Two spaces homeomorphic to $Seq(p)$

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Abstract. We consider the spaces called $Seq(u_t)$, constructed on the set Seq of all finite sequences of natural numbers using ultrafilters u_t to define the topology. For such spaces, we discuss continuity, homogeneity, and rigidity. We prove that $S(u_t)$ is homogeneous if and only if all the ultrafilters u_t have the same Rudin-Keisler type. We proved that a space of Louveau, and in certain cases, a space of Sirota, are homeomorphic to $Seq(p)$ (i.e., $u_t = p$ for all $t \in Seq$). It follows that for a Ramsey ultrafilter p, $Seq(p)$ is a topological group.

Keywords: ultrafilters, continuity, homeomorphisms, homogeneous, rigid, topological group, Ramsey ultrafilters, selective ultrafilters

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1. Introduction

By Seq we mean the set of all finite sequences of natural numbers. More precisely, for each natural number $n \in \omega$, let ${}^n\omega = \{t : t$ is a function and $t : n \to \omega\}$. Then $Seq = \cup_{n\in\omega} {}^n\omega$. If $t \in Seq$, with domain $k = \{0, 1, \cdots (k-1)\}$, and $n \in \omega$, let $t^\frown n$ denote the function $t \cup \{(k, n)\}$. For every $t \in Seq$ let u_t be a non-principal ultrafilter on ω. By $Seq(\{u_t : t \in Seq\})$ we denote the space with underlying set Seq and topology defined by declaring a set $U \subset Seq$ to be open if and only if

$$(\forall\, t \in U)\{n \in \omega : t^\frown n \in U\} \in u_t. \tag{1}$$

For short, we write $Seq(u_t)$ instead of $Seq(\{u_t : t \in Seq\})$. We also consider the case where there is a single nonprincipal ultrafilter p on ω such that $u_t = p$ for all $t \in Seq$, and in this case we write $Seq(p)$ instead of $Seq(u_t)$.

We use the following notation of W. Lindgren and A. Szymanski [11]; put $L_n = \{s \in Seq : dom(s) = n\}$, and for any $s \in Seq$ the *cone over* s is defined by $C(s) = \{t \in Seq : s \subset t\}$.

It is well known that for any choice of $\{u_t : t \in Seq\}$, the space $Seq(u_t)$ is a zero-dimensional, extremally disconnected, Hausdorff space with no isolated points. For these and other results about the spaces $Seq(u_t)$ (sometimes in different guises) see [8, 2.4.7, 2.4.8], [10], [17], [18], [16], [4], [11], [5], and [6]. On the other hand, we now discuss how different choices of the ultrafilters $\{u_t : t \in Seq\}$ can produce very different properties on $Seq(u_t)$.

We improve a result of Kannan-Rajagopalan by proving in §2:

Theorem 1.1. *If $f : Seq(u_t) \to Seq(v_t)$ is continuous and $v_{f(t)} \not\leq_{RK} u_t$ for all $t \in Seq$, then f is locally constant on Seq.*

It follows from this theorem (see §2) that there is a choice of ultrafilters $\{u_t : t \in Seq\}$ such that $Seq(u_t)$ is *rigid* (i.e., the only homeomorphism from $Seq(u_t)$ into itself is the identity map). In fact, $Seq(u_t)$ can have the property that the only finite-to-one continuous map from $Seq(u_t)$ into itself is the identity map (see Corollary 2.3).

In contrast to $Seq(u_t)$ being rigid, it follows from the next theorem that there is a choice of ultrafilters $\{u_t : t \in Seq\}$ such that $Seq(u_t)$ is *homogeneous* (i.e., for every $s, r \in Seq$ there exists a homeomorphism from $Seq(u_t)$ onto $Seq(u_t)$) sending s to r.

Theorem 1.2. *$Seq(u_t)$ is homogeneous if and only if all the u_t have the same Rudin-Keisler type.*

Using results of A. Louveau we get the following

Corollary 1.3. *There is a binary group operation $+$ on Seq such that $+$ is separately continuous on $Seq(p) \times Seq(p)$ for any non-principal ultrafilter p. If p is a Ramsey ultrafilter, then $+$ is continuous, i.e., $(Seq(p), +)$ is a topological group.*

The space $Seq(u_t)$ has been independently discovered by several people, sometimes in more generality, sometimes in special cases, and sometimes in different guises. We discuss briefly the instances of which we are aware. The first instance of the general space $Seq(u_t)$ is due to R. Levy [10] in 1977. At about the same time, V. Kannan and M. Rajagopalan [8] consider this space as a special case of a certain direct limit construction. A.G. El'kin [6] defined several topologies on arbitrary sets using filters and ultrafilters, and although he did not consider Seq specifically, it is easy to see that $Seq(u_t)$ can be included in his constructions. El'kin also stated a general theorem which yields half of our Theorem 1.2, but as is common in several Soviet journals, he gave no proofs of his theorems (for completeness, we include a proof using a theorem of van Douwen). V. Trnková [17] constructed special cases of $Seq(u_t)$ (using only finitely many ultrafilters) by direct limit, and other ways [17], [18]. A. Szymanski [16] defined the analogous space for the case of finite sequences of an arbitrary cardinal κ. The elegant type of definition of the topology in (1) was given by El'kin and Szymanski, and the same type of elegant definition was also given in the special case of one ultrafilter by A. Louveau [12], not for Seq, but for the set of all finite subsets of ω. We show that Louveau's space $L(p)$ is homeomorphic to $Seq(p)$ (see §4). Preceding the work mentioned already, A. Arhangel'skii and S. Franklin [1] in 1968 undoubtedly contributed to the development of $Seq(p)$ with their space, constructed by direct limit, called S_ω, which is $Seq(\mathcal{F})$, where $\mathcal{F}$ is the Fréchet filter on ω. In 1969, S. Sirota constructed a class of spaces $S(p)$ for any ultrafilter p, such that if p is a Ramsey ultrafilter then $S(p)$ is a topological group with respect to a certain group operation. Louveau proved that (for p Ramsey) $S(p)$ is homeomorphic to

$L(p)$ (see §5), and by our theorem, $S(p)$ is homeomorphic to $Seq(p)$ (see Theorem 5.1). As far as we can determine, $S(p)$ is the earliest version of $Seq(p)$ using ultrafilters.

2. Continuity among the $Seq(u_t)$

Continuity among spaces of the form $Seq(u_t)$ is related to the Rudin-Keisler order on types of ultrafilters in ω^*. The type of an ultrafilter u is defined by $\tau(u) = \{\overline{\phi}(u) : \phi \in {}^\omega\omega$ such that ϕ is a bijection$\}$, where $\overline{\phi}$ denotes the Stone extension $\overline{\phi} : \beta\omega \to \beta\omega$ of ϕ. The Rudin-Keisler order is defined by $\tau(v) \leq_{RK} \tau(u)$ provided there exists $\phi \in {}^\omega\omega$ such that $\bar{\phi}(u) = v$ (this is equivalent to both (i) for every $V \in v$, $\phi^{-1}(V) \in u$, and (ii) for every $U \in u$, $\phi(U) \in v$ [13, p. 539]), and we say that $\phi \in {}^\omega\omega$ *witnesses* the inequality. It is well known that the partial order $\leq_{RK}$ is antisymmetric (i.e., if $\tau(u) \leq_{RK} \tau(v)$ and $\tau(v) \leq_{RK} \tau(u)$ then $\tau(u) = \tau(v)$ [2, 9.3]). When discussing types, we will often write u instead of $\tau(u)$. Kannan and Rajagopalan [8, 2.4.5, 2.1.4] were the first to investigate the relationships among Rudin-Keisler order, continuity, and locally constant functions among the spaces $Seq(u_t)$.

Definition 2.1 (Kannan-Rajagopalan [8, p. 104]). *A function $f : X \to Y$ is said to be* locally constant at $x \in X$ *provided f maps some neighborhood of x to a single point. If f is locally constant at all points of X we say that f is* locally constant (on X).

We delve further into the relation among the three notions RK-type, continuity and locally constant functions. Using the El'kin-Szymanski definition of the topology on Seq, we can give rather short proofs of our results.

Lemma 2.2. *If $f : Seq((u_t)) \to Seq((v_t))$ is continuous at s and $\{n : f(s^\frown n) = f(s)\} \notin u_s$ then $v_{f(s)} \leq_{RK} u_s$.*

Proof: Define a function $\phi = \phi_s$ by

$$\phi(n) = \begin{cases} m & \text{if } f(s^\frown n) \in C(f(s)^\frown m) \\ 0 & \text{otherwise.} \end{cases}$$

This function is well-defined since each $f(s^\frown n)$ is in at most one $C(f(s)^\frown m)$. We now show that for every $H \in v_{f(s)}$, $\phi^{-1}(H) \in u_s$, i.e., ϕ witnesses the inequality $v_{f(s)} \leq_{RK} u_s$. Let $H \in v_{f(s)}$, then

$$W = \{f(s)\} \cup \bigcup_{m \in H} C(f(s)^\frown m)$$

is a neighborhood of $f(s)$ in $Seq((v_t))$. By continuity of f at s, $f^{-1}(W)$ is a neighborhood of s, hence

$$\{n : s^\frown n \in f^{-1}(W)\} = \{n : f(s^\frown n) \in W\} \in u_s.$$

Hence

$$\{n : f(s^\frown n) = f(s)\} \cup \{n : \exists\, m \in H(f(s)^\frown n) \in C(f(s)^\frown m)\} \in u_s.$$

Since the first set in the preceding union is not in u_s, and $\phi^{-1}(H)$ contains the second set, we get that $\phi^{-1}(H) \in u_s$. □

Corollary 2.3. *If $f : Seq((u_t)) \to Seq((v_t))$ is continuous and finite-to-one (in particular if f is a homeomorphic embedding), then $v_{f(s)} \leq_{RK} u_s$ for all $s \in Seq$.*

PROOF: For a finite-to one function, $\{n : f(s^\frown n) = f(s)\}$ is finite, hence not a member of a non-principal ultrafilter. □

Proof of Theorem 1.1. First we note the following equivalent statements:

$$\begin{aligned} f \text{ is locally constant} &\Leftrightarrow \forall\, s, f^{-1}(f(s)) \text{ is open in } Seq(u_t) \\ &\Leftrightarrow \forall\, s, \{n \in \omega : s^\frown n \in f^{-1}(f(s))\} \in u_s \\ &\Leftrightarrow \forall\, s, \{n : f(s^\frown n) = f(s)\} \in u_s. \end{aligned}$$

Thus it suffices to show that $\{n : f(s^\frown n) = f(s)\} \in u_s$ for each $s \in Seq$. This, however, is the contrapositive of Lemma 2.2. □

Corollary 2.4 (Kannan-Rajagopalan [8, Remark 2.4.9])**.** *There exists a family of $2^{\mathfrak{c}}$ rigid spaces of the form $Seq(u_t)$ such that every continuous function between any two elements of the family is locally constant.*

PROOF: Let $\mathcal{P}$ be a set of non-principal ultrafilters that are pairwise unrelated by the Rudin-Keisler order with $|\mathcal{P}| = 2^{\mathfrak{c}}$ (see [14]). Partition the set $\mathcal{P}$ into countably infinite subsets $\{\{u_t^\alpha : t \in Seq\} : \alpha < 2^{\mathfrak{c}}\}$, and take the family of spaces $\{Seq(u_t^\alpha) : \alpha < 2^{\mathfrak{c}}\}$. If $\alpha \neq \beta$ and $f : Seq(u_t^\alpha) \to Seq(u_t^\beta)$ is continuous, then f is locally constant by Theorem 1.1. If $f : Seq(u_t^\alpha) \to Seq(u_t^\alpha)$ is continuous and not the identity map, then there exists $t \in Seq$ such that $f(t) \neq t$. By Lemma 2.2, $\{n : f(t^\frown n) = f(t)\} \in u_t$, hence f is not finite-to-one. In particular, f is not a homeomorphism. Thus each $Seq(u_t^\alpha)$ is rigid. □

Theorem 1.1 corrects an error in the discussion given by Kannan-Rajagopalan [8, Remark 2.4.6, 2.4.9]. They state that if the ultrafilters in the set $\{u_t, v_t : t \in Seq\}$ have pairwise different types, then every continuous function $f : Seq(u_t) \to Seq(v_t)$ is locally constant. Rather than pairwise different types, one needs pairwise incomparable types, or the weaker hypothesis of Theorem 1.1. The following example makes this clear.

Example 2.5. *There exists a set of ultrafilters $\{u_t, v_t : t \in Seq\}$ having pairwise different types and $f : Seq(u_t) \to Seq(v_t)$ such that f is continuous (and open) and not locally constant at any $t \in Seq$.*

PROOF: We will use the following Fact: Given a countable set $U \subset \omega^*$ with the ultrafilters in U having pairwise different types, there exists infinite sets $V, T \subset \omega^*$

and family $\{S_v : v \in V\}$ of infinite sets $S_v \subset \omega^*$ such that (1) for each $v \in V$ and all $s \in S_v$, $v < s$, and (2) the ultrafilters in $U \cup V \cup (\cup\{S_v : v \in V\}) \cup T$ have pairwise different types. The Fact holds because there are $2^{\mathfrak{c}}$ types of ultrafilters ([2, (c) p. 206]); so we can pick an infinite set $V \subset \omega^*$ such that the ultrafilters in $U \cup V$ have pairwise different type. There are $2^{\mathfrak{c}}$ ultrafilters above any ultrafilter (in the RK-order) [2, (d) p. 206]; so we can pick for each $v \in V$ a countable set S_v so that $v < s$ for all $s \in S_v$, and so that the ultrafilters in $U \cup V \cup (\cup\{S_v : v \in V\})$ have pairwise different types. Now we pick another infinite set T such that $U \cup V \cup (\cup\{S_v : v \in V\}) \cup T$ have pairwise different types, and this completes the choice of ultrafilters.

We construct f, and ultrafilters u_t and v_t by induction. We start by letting $f(\emptyset) = \emptyset$, and pick $u_\emptyset, v_\emptyset \in \omega^*$ so that $v_\emptyset < u_\emptyset$. Let $\phi_\emptyset$ witness this inequality. Assume we have defined $f(t)$, ultrafilters u_t, v_t, and maps $\phi_t \in {}^\omega\omega$ (for $t \in \cup_{i \leq n} L_i$) satisfying

(1) $v_{f(t)} < u_t$ and the inequality is witnessed by ϕ_t,

(2) $f(t^\frown j) = f(t)^\frown \phi_t(j)$ for all $t \in \cup_{i<n} L_i$.

Define f on L_{n+1} as follows: For all $t \in L_n$ and all $j \in \omega$ put $f(t^\frown j) = f(t)^\frown \phi_t(j)$, Thus (2) holds for $t \in L_n$. Next define ultrafilters u_t, v_t for $t \in L_{n+1}$ using the partition of L_{n+1} given by $\{f^{-1}(r) : r \in L_{n+1} \cap f(L_{n+1})\} \cup (L_{n+1}) \setminus f(L_{n+1})$ as follows: By the above Fact, for every $r \in L_{n+1} \cap f(L_{n+1})$ there exist ultrafilters v_r, and u_s for $s \in f^{-1}(r)$, and v_x for $x \in (L_{n+1}) \setminus f(L_{n+1})$ such that $v_r < u_s$ for all $s \in f^{-1}(r)$, and such that all ultrafilters picked so far have pairwise different types. Thus for each $s \in f^{-1}(r)$ we have $v_{f(s)} < u_s$. Let ϕ_s witness this inequality. Now (1) holds for $t \in L_{n+1}$. This completes the induction.

To see that f is not locally constant, we first observe that for every non-empty open set $U \subset Seq(u_t)$, there exists $N \in \omega$ such that for all $n \geq N$, $U \cap L_n \neq \emptyset$. Next we observe that the function f we constructed can easily be shown (using (2)) to have the property that $t \in L_n$, if and only if $f(t) \in L_n$, and therefore if $t \in L_n$ then $f^{-1}(f(t)) \subset L_n$.

To see that f is continuous (and open), we use the next lemma which is also used in the next section.

Lemma 2.6. *If $f : Seq(u_t) \to Seq(v_t)$, and for all $s \in Seq$, $v_{f(s)} \leq_{RK} u_s$, with ϕ_s witnessing the inequality, and $f(s^\frown n) = f(s)^\frown \phi_s(n)$ for all $n \in \omega$, then f is continuous and open. In particular, $f(Seq)$, the image of f, is open in $Seq(v_t)$.*

First we prove that f is continuous. Let W be open in $Seq(v_t)$, and let $x \in f^{-1}(W)$. We need to show that

$$\{n : x^\frown n \in f^{-1}(W)\} = \{n : f(x^\frown n) \in W\} \in u_x.$$

Since $f(x) \in W$, we have $B = \{m : f(x)^\frown m \in W\} \in v_{f(x)}$. It follows that $\phi_x^{-1}(B) \in u_x$. Thus it suffices to prove

$$\phi_x^{-1}(B) \subset \{n : x^\frown n \in f^{-1}(W)\}.$$

Now let $n \in \phi_x^{-1}(B)$. Then $\phi_x(n) \in B$; so $f(x)^\frown\phi_x(n) = f(x^\frown n) \in W$, hence $x^\frown n \in f^{-1}(W)$.

To see that f is an open mapping, let U be open in $Seq(u_t)$, and let $y \in f(U)$. Pick $x \in U$ such that $f(x) = y$. Then $S = \{n : x^\frown n \in U\} \in u_x$. Since ϕ_x witnesses the inequality $v_{f(x)} \leq u_x$, we have $\phi_x(S) \in v_y$. We must show that $\{n : y^\frown n \in f(U)\} \in v_y$; so it suffices to prove that $\phi_x(S) \subset \{n : y^\frown n \in f(U)\}$. Let $n \in \phi_x(S)$. There exists $m \in S$ such that $\phi_x(m) = n$. Now we have $x^\frown m \in U$, hence $f(x^\frown m) \in f(U)$; so $f(x)^\frown\phi_x(m) \in f(U)$, which says that $y^\frown n \in f(U)$. This completes the proof that $f(U)$ is open. □

3. Homogeneity and $Seq(u_t)$

In this section we prove Theorem 1.2 which generalizes the well-known result that $Seq(p)$ is homogeneous for any ultrafilter p. The statement "if all the u_t have the same type then the space $Seq(u_t)$ is homogeneous" was stated without proof in more generality by El'kin [6]. A. Kato [9] gave a proof that $Seq(p)$ is homogeneous using the following lemma of van Douwen. Our proof of Theorem 1.2 uses van Douwen's lemma in a similar way.

Lemma 3.1 Homogeneity Lemma (van Douwen [3, 1.4])**.** *Let X be a countable T_3-space with no isolated points. Then the following are equivalent:*

(1) *X is homogeneous,*
(2) *every non-empty open subset of X is homeomorphic to X,*
(3) *for every $x, y \in X$ there exist a clopen neighborhood U of x and a clopen neighborhood V of y such that U is homeomorphic to V by a homeomorphism that carries x to y.*

Proof of Theorem 1.2. If $Seq(u_t)$ is homogeneous, then all the u_t have the same type by Corollary 2.3 and the antisymmetric property of the Rudin-Keisler order. We prove the converse. Assume for every $s, t \in Seq$ that $\phi_{s,t} \in {}^\omega\omega$ is a bijection whose Stone extension satisfies $\overline{\phi_{s,t}}(u_s) = u_t$. By van Douwen's Homogeneity Lemma, it suffices to show that $C(s)$ is homeomorphic to $C(t)$ by a homeomorphism that takes s to t. We define such a homeomorphism h by induction. Define $h(s) = t$, and by induction define

$$\forall\, x \in C(s)(h(x^\frown n) = h(x)^\frown\phi_{x,h(x)}(n)).$$

By Lemma 2.6, h is continuous and open. We need to prove that h is one-to-one and onto. This we do in the following lemma.

Lemma 3.2. *Let $h : Seq(u_t) \to Seq(v_t)$, such that $h(s) = t$, and for all $x \in Seq$, $v_{h(x)} \leq_{RK} u_x$, with ϕ_x witnessing the inequality, and $h(x^\frown n) = h(x)^\frown\phi_x(n)$ for all $x \in C(s)$ and all $n \in \omega$. If all the ϕ_x are one-to-one then h is one-to-one, and if all the ϕ_x are onto then h is onto.*

First we show that h is one-one. Obviously s is the only element that h maps to t. Assume we have shown that h is one-one for all $x \in C(s)$ with $dom(x) < k$.

If $x, y \in C(s)$ and $dom(x) = dom(y) = k$, then there exist $a, b \in C(s)$ with $dom(a) = dom(b) = k-1$ such that $x = a^\frown n$ and $y = b^\frown m$ for some n, m. If $f(x) = f(y)$, then $f(a)^\frown \phi_{a,f(a)}(n) = f(b)^\frown \phi_{b,f(b)}(m)$. Since $f(a), f(b)$ have the same domain, $f(a) = f(b)$, so by the induction hypothesis $a = b$; thus $\phi_{a,f(a)} = \phi_{b,f(b)}$, and since $\phi_{a,f(a)}$ is one-one, we have $n = m$, hence $x = y$.

To see that h is onto $C(t)$, it is obvious that h maps s to t; so assume that h is onto all $y \in C(t)$ with $dom(y) < k$, and suppose $z \in C(t)$ and $dom(z) = k$. Let $w = z \restriction (k-1)$. By the induction hypothesis there exists $a \in C(s)$ such that $f(a) = w$. Let $m = z(k-1)$, and since $\phi_{a,f(a)}$ is onto, there exists n such that $\phi_{a,f(a)}(n) = m$. Then $f(a^\frown n) = f(a)^\frown \phi_a(n) = w^\frown m = z$.

Corollary 3.3 ([9, Lemma 2.1]). *For every non-principal ultrafilter p on ω, the space $Seq(p)$ is homogeneous.*

4. Louveau's space

The underlying set of Louveau's space $L(p)$ is the set $[\omega]^{<\omega}$ of all finite subsets of ω. Let Δ denote the symmetric difference operator (i.e., for $F, G \in [\omega]^{<\omega}$, $F\Delta G = (F\backslash G)\cup(G\backslash F)$). It is well known that $([\omega]^{<\omega}, \Delta)$ is a commutative group. A. Louveau [12] defined, for each non-principal ultrafilter p on ω, a topology $\mathcal{T}_L(p)$ on the set $[\omega]^{<\omega}$, by declaring a set $U \subset [\omega]^{<\omega}$ to be open if and only if

$$(\forall F \in U)(\{n \in \omega : F\Delta\{n\} \in U\} \in p). \tag{2}$$

For $n > \max F$, $F\Delta\{n\} = F \cup \{n\}$, thus the topology on $L(p)$ can also be defined by declaring that a set U is open if and only if

$$(\forall F \in U)(\{n \in \omega : F \cup \{n\} \in U\} \in p). \tag{3}$$

In this section, we show that $L(p)$ and $Seq(p)$ are homeomorphic. Despite the obvious similarity in the definition of the topologies on $Seq(p)$ and $L(p)$ displayed in (1) and (3), a homeomorphism between these spaces is not readily apparent.

We say that t is *increasing* provided for every $i, j \in dom(t)$, if $i < j$ then $t(i) < t(j)$.

Lemma 4.1. *Let $X = \{t \in Seq : t$ is increasing$\}$. Then X is open in $Seq(p)$.*

Proof: Let $t \in X$. It is obvious that if t is increasing and $n > m(t) = \max\{t(i) : i \in dom(t)\}$ then $t^\frown n$ is increasing. Thus $\{n > m(t) : t^\frown n \in X\}$ is cofinal in ω hence is a member of the ultrafilter p. Thus X is open. $\square$

For any $F \in [\omega]^{<\omega}$ let ψ_F denote the enumerating function of F, i.e., if F contains exactly n points, then F is listed by $\psi_F(0) < \psi_F(1) < \cdots < \psi_F(n-1)$ (the enumerating function of the empty set is defined to be the empty function in Seq). Note that all ψ_F are increasing functions.

Theorem 4.2. *$Seq(p)$ is homeomorphic to $L(p)$.*

Proof: We need only prove that the function $f : L(p) \to Seq(p)$ defined by $f(F) = \psi_F$ is a homeomorphism onto $X = \{t \in Seq : t \text{ is increasing}\}$. Clearly f is a one-one function into X. To see that f is onto X, let $t \in X$, and put $F = range(t) = \{t(i) : i \in dom(t)\}$. Clearly $f(F) = t$ since t is increasing. We now show that f is continuous. Let U be open in X. Since X is open in $Seq(p)$, U is also open in Seq. We need to show that $f^{-1}(U)$ is open in $L(p)$. Let $F \in f^{-1}(U)$, i.e., $\psi_F \in U$; so $P = \{n \in \omega : (\psi_F)^\frown n \in U\} \in p$. For $n > \max F$, $(\psi_F)^\frown n$ is the enumerating function of the set $F \cup \{n\}$ (i.e., $\psi_{F\cup\{n\}} = (\psi_F)^\frown n$), thus $\{n \in \omega : F \cup \{n\} \in f^{-1}(U)\}$ differs from P by a finite set; hence is a member of p; so $f^{-1}(U)$ is open in $L(p)$. To see that f is an open map, let U be open in $L(p)$, and let $t \in f(U)$; say $t = \psi_F$ where $F \in U$. Then $\{n \in \omega : t^\frown n \in f(U)\}$ differs from $\{n \in \omega : F \cup \{n\} \in U\}$ by a finite set; so $f(U)$ is open in Seq, hence in X. This completes the proof. □

By 3.3, $Seq(p)$ is homogeneous for any non-principal ultrafilter p. If we take p to be a Ramsey ultrafilter (see [2, Theorem 9.6]) we can get the stronger result that $Seq(p)$ is a topological group. We call on theorems of Louveau and therefore do not need the definition of a Ramsey (also called *selective*) ultrafilter (see [2], [12]).

Proof of Corollary 1.3. Louveau proved that the symmetric difference operator Δ is separately continuous on $L(p) \times L(p)$ for any p. Using the homeomorphism $h : Seq(p) \to L(p)$ given by Theorem 4.2, the operation defined by $s + t = h^{-1}(h(s)\Delta h(t))$ is also separately continuous. If p is Ramsey, by Louveau [12, Theorem 6], $(L(p), \Delta)$ is a topological group. Thus $(Seq(p), +)$ is also a topological group. □

5. Sirota's space

We define Sirota's space $S(p)$. Let u be a filter on ω finer than the Frechét filter, let

$$X = \{x \in {}^\omega 2 : (\exists\, n_x)(\forall\, i \geq n_x)(x(i) = 0)\},$$

and for each $x \in X$ let $m(x) = \{i \in \omega : x(i) = 1\}$, a finite set. In [15], S.M. Sirota defined a topology $\mathcal{T}(u)$ on X by taking as basic neighborhoods of $y \in X$ all sets of the form

$$H(y, A) = \{x \in X : m(x) \setminus A = m(y) \setminus A\},$$

where $A \in u$. Let $\mathcal{T}_S(p)$ denote the topology on X generated by the above base, and let $S(p) = (X, \mathcal{T}_S(p))$.

Theorem 5.1. *Let p be an ultrafilter on ω. Then $S(p)$ is homeomorphic to $Seq(p)$ if and only if p is a Ramsey ultrafilter.*

Proof: Since the underlying set of Sirota's space can be considered as the set of all characteristic functions of finite subsets of ω, the topology $\mathcal{T}_S(p)$ can be considered as a topology on $[\omega]^{<\omega}$, as done in [12]. Louveau denoted Sirota's topology on $[\omega]^{<\omega}$ by $\mathcal{T}_1(p)$ ([12, Definition 5]), and proved that the Sirota topology equals the Louveau topology if and only if p is a Ramsey ultrafilter (iff $(L(p), \Delta)$ is a topological group [12, Theorem 6]). The result now follows from Theorem 4.2.

$\square$

It is interesting to note that Sirota's space is possibly the earliest (homeomorphic) version of $Seq(p)$, and that Sirota's version of $Seq(p)$ can only be constructed in models of set theory in which Ramsey ultrafilters exist such as models of the continuum hypothesis or Martin's Axiom. There are models, however, where there are no Ramsey ultrafilters (see [7, Theorem 91]).

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