On interval homogeneous orthomodular lattices

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Abstract. An orthomodular lattice L is said to be *interval homogeneous* (resp. *centrally interval homogeneous*) if it is σ-complete and satisfies the following property: Whenever L is isomorphic to an interval, $[a,b]$, in L then L is isomorphic to each interval $[c,d]$ with $c \leq a$ and $d \geq b$ (resp. the same condition as above only under the assumption that all elements a, b, c, d are central in L).

Let us denote by *Inthom* (resp. *Inthom$_c$*) the class of all interval homogeneous orthomodular lattices (resp. centrally interval homogeneous orthomodular lattices). We first show that the class *Inthom* is considerably large — it contains any Boolean σ-algebra, any block-finite σ-complete orthomodular lattice, any Hilbert space projection lattice and several other examples. Then we prove that L belongs to *Inthom* exactly when the Cantor-Bernstein-Tarski theorem holds in L. This makes it desirable to know whether there exist σ-complete orthomodular lattices which do not belong to *Inthom*. Such examples indeed exist as we than establish. At the end we consider the class *Inthom$_c$*. We find that each σ-complete orthomodular lattice belongs to *Inthom$_c$*, establishing an orthomodular version of Cantor-Bernstein-Tarski theorem. With the help of this result, we settle the Tarski cube problem for the σ-complete orthomodular lattices.

Keywords: interval in a σ-complete orthomodular lattice, center, Boolean σ-algebra, Cantor-Bernstein-Tarski theorem

Classification: 06C15, 06E05, 81P10

1. Basic notions

We shall be exclusively interested in σ-complete orthomodular lattices (abbreviated OMLs), i.e. in those OMLs which are closed under the formations of countable suprema and infima (we refer to [1], [4] and [8] for the background on OMLs). We shall frequently use the elementary fact (see [8]) that an interval in a σ-complete OML constitutes, with the operations naturally inherited from the host OML, a σ-complete OML. If L is an OML, we shall define the center of L as the Boolean sub-σ-algebra consisting of all "absolutely compatible" elements, i.e., as the set of all elements compatible to each element of L (see [8]). As known, L is a Boolean σ-algebra if and only if all its elements are central.

Recall that a sequence $(a_n)_{n\in\mathbb{N}}$ of pairwise orthogonal elements in the center of an OML is called a central partition of unity if $\bigvee_{n\in\mathbb{N}} a_n = 1$.

Let us consider two σ-complete OMLs. By an isomorphism between them we mean a bijective mapping f such that both f and f^{-1} are OML morphisms (thus, as a consequence, f and f^{-1} preserve countable infima and suprema). We shall be interested in the class of those σ-complete OMLs L which, roughly speaking, satisfy the following homogeneity condition: If an interval in L is found isomorphic to L, then it has to coincide with all its hyperintervals in L. Let us formally introduce this class in the following definition. Besides the natural meaning of this class within the theory of OMLs, it may be of significance in the logico-algebraic foundations of quantum theories, too (see also Theorem 2.1 in the next paragraph).

Definition 1.1. *Let L be an OML. Then L is said to be* interval homogeneous *if it is σ-complete and enjoys the following property: If, for some $a, b \in L$, $a \leq b$, the interval $[a, b]$ is isomorphic to the entire L, then L is isomorphic to each interval $[c, d]$ with $c \leq a$ and $d \geq b$ ($c, d \in L$).*

If all the elements $a, b, c, d \in L$ from the previous definition are supposed to be taken from the center of L, then L is called centrally interval homogeneous.

Let us denote the class of all interval homogeneous OMLs (resp. centrally interval homogeneous OMLs) by *Inthom* (resp. by $Inthom_C$). Obviously, *Inthom* $\subseteq$ $Inthom_C$. Before we exhibit basic examples of the σ-complete OMLs which belong to *Inthom*, let us observe that our definition can be rephrased in a slightly simplified form.

Proposition 1.2. *An OML L belongs to Inthom if and only if L enjoys the following property: If, for some $a \in L$, the interval $[0, a]$ is isomorphic to the entire L, then L is isomorphic to the interval $[0, b]$ for each $b \geq a$ ($b \in L$).*

Proof: Let L satisfy the property stated in Proposition 1.2. We want to show that L belongs to *Inthom*. Assume that, for some $a, b \in L$, $a \leq b$, the interval $[a, b]$ is isomorphic to L. Since $[a, b]$ is isomorphic to $[0, b \wedge a']$ (see e.g. [8, Proposition 1.3.12]), we infer that L is isomorphic to $[0, b \wedge a']$. Take arbitrary elements $c, d \in L$ with $c \leq a$ and $d \geq b$. The relations $d \wedge c' \geq b \wedge a'$ and $[0, d \wedge c'] \cong [c, d]$ then imply that $L \cong [0, d \wedge c'] \cong [c, d]$. $\square$

2. Interval homogeneous OMLs and the Cantor-Bernstein-Tarski theorem

In our first result we list basic examples of OMLs that belong to *Inthom*. As usual, let us call a maximal Boolean subalgebra of an OML a *block*. As we assume σ-completeness of the OML, each block is σ-complete, too. Let us use the following notations: let us denote by $\mathbb{N}$ the set of all positive integers, and let us further set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$.

Theorem 2.1. *Let L be a σ-complete OML. Each of the following conditions guarantees that L belongs to Inthom:*

(a) *Each block of L is finite.*
(b) *L is a Boolean σ-algebra.*
(c) *L is the lattice of projections in a Hilbert space.*
(d) *L possesses only finitely many blocks (i.e., L is made up of finitely many Boolean σ-algebras).*

Proof: (a) Let $i\colon L \to [0,a]$ be an isomorphism for some element $a \in L$, $a < 1$. Define a sequence, $(a_n)_{n\in\mathbb{N}_0}$, by putting $a_0 = 1$ and $a_{n+1} = i(a_n)$. Thus $(a_n)_{n\in\mathbb{N}_0}$ is a strictly decreasing sequence which is contained in a block of L — a contradiction. Thus, $L \in$ *Inthom*.

(b) If L is a Boolean σ-algebra, then it is known from the Boolean algebra theory that L belongs to *Inthom* ([6, Theorem 12.4, p. 180]). A proof can be also obtained from our more general result of Theorem 3.1 which follows. (It is perhaps worthwhile observing that yet another proof can be provided via the Loomis-Sikorski theorem. Here is a sketch of the argument. One first applies the famous Cantor-Bernstein iterating mechanism to set-representable Boolean σ-algebras — the isomorphism of L and $[a,b]$ can be made pointwise by [9] — and then, for general (possibly non set-representable) Boolean σ-algebras, one completes the proof by using the Loomis–Sikorski theorem.)

(c) Let H be a Hilbert space. Let us denote by $\mathcal{L}(H)$ the lattice of projections in H. Set $L = \mathcal{L}(H)$. If $\dim H < \aleph_0$, then all blocks are finite and the case (a) applies. If $\dim H = \aleph_0$, then the result is easy — for each infinite-dimensional subspace M of L we obviously have $\mathcal{L}(H) \cong \mathcal{L}(M)$. If $\dim H > \aleph_0$, then $\mathcal{L}(H) \in$ *Inthom* by a simple cardinality argument.

(d) Let $i\colon L \to [0,a]$ be an isomorphism for some element $a \in L$, $a < 1$. Define a sequence, $(a_n)_{n\in\mathbb{N}_0}$, by putting $a_0 = 1$ and $a_{n+1} = i(a_n)$. Thus, $a_1 = a$ and $(a_n)_{n\in\mathbb{N}_0}$ is a strictly decreasing sequence. Due to the σ-completeness of L, there exists the infimum of $(a_n)_{n\in\mathbb{N}_0}$, $c = \bigwedge_{n\in\mathbb{N}_0} a_n$. We distinguish two cases depending on whether or not the element a is central.

(i) Suppose that a is central in L. Then $L \cong [0,a] \times [0,a']$ and the property of being central in $[0,a]$ implies being central in L.

As L has finitely many blocks, according to [4, Theorem 4, p. 40] (see also [2]) it is isomorphic to the product $L \cong B \times K$, where B is a Boolean σ-algebra which is maximal in the sense that K cannot be decomposed into a product of a nontrivial Boolean σ-algebra and a (possibly trivial) OML. The decomposition $L \cong B \times K$ corresponds to the existence of a central element $k \in L$ such that $K \cong [0,k]$, $B \cong [0,k']$. The image of $[0,k']$ under i is the interval $[0,i(k')]$ and it is a Boolean σ-algebra. Due to the maximality of B, $[0,k] \cong K$ has no factor which is a nontrivial Boolean σ-algebra, therefore $i(k') \le k'$. The image of $[0,k]$ under i is the interval $[0,i(k)]$, where $i(k)$ is central in L, and no nontrivial factor of $[0,i(k)]$ is a Boolean σ-algebra. Thus it is a subinterval of $[0,k]$ and $i(k) \le k$. We obtained

$$i(k) \vee i(k') = i(k \vee k') = i(1) = a = (a \wedge k) \vee (a \wedge k'),$$

where all the joins in the latter equality are orthogonal. As $i(k) \le k$, $i(k') \le k'$, the two decompositions of a coincide, i.e.,

$$i(k) = a \wedge k\ , \quad i(k') = a \wedge k'.$$

Since any isomorphism maps central elements onto central elements, all elements a_n ($n \in \mathbb{N}_0$) as well as the element c are central in $[0, a]$ and in L. We therefore have a central partition of unity in L, $(c, a_0 \wedge a_1', a_1 \wedge a_2', \dots , a_n \wedge a_{n+1}', \dots)$, which gives us the isomorphism

$$L \cong [0, c] \times \prod_{n \in \mathbb{N}_0} [0, a_n \wedge a_{n+1}'].$$

Moreover, all the factors $[0, a_n \wedge a_{n+1}']$ in the latter decomposition are isomorphic. If $[0, a_n \wedge a_{n+1}']$ has more than one block, then L has infinitely many blocks since the blocks of L correspond to the products of blocks of the factors (see e.g. [2]). This contradicts the hypothesis on L. Thus $[0, a_n \wedge a_{n+1}']$ is a Boolean σ-algebra for all $n \in \mathbb{N}_0$. In particular, $[0, a_0 \wedge a_1'] = [0, a']$ is a Boolean σ-algebra, hence $a' \leq k'$. We proved that $i(k) = a \wedge k = k$, so i restricted to $[0, k]$ is an automorphism. The restriction of i to $[0, k']$ is an isomorphism of a Boolean σ-algebra $[0, k']$ and its subinterval $[0, a \wedge k']$. The standard Cantor-Bernstein-Tarski theorem for Boolean σ-algebras then completes the proof.

(ii) Suppose that there is an element b in L which is not compatible to a. Define a sequence, $(b_n)_{n \in \mathbb{N}}$, in L by setting $b_1 = b$, and $b_{n+1} = i(b_n)$ ($n \in \mathbb{N}$). Obviously $b < 1$, i.e., $b_1 < a$, so we have also $b_{n+1} < a_n$ for each $n \in \mathbb{N}$, and we obtain the chain

$$b_{n+1} < a_n < a_{n-1} < \cdots < a_1.$$

Therefore there is a Boolean sub-σ-algebra of L which contains the set $\{b_{n+1}, a_n, a_{n-1}, \dots a_1\}$. But b_{n+1} is not compatible to a_{n+1}. As a result of the previous considerations, for each $n \in \mathbb{N}$ there exists a block in L containing the set $\{b_{n+1}, a_n, a_{n-1}, \dots , a_1\}$ but not containing the element b_n. This means that there exist infinitely many distinct blocks in L. This is a contradiction. It follows that the case (i) above applies and therefore $L \in$ *Inthom.* $\square$

In the next result we observe that the relation of *Inthom* to Cantor-Bernstein-Tarski theorem known for Boolean σ-algebras can be generalized to σ-complete OMLs.

Proposition 2.2. *The following statements on L are equivalent:*

i) *$L \in$ Inthom.*
ii) *The Cantor-Bernstein-Tarski theorem holds true for L: If M is a σ-complete OML such that L is isomorphic to an interval $[0, b]_M$ in M, and M is isomorphic to an interval $[0, a]_L$ in L, then L is isomorphic to M.*

PROOF: i) $\Rightarrow$ ii): Let us assume that $L \in$ *Inthom* and that there exists a σ-complete orthomodular lattice M with two isomorphisms $\alpha: L \to [0, b]_M$ and $\beta: M \to [0, a]_L$. Since the restriction, $\tilde{\beta}$, of β to the interval $[0, b]_M$ is an isomorphism between $[0, b]_M$ and $[0, \beta(b)]_L$, we see that $\tilde{\beta} \circ \alpha: L \to [0, \beta(b)]_L$ is an isomorphism. The assumption on L plus the obvious relation $\beta(b) \leq a$ then imply that L is isomorphic to $[0,a]_L$. Thus, $L \cong M$.

ii) $\Rightarrow$ i): This implication is obviously true — it suffices to take $M = [0,b]_L$ in Proposition 1.2. □

The previous result allows us to easily exhibit OMLs which do not belong to *Inthom*.

Theorem 2.3. *The class Inthom is not closed under the formation of products. A consequence: There exist σ-complete OMLs which do not belong to Inthom.*

Proof: Let $K = \{0, 1, x, x', y, y'\}$ (this lattice is often denoted by MO_2 – see [4]). Then K obviously belongs to *Inthom*. Let $K_n = K$ for each $n \in \mathbb{N}$. Take $L = \prod_{n=1}^{\infty} K_n$. Then the Cantor-Bernstein-Tarski theorem does not hold true for L. Indeed, let $M = \{0,1\} \times L$. Then we can easily find an isomorphism of L onto an interval in M (take e.g. the interval $\{0\} \times L = [0,(0,1)]_M$), and we can also find an isomorphism of M onto an interval in L (take e.g. the interval $\{0,x\} \times \prod_{n=2}^{\infty} K_n = [0,(x,1,1,\dots)]_L$). But L is obviously not isomorphic to M since M possesses a central atom — a minimal nonzero element in the center — here it is the element $(1,0,0,\dots)$ but L does not. The proof is complete. □

Let us comment on the previous result. It implies that there are in fact modular set-representable complete OMLs which do not belong to *Inthom* — we have just constructed one. This result can be understood in such a way that there are OMLs which are intrinsically fairly close to Boolean σ-algebras and yet do not belong to *Inthom*. It should be noted that there are also examples of OMLs which are intrinsically fairly close to $\mathcal{L}(H)$ and do not belong to *Inthom* either. Indeed, it is easily seen that if we take the lattice $\mathcal{L}(\mathbb{R}^3)$ for K_n ($n \in \mathbb{N}$) in the above construction, we obtain an OML, L, such that $L \notin$ *Inthom*. A quantum logic reformulation of this fact is this (see e.g. [8] for the investigation of the Jauch-Piron property): There are Jauch-Piron OMLs which do not belong to *Inthom*.

3. Centrally interval homogeneous OMLs and the Tarski cube problem

In the final part of this paper we shall investigate the class $Inthom_c$. Making use of Proposition 1.2, $L \in Inthom_c \Leftrightarrow$ if L is isomorphic to $[0,a]$, a central in L, then L is isomorphic to $[0,b]$ for each central $b \in L$, $b \geq a$.

Theorem 3.1. *Each σ-complete OML belongs to $Inthom_c$. A corollary (the <u>Cantor-Bernstein-Tarski theorem in OMLs</u>): Let L, M be σ-complete OMLs and let L be isomorphic to $[0,b]_M$ for a central element $b \in M$ and M be isomorphic to $[0,a]_L$ for a central element $a \in L$. Then L is isomorphic to M.*

Proof: Let L be a σ-complete OML. Let a, b be two central elements in L and let $a \leq b$. Let $i\colon L \to [0,a]$ be an isomorphism. Define the sequences $(a_n)_{n\in\mathbb{N}_0}$

and $(b_n)_{n\in\mathbb{N}_0}$ by induction:

$$\begin{aligned} a_0 &= 1, & b_0 &= b, \\ a_{n+1} &= i(a_n), & b_{n+1} &= i(b_n). \end{aligned}$$

Note that

$a_0 \geq b_0 \geq a_1 \geq b_1 \geq a_2 \geq b_2 \geq \dots,$

where

$a_1 = i(1) = a,$
$a_n = i^n(1) = i^{n-1}(a),$
$b_n = i^n(b).$

We chose a, b central in L. In other words, a_1, b_0 are central in $[0, a_0]$. An isomorphism maps central elements onto central elements, therefore by induction $i^n(a_1) = a_{n+1}$, $i^n(b_0) = b_n$ are central in $[0, i^n(a_0)] = [0, a_n]$. As a consequence, all a_n, $n \in \mathbb{N}$, as well as all b_n, $n \in \mathbb{N}$, are central in L. Due to the σ-completeness of L, there exists the infimum,

$$c = \bigwedge_{n\in\mathbb{N}_0} a_n = \bigwedge_{n\in\mathbb{N}_0} b_n.$$

Moreover, c is also central. Thus, we have a central partition of unity

$$(c, a_0 \wedge b_0', b_0 \wedge a_1', a_1 \wedge b_1', b_1 \wedge a_2', \dots)$$

in L and each $x \in L$ admits a unique decomposition with respect to it,

$$x = (x \wedge c) \vee \bigvee_{n\in\mathbb{N}_0} (x \wedge a_n \wedge b_n') \vee \bigvee_{n\in\mathbb{N}_0} (x \wedge b_n \wedge a_{n+1}').$$

It is easy to see that c is a fixed point of the mapping i. The restriction of the isomorphism i to the interval $[0, c]$ is obviously an isomorphism.

In the final step, one only checks that the function φ defined by

$$\varphi(x) = (x \wedge c) \vee \bigvee_{n\in\mathbb{N}_0} i(x \wedge a_n \wedge b_n') \vee \bigvee_{n\in\mathbb{N}_0} (x \wedge b_n \wedge a_{n+1}')$$

is an isomorphism of L onto $[0, b]$. Indeed, φ restricts to the identity on the intervals

$$[0, c],\ [0, b_n \wedge a_{n+1}'],\ n \in \mathbb{N}_0,$$

and to isomorphisms

$$[0, a_n \wedge b_n'] \to [0, a_{n+1} \wedge b_{n+1}'],\ n \in \mathbb{N}_0.$$

For the range $\varphi(L)$ we may write

$$\begin{aligned}\varphi(L) &\cong [0,c] \times \prod_{n\in\mathbb{N}_0} [0, b_n \wedge a'_{n+1}] \times \prod_{n\in\mathbb{N}_0} [0, a_{n+1} \wedge b'_{n+1}] \\ &\cong [0, (a_0 \wedge b'_0)'] = [0,b].\end{aligned}$$

The proof is complete. □

Let us make final remarks. First, it should be noted that the σ-complete setup of the problem pursued here seems well justified by the Boolean algebra results. To demonstrate that, let us tentatively denote by *inthom* the class of OMLs defined in the category of (generally non σ-complete) OMLs in the full analogy with *Inthom*. Then the fact is that there is even a Boolean algebra which does not lie in *inthom*. An example can be constructed easily on the ground of the so-called "Tarski cube phenomenon": There is a Boolean algebra A such that A^2 is not Boolean isomorphic to A but A is Boolean isomorphic to A^3 (see [3] and [5]). This Boolean algebra obviously does not belong to *inthom*. An interesting question arises: Since the phenomenon $A \not\cong A^2$, $A \cong A^3$ obviously cannot occur for Boolean σ-algebras, can it occur for σ-complete OMLs? It cannot as Theorem 3.1 implies — if there is a σ-complete OML with $A \not\cong A^2$ and $A \cong A^3$, then $A \cong [(0,0,0),(0,0,1)]_{A^3}$ and $A^2 \cong [(0,0,0),(0,1,1)]_{A^3}$. This is in contradiction with Theorem 3.1 because the elements $(0,0,1)$ and $(0,1,1)$ are central in A^3. Let us explicitly record the latter result.

Theorem 3.2. *Let L be a σ-complete OML. If $L \not\cong L^2$, then $L \not\cong L^n$ for any $n \in \mathbb{N}$, $n > 1$.*

Acknowledgment. This research was carried on while the first author was on a study leave in Prague. She would like to gratefully acknowledge the financial support of C.N.R., Italy, which made the stay possible. The authors also acknowledge the support of Research Programme J04/98:212300013 of the Czech Ministry of Education and the grant no. 201/98/1153 of the Grant Agency of the Czech Republic. The authors thank the referee for valuable remarks.

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(*Received* January 12, 2000)