Remark on regularity of weak solutions to the Navier-Stokes equations

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Abstract. Some results on regularity of weak solutions to the Navier-Stokes equations published recently in [3] follow easily from a classical theorem on compact operators. Further, weak solutions of the Navier-Stokes equations in the space $L^2(0,T,W^{1,3}(\Omega)^3)$ are regular.

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Introduction

Let Ω be a bounded domain in R^3 with C^2-boundary $\partial\Omega$, let $T>0$ and $Q_T=\Omega\times(0,T)$. We consider the Navier-Stokes initial-boundary value problem describing the evolution of the velocity $\boldsymbol{u}(\boldsymbol{x},t)$ and the pressure $p(\boldsymbol{x},t)$ in Q_T:

$$\frac{\partial \boldsymbol{u}}{\partial t}-\nu\Delta\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p=\boldsymbol{f}, \tag{1}$$

$$\nabla\cdot\boldsymbol{u}=0, \tag{2}$$

$$\boldsymbol{u}=\boldsymbol{0} \qquad \text{on} \quad \partial\Omega\times(0,T), \tag{3}$$

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0, \tag{4}$$

where $\nu>0$ is the viscosity coefficient and $\boldsymbol{f}$ is the external body force. The initial data $\boldsymbol{u}_0$ should satisfy the compatibility conditions $\boldsymbol{u}_0|_{\partial\Omega}=\boldsymbol{0}$ and $\nabla\cdot\boldsymbol{u}_0=0$.

The definition and the proof of the existence of weak solutions of the equations (1)–(4) can be found for example in [3] or [6]. In general, it is unknown whether weak solutions are regular or not. Serrin ([5]) proved that if a weak solution $\boldsymbol{u}$ of (1)–(4) belongs to $L^\alpha(0,T,L^q(\Omega))$ for $2/\alpha+3/q=1$ and $q\in(3,\infty]$ then $\boldsymbol{u}$ is regular. Kozono ([3]) generalized this result to a certain class of functions characterized by means of local singularities in the weak-L^3 space. He further showed that there exists an absolute constant $\varepsilon>0$ such that if $\boldsymbol{u}$ is a weak solution of (1)–(4) in $L^\infty(0,T,L^3(\Omega)^3)$ and $\limsup_{t\to t_*-}\ \|\boldsymbol{u}(t)\|_{L^3(\Omega)}<\|\boldsymbol{u}(t_*)\|_{L^3(\Omega)}+\varepsilon$, then $\boldsymbol{u}$ is necessarily regular in $\Omega\times(t_*-\sigma,t_*+\sigma)$ for some $\sigma>0$. Let us mention here that the Kozono's results were applied in [4] where partial regularity of weak solutions to the Navier-Stokes equations in the class $L^\infty(0,T,L^3(\Omega))$ was shown.

The main goal of this paper is to show that the results stated above can be easily derived from the following well known theorem on compact operators ([2]):

Theorem A. *Let X, Y be Banach spaces. Let S be a one to one continuous linear operator from X onto Y and K a linear compact operator from X to Y. If $Ker(S+K)=o$ then $(S+K)(X)=Y$.*

Let $p>1$. $L^p(\Omega)$ is the Lebesgue space with the norm $\|\cdot\|_p$. $C_0^\infty(\Omega)$ denotes the set of all infinitely differentiable vector-functions defined in Ω, with a compact support in Ω. $C_{0,\sigma}^\infty(\Omega)$ is a subset of $C_0^\infty(\Omega)$ which contains only the divergence-free vector functions. H is the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)^3$ with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|_2$. $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ $(m\in N)$ are the usual Sobolev spaces. V denotes the completion of $C_{0,\sigma}^\infty(\Omega)$ in the norm of $W_0^{1,2}(\Omega)^3$ with the scalar product $((\boldsymbol{u},\boldsymbol{v}))=\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,d\boldsymbol{x}$ and the norm $\|\cdot\|$. P_H is the projection operator from $L^2(\Omega)^3$ onto H.

$L_w^p(\Omega)$ denotes the weak Lebesgue space over Ω with the quasi-norm $\|\cdot\|_{p,w}$ defined by $\|\phi\|_{p,w}=\sup_{R>0} R\mu\{\boldsymbol{x}\in\Omega;|\phi(\boldsymbol{x},t)|>R\}^{1/p}$, where μ is the Lebesgue measure. There is another equivalent norm to the above $\|\cdot\|_{p,w}$ (see [3]), so we may understand $L_w^p(\Omega)$ as a Banach space. Let us note that $L^p(\Omega)\subseteq L_w^p(\Omega)$ and $\|\phi\|_{p,w}\le\|\phi\|_p$ for every $\phi\in L^p(\Omega)$.

Let $D(A)=\{\boldsymbol{u}\in V;\exists \boldsymbol{f}\in H;((\boldsymbol{u},\boldsymbol{v}))=(\boldsymbol{f},\boldsymbol{v})\ \forall \boldsymbol{v}\in V\}$. A is the Stokes operator from $D(A)$ onto H defined for every $\boldsymbol{u}\in D(A)$ by the equation $((\boldsymbol{u},\boldsymbol{v}))=(A\boldsymbol{u},\boldsymbol{v})\ \forall\boldsymbol{v}\in V$. $D(A)$ is endowed with the norm $\|\boldsymbol{u}\|_{D(A)}=\|A\boldsymbol{u}\|_2$ and $D(A)\hookrightarrow\hookrightarrow V$. Since $\Omega\in C^2$, $D(A)=W^{2,2}(\Omega)^3\cap V$ and the norm $\|\boldsymbol{u}\|_{D(A)}$ on $D(A)$ is equivalent to the norm induced by $W^{2,2}(\Omega)^3$ (see [6, Lemma 3.7]). We often use this fact throughout the paper. Let us define the Banach spaces $X=\{\boldsymbol{u}\in L^2(0,T,D(A)),\boldsymbol{u}_t\in L^2(0,T,H)\}$ and $Y=L^2(0,T,H)\times V$ with $\|\boldsymbol{u}\|_X=\|\boldsymbol{u}\|_{L^2(0,T,D(A))}+\|\boldsymbol{u}_t\|_{L^2(0,T,H)}$ and $\|(\boldsymbol{f},\boldsymbol{v}_0)\|_Y=\|\boldsymbol{f}\|_{L^2(0,T,H)}+\|\boldsymbol{v}_0\|_V$.

Throughout the paper, we suppose that in (1)–(4) $\boldsymbol{f}\in L^2(0,T,H)$ and $\boldsymbol{u}_0\in H$. For simplicity, we use the following notation: If F is a space of real functions then $\boldsymbol{u}\in F$ means that every component of $\boldsymbol{u}$ is from F, e.g. $\boldsymbol{u}\in W^{1,2}(\Omega)$ means in fact that $\boldsymbol{u}\in W^{1,2}(\Omega)^3$. Similarly, $\|\boldsymbol{u}\|_F$ means $\|\boldsymbol{u}\|_{F^3}$.

Proof of regularity results

At first, we prove two basic propositions. The results mentioned in Introduction will then be their straightforward consequences.

Proposition 1. *Let $\boldsymbol{u}\in L^\alpha(0,T,L^q(\Omega))$ for $2/\alpha+3/q\le 1$ and $q\in(3,\infty]$. Then the operator $\boldsymbol{w}\longmapsto P_H(\boldsymbol{u}\cdot\nabla\boldsymbol{w})$ is compact from X to $L^2(0,T,H)$.*

PROOF: Firstly, suppose that $2/\alpha+3/q<1$ and $\alpha,q<\infty$. Using the Hőlder inequality we have for almost every $t\in(0,T)$ and every $\boldsymbol{v}\in H$:

$$|\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{v}\,d\boldsymbol{x}|\le\|\boldsymbol{v}\|_2\|\boldsymbol{u}\|_q\|\nabla\boldsymbol{w}\|_{2q/(q-2)}.$$

It follows further that

$$\int_0^T \|\boldsymbol{u}\|_q^2 \|\nabla \boldsymbol{w}\|_{2q/(q-2)}^2\ dt \le \|\boldsymbol{u}\|^2_{L^\alpha(0,T,L^q(\Omega))} \big(\int_0^T \|\nabla \boldsymbol{w}\|_{2q/(q-2)}^{2\alpha/(\alpha-2)}\ dt\big)^{(\alpha-2)/\alpha} \le$$
$$\|\boldsymbol{u}\|^2_{L^\alpha(0,T,L^q(\Omega))} \big(\int_0^T [\|\nabla \boldsymbol{w}\|_2^{2/\alpha} \|\nabla \boldsymbol{w}\|^{(\alpha-2)/\alpha}_{(2\alpha q-4q)/(\alpha q-2\alpha-2q)}]^{2\alpha/(\alpha-2)}\ dt\big)^{(\alpha-2)/\alpha} \le$$
$$\|\boldsymbol{u}\|^2_{L^\alpha(0,T,L^q(\Omega))} \|\boldsymbol{w}\|^{4/\alpha}_{L^\infty(0,T,W^{1,2}(\Omega))} \big(\int_0^T \|\nabla \boldsymbol{w}\|^2_{(2\alpha q-4q)(\alpha q-2\alpha-2q)}\ dt\big)^{(\alpha-2)/\alpha} \le$$
$$\|\boldsymbol{u}\|^2_{L^\alpha(0,T,L^q(\Omega))} \|\boldsymbol{w}\|^{4/\alpha}_{L^\infty(0,T,W^{1,2}(\Omega))} \|\boldsymbol{w}\|^{2(\alpha-2)/\alpha}_{L^2(0,T,W^{1,(2\alpha q-4q)/(\alpha q-2\alpha-2q)}(\Omega))}$$

and, therefore,

$$(5)\quad \|P_H(\boldsymbol{u}\cdot\nabla \boldsymbol{w})\|_{L^2(0,T,H)} \le \|\boldsymbol{u}\|_{L^\alpha(0,T,L^q(\Omega))} \|\boldsymbol{w}\|_X^{2/\alpha} \|\boldsymbol{w}\|^{(\alpha-2)/\alpha}_{L^2(0,T,W^{1,(2\alpha q-4q)/(\alpha q-2\alpha-2q)}(\Omega))},$$

where we used the fact that X is embedded continuously into $L^\infty(0,T,W^{1,2}(\Omega))$. Since $(2\alpha q-4q)/(\alpha q-2\alpha-2q) < 6$ it follows e.g. from [5, Theorem 2.1, Chapter III] that the injection of X into $L^2(0,T,W^{1,(2\alpha q-4q)/(\alpha q-2\alpha-2q)}(\Omega))$ is compact. The proof now follows immediately from (5) and the definition of compact operators.

Secondly, let $\boldsymbol{u} \in L^\alpha(0,T,L^\infty(\Omega))$, $\alpha > 2$. Then $|\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{v}\ d\boldsymbol{x}| \le \|\boldsymbol{v}\|_2\|\boldsymbol{u}\|_\infty\|\boldsymbol{w}\|_{W^{1,2}}$ for almost every $t\in(0,T)$ and every $\boldsymbol{v}\in H$ and

$$\int_0^T \|\boldsymbol{u}\|_\infty^2 \|\boldsymbol{w}\|^2_{W^{1,2}}\ dt \le \|\boldsymbol{u}\|^2_{L^\alpha(0,T,L^\infty(\Omega))} \big(\int_0^T \|\boldsymbol{w}\|^{2\alpha/(\alpha-2)}_{W^{1,2}(\Omega)}\ dt\big)^{(\alpha-2)/\alpha} =$$
$$\|\boldsymbol{u}\|^2_{L^\alpha(0,T,L^\infty(\Omega))} \big(\int_0^T \|\boldsymbol{w}\|^{4/(\alpha-2)}_{W^{1,2}(\Omega)} \|\boldsymbol{w}\|^2_{W^{1,2}(\Omega)}\ dt\big)^{(\alpha-2)/\alpha} \le$$
$$\|\boldsymbol{u}\|^2_{L^\alpha(0,T,L^\infty(\Omega))} \|\boldsymbol{w}\|^{4/\alpha}_{L^\infty(0,T,W^{1,2}(\Omega))} \big(\int_0^T \|\boldsymbol{w}\|^2_{W^{1,2}}\ dt\big)^{(\alpha-2)/\alpha} =$$
$$\|\boldsymbol{u}\|^2_{L^\alpha(0,T,L^\infty(\Omega))} \|\boldsymbol{w}\|^{4/\alpha}_{L^\infty(0,T,W^{1,2}(\Omega)} \|\boldsymbol{w}\|^{2(\alpha-2)/\alpha}_{L^2(0,T,W^{1,2}(\Omega))}.$$

Therefore,

$$(6)\quad \|P_H(\boldsymbol{u}\cdot\nabla \boldsymbol{w})\|_{L^2(0,T,H)} \le \|\boldsymbol{u}\|_{L^\alpha(0,T,L^\infty(\Omega))} \|\boldsymbol{w}\|_X^{2/\alpha} \|\boldsymbol{w}\|^{(\alpha-2)/\alpha}_{L^2(0,T,W^{1,2}(\Omega))}.$$

The injection of X into $L^2(0,T,W^{1,2}(\Omega))$ is compact and the proof follows immediately from (6) and the definition of compact operators.

If $\boldsymbol{u}\in L^\infty(0,T,L^q(\Omega))$ and $q>3$ then the proof proceeds in the same way as in the previous paragraphs and we will skip it.

Finally, let $\boldsymbol{u} \in L^\alpha(0,T,L^q(\Omega))$ for $2/\alpha + 3/q = 1$, $q \in (3,\infty]$. Let $M_n = \{t \in (0,T); \|\boldsymbol{u}(t)\|_q > n\}, n \in N$ and define $\boldsymbol{u}_n$ on $(0,T)$ as:

$$\begin{aligned} \boldsymbol{u}_n(t) &= \boldsymbol{u}(t) \quad \text{if} \quad t \notin M_n, \\ \boldsymbol{u}_n(t) &= \boldsymbol{0} \qquad \text{if} \quad t \in M_n. \end{aligned}$$

Obviously, $\boldsymbol{u}_n \in L^\infty(0,T,L^q(\Omega))$ and according to the previous paragraphs the operators $\boldsymbol{w} \longmapsto P_H(\boldsymbol{u}_n \cdot \nabla \boldsymbol{w})$ are compact from X to $L^2(0,T,H)$. Further, the Lebesgue measure of M_n goes to zero for $n \to \infty$ so that $\|\boldsymbol{u} - \boldsymbol{u}_n\|_{L^\alpha(0,T,L^q(\Omega))} = (\int_{M_n} \|\boldsymbol{u}\|_q^\alpha \, dt)^{1/\alpha} \longmapsto 0$. Therefore, the operator $\boldsymbol{w} \longmapsto P_H(\boldsymbol{u} \cdot \nabla \boldsymbol{w})$ is compact from X to $L^2(0,T,H)$ as a limit of compact operators $\boldsymbol{w} \longmapsto P_H(\boldsymbol{u}_n \cdot \nabla \boldsymbol{w})$ in the usual norm of the space of all linear bounded operators from X to $L^2(0,T,H)$.

□

Let us consider the following Stokes equations with the perturbed convection term $P_H(\boldsymbol{u} \cdot \nabla \boldsymbol{w})$:

$$\boldsymbol{w}_t + \nu A \boldsymbol{w} + P_H(\boldsymbol{u} \cdot \nabla \boldsymbol{w}) = \boldsymbol{f}, \tag{7}$$

$$\boldsymbol{w}(0) = \boldsymbol{w}_0. \tag{8}$$

Proposition 2. *Let $2/\alpha + 3/q = 1$ with $q \in (3,\infty]$. Then there exists $\varepsilon > 0$ with the following property: if $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1$ in $(0,T)$, $\boldsymbol{u}(t) \in V$ for almost every $t \in (0,T)$, $\boldsymbol{u}_0 \in L^\infty(0,T,L^3_w(\Omega))$, $\boldsymbol{u}_1 \in L^\alpha(0,T,L^q(\Omega))$ and $\sup_{0<t<T} \|\boldsymbol{u}_0(t)\|_{3,w} < \varepsilon$, then for every $\boldsymbol{w}_0 \in V$ and $\boldsymbol{f} \in L^2(0,T,H)$ there exists a unique solution $\boldsymbol{w}$ of (7), (8) in X.*

PROOF: The operator $\boldsymbol{w} \longmapsto (\boldsymbol{w}_t + \nu A\boldsymbol{w}, \boldsymbol{w}(0))$ is a one to one continuous linear operator from X onto Y. It is possible to prove (see also [3, Lemma 2.7]) that the operator $\boldsymbol{w} \longmapsto P_H(\boldsymbol{u}_0 \cdot \nabla \boldsymbol{w})$ is linear and bounded from X to $L^2(0,T,H)$ with the norm less than $C\|\boldsymbol{u}_0\|_{L^\infty(0,T,L^3_w(\Omega))}$. Since the set of linear bounded one to one operators is open in the space of all linear bounded operators (using the usual topology) we get that the operator $\boldsymbol{w} \longmapsto (\boldsymbol{w}_t + \nu A\boldsymbol{w} + P_H(\boldsymbol{u}_0 \cdot \nabla \boldsymbol{w}), \boldsymbol{w}(0))$ is a one to one operator from X onto Y for ε being sufficiently small. Finally, it follows from Proposition 1 that the operator $\boldsymbol{w} \longmapsto P_H(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{w})$ is compact from X to $L^2(0,T,H)$. Moreover, the operator $\boldsymbol{w} \longmapsto (\boldsymbol{w}_t + \nu A\boldsymbol{w} + P_H(\boldsymbol{u} \cdot \nabla \boldsymbol{w}), \boldsymbol{w}(0))$ is one to one from X to Y and the proof follows immediately from Theorem A.

□

Now, we present proofs of the results stated in Introduction. The proofs are based on Propositions 1 and 2. Theorem 3 is a generalization of the famous Serrin's result ([5]) on regularity of weak solutions in the subcritical case and was proved in [3]. Theorem 4 which is dealing with the partial regularity of weak solutions in the supercritical case $L^\infty(0,T,L^3(\Omega))$ was also proved in [3]. We present these theorems in a little more general way.

Theorem 3. *There exists a constant ε with the following property. If $\boldsymbol{u}$ is a weak solution of* (1)–(4) *and there exists a non-negative L^2-function $M = M(t)$ on $(0,T)$ such that*

$$\sup_{R \geq M(t)} R\mu\{\boldsymbol{x} \in \Omega; |\boldsymbol{u}(\boldsymbol{x},t)| > R\}^{1/3} \leq \varepsilon \tag{9}$$

for almost every $t \in (0,T)$, then $\boldsymbol{u}$ is regular, that is $\frac{\partial \boldsymbol{u}}{\partial t}, D^\alpha_{\boldsymbol{x}} \boldsymbol{u} \in C(\Omega \times (0,T))$ for every multi-index α with $|\alpha| \leq 2$.

Proof: Due to the condition (9) $\boldsymbol{u}$ can be easily decomposed as $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1$, where $\boldsymbol{u}_0 \in L^\infty(0,T,L^3_w(\Omega))$, $\boldsymbol{u}_1 \in L^2(0,T,L^\infty(\Omega))$ and $\sup_{0<t<T} \|\boldsymbol{u}_0(t)\|_{3,w} < \varepsilon$ (see [3]). Let $\sigma \in (0,T)$ be an arbitrary number. Since the weak solution $\boldsymbol{u} \in L^2(0,T,V)$, there exists a $t_0 \in (0,\sigma)$ such that $\boldsymbol{u}(t_0) \in V$. If ε is sufficiently small it follows from Proposition 2 that there exists a unique solution $\boldsymbol{w} \in X$ of (7), (8) on (t_0,T) with $\boldsymbol{w}(t_0) = \boldsymbol{u}(t_0)$. It is easy to show that $\boldsymbol{u} = \boldsymbol{w}$ on (t_0,T) and therefore $\boldsymbol{u} \in X$ on (t_0,T). Since σ was chosen arbitrarily the theorem follows immediately using the results on interior regularity of weak solutions proved in [5]. □

Theorem 4. *There exists a positive constant ε with the following property. If $\boldsymbol{u}$ is a weak solution of* (1)–(4) *and there exists $\boldsymbol{w} \in L^3(\Omega)$ such that $\|\boldsymbol{u}(t) - \boldsymbol{w}\|_{3,w} < \varepsilon$ for almost every $t \in (a,b) \subset (0,T)$, then $\frac{\partial \boldsymbol{u}}{\partial t}, D^\alpha_{\boldsymbol{x}} \boldsymbol{u} \in C(\Omega \times (a,b))$ for every multi-index α with $|\alpha| \leq 2$.*

Proof: There exists $\boldsymbol{w}_1 \in L^4(\Omega)$ such that $\|\boldsymbol{w} - \boldsymbol{w}_1\|_3 < \varepsilon$. If we put $\boldsymbol{u}_0 = \boldsymbol{u} - \boldsymbol{w}_1$ and $\boldsymbol{u}_1 = \boldsymbol{w}_1$, then $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1$ on (a,b), $\boldsymbol{u}_0 \in L^\infty(a,b,L^3_w(\Omega))$, $\boldsymbol{u}_1 \in L^\infty(a,b,L^4(\Omega))$ and $\sup_{a<t<b} \|\boldsymbol{u}_0(t)\|_{3,w} < 2\varepsilon$. Now, applying again Propositions 1 and 2 on (a,b) and using the same arguments as in Theorem 3, Theorem 4 follows immediately. □

It was proved in [1] and [3] that if $\boldsymbol{u}$ is a weak solution of (1)–(4) and $\boldsymbol{u} \in C([0,T),L^3(\Omega))$ or $\boldsymbol{u} \in BV([0,T),L^3(\Omega))$ — the set of all functions of bounded variation on $[0,T)$ with values in $L^3(\Omega)$ — then $\boldsymbol{u}$ is regular. These results are consequences of Theorem 4.

The following theorem is another example of the use of Theorem A in the regularity theory of the Navier-Stokes equations. Let us note here that the space $L^2(0,T,W^{1,3}(\Omega))$ is not imbedded into any $L^\alpha(0,T,L^q(\Omega))$ with $2/\alpha + 3/q = 1$ and $q \in (3,\infty]$.

Theorem 5. *Let $\boldsymbol{u}$ be a weak solution of* (1)–(4) *and $\boldsymbol{u} \in L^2(0,T,W^{1,3}(\Omega))$. Then $\frac{\partial \boldsymbol{u}}{\partial t}, D^\alpha_{\boldsymbol{x}} \boldsymbol{u} \in C(\Omega \times (0,T))$ for every multi-index α with $|\alpha| \leq 2$.*

Proof: Firstly, let us show that the operator $\boldsymbol{w} \longmapsto P_H(\boldsymbol{w} \cdot \nabla \boldsymbol{u})$ is compact from X to $L^2(0,T,H)$. Using the Hőlder inequality we have for almost every $t \in (0,T)$ and every $v \in H$:

$$\Big|\int_\Omega \boldsymbol{w} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x}\Big| \leq c\|\boldsymbol{v}\|_2 \|\boldsymbol{w}\|_{W^{1,2}(\Omega)} \|\boldsymbol{u}\|_{W^{1,3}(\Omega)}.$$

It follows easily as in the first paragraph of Proposition 1 that $\|P_H(\boldsymbol{w}\cdot\nabla\boldsymbol{u})\|_{L^2(0,T,H)} \le c\|\boldsymbol{w}\|_X\|\boldsymbol{u}\|_{L^2(0,T,W^{1,3}(\Omega))}$ so that $\boldsymbol{w}\longmapsto P_H(\boldsymbol{w}\cdot\nabla\boldsymbol{u})$ is a linear bounded operator from X to $L^2(0,T,H)$. As in the last paragraph of Proposition 1 it is possible to construct $\boldsymbol{u}_n\in L^\infty(0,T,W^{1,3}(\Omega))$ such that $\|\boldsymbol{u}-\boldsymbol{u}_n\|_{L^2(0,T,W^{1,3}(\Omega))}\longmapsto 0$ and the compactness of the operator $\boldsymbol{w}\longmapsto P_H(\boldsymbol{w}\cdot\nabla\boldsymbol{u})$ follows now from this and from the fact that the operators $\boldsymbol{w}\longmapsto P_H(\boldsymbol{w}\cdot\nabla\boldsymbol{u}_n)$ are compact.

It follows from the standard estimates in Sobolev spaces, the Gronwall lemma and Theorem A that for every $\boldsymbol{w}_0\in V$ and $\boldsymbol{f}\in L^2(0,T,H)$, the following problem has a unique solution $\boldsymbol{w}\in X$:

$$\boldsymbol{w}_t+\nu A\boldsymbol{w}+P_H(\boldsymbol{w}\cdot\nabla\boldsymbol{u})=\boldsymbol{f}, \tag{12}$$

$$\boldsymbol{w}(0)=\boldsymbol{w}_0. \tag{13}$$

The proof is concluded using the same arguments as in the proof of Theorem 3. □

Remark 6. If e.g. $\boldsymbol{f}\in H$ ($\boldsymbol{f}$ independent of time) then in Theorem 3 and Theorem 5, resp. Theorem 4 $\boldsymbol{u}$ is analytic in time, in a neighborhood of the interval $(0,T)$, resp. (a,b), as a $D(A)$-valued function (see [7]). It follows that $\boldsymbol{u}\in C^\infty(0,T,C(\overline{\Omega}))$, resp. $\boldsymbol{u}\in C^\infty(a,b,C(\overline{\Omega}))$. Therefore, $\boldsymbol{u}$ has no singular points in $\overline{\Omega}\times(0,T)$, resp. $\overline{\Omega}\times(a,b)$. Also, $\boldsymbol{u}(\boldsymbol{x},\cdot)$ is an infinitely differentiable function in $(0,T)$, resp. (a,b), for every $\boldsymbol{x}\in\Omega$.

Remark 7. If $\Omega\in C^{0,1}$ then the information from the Introduction — $D(A)=W^{2,2}(\Omega)^3\cap V$ and the norm $\|\boldsymbol{u}\|_{D(A)}$ on $D(A)$ is equivalent to the norm induced by $W^{2,2}(\Omega)^3$ — cannot be used. We do not even know in this case whether $D(A)\hookrightarrow W^{1,2+\varepsilon}(\Omega)^3$ for a positive ε or not. What we only have here is that $D(A)\hookrightarrow\hookrightarrow V$ and also $X\hookrightarrow L^\infty(0,T,V)$. As a consequence, Propositions 1 and 2 can be proved only if $\boldsymbol{u}\in L^2(0,T,L^\infty(\Omega))$ and the proofs of Theorems 3 and 4 fail totally. On the other hand, it is interesting that Theorem 5 can be stated and proved without any change.

Remark 8. If Ω is the half-space or R^3 (or possibly some other special unbounded domain) then we are able to obtain almost the same results as in the case of a bounded domain. Let us discuss it briefly. V denotes the completion of $C^\infty_{0,\sigma}(\Omega)$ in the norm of $W^{1,2}(\Omega)^3$ with the scalar product $((\boldsymbol{u},\boldsymbol{v}))_V=\int_\Omega(\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}+u_iv_i)\,d\boldsymbol{x}$. $D(A)$ is then defined as $\{\boldsymbol{u}\in V;\exists\boldsymbol{f}\in H;((\boldsymbol{u},\boldsymbol{v}))_V=(\boldsymbol{f},\boldsymbol{v})\ \forall\boldsymbol{v}\in V\}$ and using the cut-off method it is possible to show that $D(A)\hookrightarrow W^{2,2}(\Omega)$. It implies that $X\hookrightarrow L^2(0,T,W^{2,2}(\Omega))$ and, consequently, $X\hookrightarrow\hookrightarrow L^2(0,T,W^{1,6-\varepsilon}(\Theta))$ for every small $\varepsilon>0$ and every smooth domain $\Theta\subseteq\Omega$. As a result, Proposition 1 can be proved in a similar way as in the case of a bounded domain and Proposition 2 holds with only one change: the weak Lebesgue space $L^3_w(\Omega)$ is replaced by the Lebesgue space $L^3(\Omega)$. In Theorem 3 the condition (9) is replaced by the assumption $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1$ and $\boldsymbol{u}_0 \in L^\infty(0,T,L^3(\Omega))$, $\boldsymbol{u}_1 \in L^\alpha(0,T,L^q(\Omega))$, $\sup_{0<t<T} \|\boldsymbol{u}_0(t)\|_3 < \varepsilon$ and $2/\alpha + 3/q = 1$ with $q \in (3,\infty]$. In Theorem 4, the space $L^3(\Omega)$ is used instead of the space $L^3_w(\Omega)$. Theorem 5 can be stated without any change.

Conclusion

The results on regularity of weak solutions to the Navier-Stokes equations presented in this paper have been proved recently in [3]. It is interesting, however, that an easy proof of these results can be based on a well known classical theorem on compact operators. Further, weak solutions of the Navier-Stokes equations in the space $L^2(0,T,W^{1,3}(\Omega)^3)$ are regular (Theorem 5), which is interesting in connection with the famous Prodi-Serrin's conditions (see [3]).

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