n}{1 + a_n^2} \,. \tag{28}$$

On the other hand (27) gives for $x = \bar{a} - e_i$

$$0 = b_n - r \frac{R_{ni}}{1 + a_n^2} - r \frac{R_{nn} a_n}{1 + a_n^2} . \tag{29}$$

Equations (28), (29) give $R_{ni} = 0$ for $i = 1, \dots, n-1$ and $R_{nn} = \pm 1$ (note that $\sum_{j=1}^n R_{nj}^2 = 1$). Let $x = \bar{a} + c e_1$ for $c \in \mathbb{R}$. Then equation (27) has the form

$$r \frac{R_{nn} a_n}{c^2 + a_n^2} = b_n .$$

Since c can be an arbitrary number it should be $b_n = R_{nn}a_n = 0$. But since $R_{nn} = \pm 1$ it should be $a_n = 0$. Relation (23) has therefore the form

$$\psi_n(x) = \frac{rR_{nn}x_n}{\|x-a\|^2}$$

with $R_{nn} = \pm 1$. But as ψ_n should be positive for $x_n > 0$ we have $R_{nn} = 1$ and

$$\psi_n(x) = \frac{rx_n}{\|x-a\|^2}$$

which together with (25) gives $\psi_n(x) = x_n \varkappa(x)$.
Consider the case (24) now. Again $\psi_n = 0$ whenever $x_n = 0$ and hence

$$b_n + r\sum_{j=1}^{n-1} R_{nj}x_j = 0$$

for an arbitrary point $(x_1, \dots, x_{n-1}, 0)$ which gives again $b_n = 0$, $R_{nj} = \delta_{nj}$, $\psi_n(x) = rx_n$ and hence $\psi_n = x_n\varkappa$. □

Lemma 2. *Let $U \subset \mathbb{R}^n$ be a domain, $r > 0$, R_{ij} be an orthonormal matrix $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{R}^n$, $a \notin U$ and $\psi_1, \dots, \psi_n$ be defined by (23) or by (24), respectively, on U. Let $\varkappa$ be defined by (25) or by (26), respectively, on U. Then the following identity*

$$\frac{n-2}{2}\frac{\partial}{\partial x_i}(\log\varkappa) = -\frac{1}{2\varkappa^2}\sum_{l=1}^{n}\Delta\psi_l\frac{\partial\psi_l}{\partial x_i} \tag{30}$$

is valid for $x \in U$ and for $i = 1, \dots, n$.

Proof: We consider the case (23), (25) first. The left hand side of (30) is

$$\frac{n-2}{2}\frac{\partial}{\partial x_i}(\log\varkappa) = \frac{n-2}{2}\frac{\partial}{\partial x_i}(\log r - 2\log\|x-a\|) = (2-n)\frac{x_i-a_i}{\|x-a\|^2}. \tag{31}$$

To compute the right hand side we need to differentiate (23)

$$\frac{\partial\psi_l}{\partial x_i} = \frac{rR_{li}}{\|x-a\|^2} - 2r\sum_{j=1}^{n}\frac{R_{lj}(x_j-a_j)(x_i-a_i)}{\|x-a\|^4}$$

$$\frac{\partial^2\psi_l}{\partial x_i^2} = -\frac{4rR_{li}(x_i-a_i)}{\|x-a\|^4} + \sum_{j=1}^{n}\left(-\frac{2rR_{lj}(x_j-a_j)}{\|x-a\|^4} + \frac{8rR_{lj}(x_j-a_j)(x_i-a_i)^2}{\|x-a\|^6}\right)$$

and compute the Laplace operator

$$\Delta\psi_l = \sum_{i=1}^{n}\frac{rR_{li}(x_i-a_i)}{\|x-a\|^4}(-4-2n+8) = \sum_{i=1}^{n}\frac{(4-2n)rR_{li}(x_i-a_i)}{\|x-a\|^4}. \tag{32}$$

Now we are ready to compute the right hand side of (30)

$$
\begin{aligned}
&-\frac{1}{2\varkappa^2}\sum_{l=1}^{n}\Delta\psi_l\frac{\partial\psi_l}{\partial x_i}= \\
&-\frac{\|x-a\|^4}{2r^2}\sum_{l,p=1}^{n}\Bigg[\frac{(4-2n)rR_{lp}(x_p-a_p)}{\|x-a\|^4} \\
&\qquad\qquad\left(\frac{rR_{li}}{\|x-a\|^2}-\sum_{j=1}^{n}\frac{2rR_{lj}(x_j-a_j)(x_i-a_i)}{\|x-a\|^4}\right)\Bigg]= \\
&\sum_{l,p=1}^{n}\Bigg(\frac{(n-2)R_{li}R_{lp}(x_p-a_p)}{\|x-a\|^2} \\
&\qquad\qquad-\sum_{j=1}^{n}\frac{2(n-2)R_{lj}(x_j-a_j)R_{lp}(x_p-a_p)(x_i-a_i)}{\|x-a\|^4}\Bigg)= \\
&-\frac{(n-2)(x_i-a_i)}{\|x-a\|^2}
\end{aligned}
$$

and we see that it is the same as the left hand side (compare it with (31)). In the case (24), (26) both left hand side and right hand side are identically equal to zero on U. □

Lemma 3. *Denote the result of both sides of* (30) *by* B_i. *That means*

$$
B_i=\begin{cases}(2-n)\frac{x_i-a_i}{\|x-a\|^2} & \textit{in the case } (23)\\ 0 & \textit{in the case } (24).\end{cases}\tag{33}
$$

Then the following identity

$$
\sum_{i=1}^{n}\left(B_i^2+\frac{\partial B_i}{\partial x_i}\right)=0\tag{34}
$$

holds for all $x\in\mathbb{R}^n-\{a\}$ *in the case* (23) *and for all* $x\in\mathbb{R}^n$ *in the case* (24).

Proof: can be obtained by a straightforward calculation and we omit it. □

4. Describing of W-morphisms

In this section we give a description of all W-morphisms for an arbitrary $k\in\mathbb{R}$ and $n\geq 3$. We do it by solving the equations (3)–(6). In the previous section we saw that the mapping Ψ should be a Möbius transformation and can be expressed as in (23) or (24). Equations (6) are used to compute φ. From (6) we can compute the scalar product

$$
\nabla\varphi\cdot\nabla\psi_i=\frac{k\varphi\varkappa^2}{2\psi_n}\delta_{in}-\frac{1}{2}\varphi W(\psi_i).
$$

Since the vectors $\nabla\psi_i$ for $i = 1, \ldots, n$ form an orthogonal basis with the same length $\|\nabla\psi_i\| = \varkappa$ we can use the relation

$$\nabla\varphi = \sum_{i=1}^{n} \frac{\nabla\varphi \cdot \nabla\psi_i}{\varkappa^2} \nabla\psi_i$$

to write

$$\nabla\varphi = \frac{k\varphi}{2\psi_n}\nabla\psi_n - \sum_{i=1}^{n} \frac{\varphi}{2\varkappa^2} W(\psi_i)\nabla\psi_i.$$

Now we use the definition of W and write the last vector relation in coordinates

$$\frac{\partial\varphi}{\partial x_j} = \frac{k\varphi}{2\psi_n}\frac{\partial\psi_n}{\partial x_j} - \frac{\varphi}{2\varkappa^2}\sum_{i=1}^{n} \Delta(\psi_i)\frac{\partial\psi_i}{\partial x_j} - \frac{\varphi}{2\varkappa^2}\frac{k}{x_n}\sum_{i=1}^{n} \frac{\partial\psi_i}{\partial x_n}\frac{\partial\psi_i}{\partial x_j}. \tag{35}$$

Further we can use $\sum_{i=1}^{n} \frac{\partial\psi_i}{\partial x_n}\frac{\partial\psi_i}{\partial x_j} = \varkappa^2\delta_{jn}$ and (30). Then we get

$$\frac{\partial\varphi}{\partial x_j} = \frac{k\varphi}{2\psi_n}\frac{\partial\psi_n}{\partial x_j} + \frac{(n-2)\varphi}{2}\frac{\partial(\log\varkappa)}{\partial x_j} - \frac{\varphi}{2}\frac{k}{x_n}\delta_{jn}$$

which can be integrated as

$$\log\varphi = \frac{k}{2}\log\psi_n + \frac{n-2}{2}\log\varkappa - \frac{k}{2}\log x_n + \log C$$

or

$$\varphi = C\left(\frac{\psi_n}{x_n}\right)^{\frac{k}{2}} \varkappa^{\frac{n-2}{2}}. \tag{36}$$

Relations (23) or (24), respectively, together with (36) give all solutions of (4)–(6). Let us check equation (3) now. To compute $W(\varphi)$ we use (35) and write $\frac{\partial\varphi}{\partial x_j} = A_j\varphi$ with

$$A_j = \frac{k}{2\psi_n}\frac{\partial\psi_n}{\partial x_j} - \frac{1}{2\varkappa^2}\sum_{i=1}^{n}\left(\Delta(\psi_i)\frac{\partial\psi_i}{\partial x_j}\right) - \frac{k}{2x_n}\delta_{jn}. \tag{37}$$

Using A_j we can write the expression $W(\varphi)$ as

$$W(\varphi) = \varphi \cdot \left(\sum_{j=1}^{n}\left(A_j^2 + \frac{\partial A_j}{\partial x_j}\right) + \frac{k}{x_n}A_n\right). \tag{38}$$

Introducing

$$C_j := \frac{k}{2\psi_n}\frac{\partial \psi_n}{\partial x_j} - \frac{k}{2x_n}\delta_{jn} \tag{39}$$

and using B_j introduced in (33) in the form

$$B_j := -\frac{1}{2\varkappa^2}\sum_{l=1}^{n}\Delta\psi_l\frac{\partial \psi_l}{\partial x_j}\,, \tag{40}$$

we can write A_j as

$$A_j = B_j + C_j.$$

Putting it to (38) and using (34) we get

$$W(\varphi) = \varphi\cdot\left(\sum_{j=1}^{n}\left(2B_jC_j + C_j^2 + \frac{\partial C_j}{\partial x_j}\right) + \frac{k}{x_n}B_n + \frac{k}{x_n}C_n\right). \tag{41}$$

Let us compute (41) part by part

$$\begin{aligned}
\sum_{j=1}^{n} 2B_jC_j &= -\frac{k}{2\psi_n}\Delta\psi_n + \frac{k}{2x_n\varkappa^2}\sum_{i=1}^{n}\Delta\psi_i\frac{\partial\psi_i}{\partial x_n}\\
\sum_{j=1}^{n} C_j^2 &= \frac{k^2}{4}\left(\frac{\varkappa^2}{\psi_n^2} + \frac{1}{x_n^2} - \frac{2}{x_n\psi_n}\frac{\partial\psi_n}{\partial x_n}\right)\\
\sum_{j=1}^{n}\frac{\partial C_j}{\partial x_j} &= \frac{k}{2}\left(-\frac{\varkappa^2}{\psi_n^2} + \frac{1}{\psi_n}\Delta\psi_n + \frac{1}{x_n^2}\right)\\
\frac{k}{x_n}B_n &= -\frac{k}{2x_n\varkappa^2}\sum_{i=1}^{n}\Delta\psi_i\frac{\partial\psi_i}{\partial x_n}\\
\frac{k}{x_n}C_n &= \frac{k^2}{2x_n}\left(\frac{1}{\psi_n}\frac{\partial\psi_n}{\partial x_n} - \frac{1}{x_n}\right).
\end{aligned} \tag{42}$$

Substituting (42) to (41) we get

$$W(\varphi) = \varphi\frac{k(k-2)}{4}\left(\frac{\varkappa^2}{\psi_n^2} - \frac{1}{x_n^2}\right). \tag{43}$$

We see that in cases $k = 0$, $k = 2$ relations (23), (36) and (24), (36) give all W-morphisms while in the cases $k \neq 0$, $k \neq 2$, transformations are admitted with $\psi_n = x_n\varkappa$ only. Note that we have proved in Lemma 1 that $\psi_n = x_n\varkappa$ is equivalent to that Möbius transformation is a bijection on $\mathbb{R}^n_+$. We can summarize these facts in the

Theorem. *Let $n \geq 3$, $\mathbb{R}^n_+$ be as above, $U, V \subset \mathbb{R}^n_+$ domains, $\Phi : U \to V$ a bijection, $\varphi : U \to (0, \infty)$ a positive function and W the Weinstein operator with $k \in \mathbb{R}$. Then the transformation $T : \mathcal{C}^2(V) \to \mathcal{C}^2(U)$ defined by $Tu := \varphi \cdot (u \circ \Psi)$ is a W-morphism iff*

1. *in cases $k \neq 0$, $k \neq 2$, the bijection Ψ is a Möbius transformation mapping $\mathbb{R}^n_+$ onto itself such that $\Psi(U) = V$ and $\varphi = \varkappa^{\frac{n+k-2}{2}}$,*
2. *in cases $k = 0$ or $k = 2$, Ψ is an arbitrary Möbius transformation with $\Psi(U) = V$ and $\varphi = \left(\frac{\psi_n}{x_n}\right)^{\frac{k}{2}} \varkappa^{\frac{n-2}{2}}$.*

5. Conclusion remarks

There are two natural questions.

1. A different result in the case $k = 0$ (i.e. $W = \Delta$) seems to be natural, but why is it in the case $k = 2$?
2. Is it possible to use Theorem to derive the Poisson formula for the operator $\Delta + \frac{2}{x_n}\frac{\partial}{\partial x_n}$?

The answer to both problems is surprisingly simple. Let $U \subset \mathbb{R}^n_+$ be a domain and take $u \in \mathcal{C}^2(U)$. Then $\Delta\left(x_n u(x)\right) = x_n \Delta u \; + 2\frac{\partial u}{\partial x_n} = x_n \left(\Delta u + \frac{2}{x_n}\frac{\partial u}{\partial x_n}\right)$. We see that when a function u is a solution of the Weinstein operator for $k = 2$, then the function $v(x) = x_n u(x)$ is harmonic on U. That means that properties of harmonic functions, including the Poisson formula, can be modified to the solutions of the Weinstein operator with the coefficient $k = 2$.

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