On maximal functions over circular sectors with rotation invariant measures

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Abstract. Given a rotation invariant measure in $\mathbb{R}^n$, we define the maximal operator over circular sectors. We prove that it is of strong type (p,p) for $p>1$ and we give necessary and sufficient conditions on the measure for the weak type $(1,1)$ inequality. Actually we work in a more general setting containing the above and other situations.

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Let X be a topological space and μ be a Borel measure on X. By $L^p_\mu(X)$ we will denote the space of all real valued measurable functions on X such that $\|f\|_{p,\mu} = (\int_X |f(x)|^p \, d\mu(x))^{1/p} < \infty$ if $1 \le p < \infty$, and $\|f\|_{\infty,\mu} = \inf\{a : \mu(\{x \in X : |f(x)| > a\}) = 0\} < \infty$, if $p = +\infty$. By $L_{\mu,loc}(X)$ we mean the set of all real valued measurable functions on X which are integrable on compact sets.

An operator T defined on $L^p_\mu(X)$ is said to be of strong type (p,p), $1 \le p \le +\infty$, if there exists a positive constant A_p such that for every $f \in L^p_\mu(X)$, $\|Tf\|_{p,\mu} \le A_p \|f\|_{p,\mu}$. The operator T is of weak type (p,p), $1 \le p < +\infty$, if there exists a constant B_p such that for every $f \in L^p_\mu(X)$ and $\lambda > 0$, $\mu(\{x \in X : |Tf(x)| > \lambda\}) \le B_p \|f\|_{p,\mu} \lambda^{-p}$.

Let $y \in S_{n-1}$, the unit sphere in $\mathbb{R}^n$, $\alpha \in (0,\pi)$ and $0 \le r \le R \le +\infty$. The set $S = S(y,\alpha,r,R) = \{x \in \mathbb{R}^n : \arg(x,y) < \alpha \text{ and } r < |x| < R\}$, where $\arg(x,y)$ is the angle between x and y, is called a circular sector in $\mathbb{R}^n$. If G is a Borel subset of S_{n-1}, $G(r,R)$ will denote the set $\{\rho x : x \in G,\ r < \rho < R\}$ and σ_r will mean the surface measure on the sphere of radius r.

Let μ be a non-negative rotation invariant Radon measure. If $f \in L_{\mu,loc}(\mathbb{R}^n)$ and $x \neq 0$ we define

$$M^s_\mu f(x) = \sup_{x \in S} \frac{1}{\mu(S)} \int_S |f(y)| d\mu(y) \tag{1}$$

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where the supremum is taken over all bounded sectors S in $\mathbb{R}^n$ with non-zero measure containing x. Set $M_\mu^s f(0) = 0$.

It is not hard to prove that if G is a Borel set in S_{n-1}, $0 \le r \le R \le +\infty$ and $f \in L^1_\mu(\mathbb{R}^n)$ then

$$\begin{aligned}\int_{G(r,R)} f(y)d\mu(y) &= \frac{1}{\sigma_1(S_{n-1})}\int_G \int_r^R f(\rho y')d\nu(\rho)d\sigma_1(y') \\ &= \frac{1}{\sigma_1(S_{n-1})}\int_r^R \int_G f(\rho y')d\sigma_1(y')d\nu(\rho),\end{aligned}$$

where ν is the measure in $\mathbb{R}^+$ defined by $\nu((a,b)) = \mu(S_{n-1}(a,b))$, with the above notation and $G = S_{n-1}$, for $0 \le a \le b \le +\infty$. So, the average of f over circular sectors is obtained by considering averages over the intervals in $\mathbb{R}$, with respect to ν, and over the "balls" in S_{n-1}, with respect to the surface measure σ_1. This fact allows us dominate $M_\mu^s f$ by an iteration of two maximal functions: one in $\mathbb{R}$ with respect to an abstract measure ν, and the other in S_{n-1} with respect to its "doubling" usual measure σ_1.

This situation is analogous to the classical one of the maximal operator over rectangles with sides parallel to the coordinate axes, with respect to the Lebesgue measure. In this case the strong type inequality holds for $p > 1$ but the weak type $(1,1)$ inequality is no longer true (see [4]).

We are interested in the boundedness of M_μ^s. The strong type (p,p) for $p > 1$ will be obtained as a consequence of the strong type inequality for the maximal functions in (S_{n-1}, σ_1) and in $(\mathbb{R}^+, \nu)$. Our main result is a characterization of the measures ν for which the weak type (1,1) inequality holds. We shall work in a more general setting that includes the above situations. The basic result of these notes concerning M_μ^s will be the following corollary of Theorems 1 and 2 below.

Given a non-negative rotation invariant Radon measure μ, we have

(a) *M_μ^s is of strong type (p,p) for $1 < p \le \infty$;*
(b) *M_μ^s is of weak type $(1,1)$ if and only if μ is a finite linear combination of surface measures; i.e., $\mu = \sum_{i=1}^N a_i \sigma_{r_i}$, with $a_i, r_i > 0$, $N \in \mathbb{N}$.*

Let X be a set; a nonnegative symmetric function d on $X \times X$ is called a quasi-distance if there exits a constant k such that

$$d(x,y) \le k\,[d(x,z) + d(z,y)]$$

for every $x, y, x \in X$, and $d(x,y) = 0$ if and only if $x = y$.

The balls $B(x,r) = \{y : d(x,y) < r\}$ form a basis of neighborhoods of x for the topology induced by d.

Consider (X,d,λ), where d is a quasi-distance on X and λ a non-negative measure defined on a σ-algebra of subsets of X which contains the balls. For $f \in L_{\lambda,loc}(X)$ we define the Hardy-Littlewood maximal function operator

$$M_\lambda f(x) = \sup_{x\in B} \frac{1}{\lambda(B)} \int_B |f(x)| d\lambda(x),$$

where B is a ball containing x with positive measure.

We shall say that (X,d,λ) is a space of homogeneous type if there exists a constant A such that

$$0 < \lambda(B(x,2r)) \leq A\,\lambda(B(x,r)) < \infty \tag{2}$$

holds for every $x \in X$ and $r > 0$.

For these spaces, the weak type $(1,1)$ and the strong type (p,p) for $p > 1$ of M_λ hold (see [2]).

Let (X,d,λ) be a space of homogeneous type and ν a Lebesgue-Stieltjes measure on $\mathbb{R}$. Consider the product space $X \times \mathbb{R}$ endowed with the product measure $\mu = \lambda \times \nu$. For $f \in L_{\mu,loc}(X \times \mathbb{R})$ we consider the maximal function

$$M_\mu f(x,t) = \sup_{(x,t)\in S} \frac{1}{\mu(S)} \int_S |f(y,s)| d\mu(y,s) \tag{3}$$

where the supremum is taken over all the cylinders $S = B \times (r,R)$ in $X \times \mathbb{R}$ with positive measure containing (x,t), B being a ball in X.

Theorem 1. *M_μ is of strong type (p,p) for $p > 1$.*

PROOF: Since the case $p = +\infty$ is trivial we consider $1 < p < \infty$. By Tonelli theorem we have that

$$\begin{aligned} M_\mu f(x,t) &= \sup_{(x,t)\in S} \frac{1}{\mu(S)} \int_S |f(y,s)| d\mu(y,s) \\ &= \sup_{(x,t)\in S} \frac{1}{\lambda(B)} \int_B \left(\frac{1}{\nu((r,R))} \int_r^R |f(y,s)| d\nu(s) \right) d\lambda(y) \\ &\leq \sup_{(x,t)\in S} \frac{1}{\lambda(B)} \int_B M_\nu(f(y,.))(t) d\lambda(y) \\ &\leq M_\lambda[(M_\nu(f(.,.)))(t)](x). \end{aligned}$$

The maximal function M_λ is $L^p_\lambda(X)$-bounded because (X,d,λ) is a space of homogeneous type and, even without any doubling condition for ν, the maximal function M_ν is $L^p_\nu(\mathbb{R})$-bounded because the basic space is one-dimensional (see [1]). Then

$$\begin{aligned}
\int_{X\times\mathbb{R}} |M_\mu f(x,t)|^p d\mu(x,t) &\leq \int_{X\times\mathbb{R}} |M_\lambda[(M_\nu(f(.,.)))(t)](x)|^p d\mu(x,t) \\
&= \int_{\mathbb{R}}\int_X |M_\lambda[(M_\nu(f(.,.)))(t)](x)|^p d\lambda(x)d\nu(t) \\
&\leq C\int_{\mathbb{R}}\int_X |M_\nu(f(x,.))(t)|^p d\lambda(x)d\nu(t) \\
&= C\int_X\int_{\mathbb{R}} |M_\nu(f(x,.))(t)|^p d\nu(t)d\lambda(x) \\
&\leq C\int_X\int_{\mathbb{R}} |f(x,t)|^p d\nu(t)d\lambda(x) \\
&= C\int_{X\times\mathbb{R}} |f(x,t)|^p d\mu(x,t).
\end{aligned}$$

$\square$

Remarks. (1) Observe that we have just used the strong type inequalities for M_λ and M_ν. In fact, if (X_1, d_1, λ_1) and (X_2, d_2, λ_2) are spaces for which their maximal operators are bounded, then Theorem 1 holds for the product space $X_1 \times X_2$.

(2) Following Zygmund, Marcinkiewicz and Jensen one can prove a stronger result: If (X_1, d_1, λ_1) and (X_2, d_2, λ_2) are spaces for which their maximal operators are of weak type $(1,1)$, for each $f \in L_{\mu,loc}(X_1 \times X_2)$ and each $\lambda > 0$ we have

$$\mu(\{(x,t) \in X_1 \times X_2 : M_\mu f(x,t) > \lambda\}) \leq c\int_{X_1\times X_2} \tfrac{|f(x,t)|}{\lambda}\left(1 + \log^+ \tfrac{|f(x,t)|}{\lambda}\right) d\mu(x,t).$$

See, for example, [4, p. 161].

The weak type $(1,1)$ inequality of M_μ is a bit more delicate. We shall now give a characterization of the measures ν for which M_μ is of weak type $(1,1)$ when dealing with a special class of spaces of homogeneous type.

Let us first observe that if ν is a finite linear combination of Dirac measures the weak type inequality holds. This easily follows from the weak type for the Hardy-Littlewood maximal operator in spaces of homogeneous type. What is important is that the converse is true in most of the spaces of homogeneous type of interest, such as the unit sphere with the surface area measure or the euclidean space with non-isotropic parabolic distances and Lebesgue measure.

The fact that every non-atomic space of homogeneous type can be normalized in such a way that each ball of radius r has measure of order r, for r small enough, was proved in [5]. In [6] it is shown that the more restrictive normalization $\lambda(B(x,r)) = r$, can be obtained in several interesting cases containing the generalized parabolic distances or some regular and smooth Vitali families.

The proof of Theorem 2 is an extension of the standard construction of a counterexample given in [4, p. 165].

Theorem 2. *Let (X, d, λ) be a space of homogeneous type normalized in such a way that $\lambda(B(x,r)) = r$ for $x \in X$ and r small enough. If M_μ is of weak type $(1,1)$ then ν is a finite linear combination of Dirac measures; in other words, $\nu = \sum_{i=1}^N a_i \delta_{r_i}$, $a_i > 0$, $r_i \in \mathbb{R}$, $N \in \mathbb{N}$.*

Proof: Let ν be a measure which is not a finite linear combination of Dirac measures. We will show that M_μ is not of weak type $(1,1)$. Fix $x_0 \in X$. We consider two cases.

Case 1. Suppose first that

$$\text{there exists } z \in \mathbb{R} \text{ such that } \nu((z, z+1/i)) \neq 0 \text{ for every } i \in \mathbb{N} \tag{4}$$

and define $\Psi(i) = 1/\nu((z, z+1/i))$. Consider, for each $N \in \mathbb{N}$, the sets

$$J_i^N = \{(x,t) \in X \times \mathbb{R} : z < t < z + 1/i,\, x \in B(x_0, \tfrac{\Psi(i)}{\Psi(N)})\} \qquad i = 1, \dots, N\,;$$

$$E_N = \bigcap_{i=1}^N J_i^N = \{(x,t) \in X \times \mathbb{R} : z < t < z + 1/N,\, x \in B(x_0, \tfrac{\Psi(1)}{\Psi(N)})\}.$$

Then, by the definitions of μ and Ψ, we have

$$\mu(J_i^N) = \lambda(B(x_0, \tfrac{\Psi(i)}{\Psi(N)}))\, \nu((z, z+1/i)) = \frac{1}{\Psi(N)}\,;$$

$$\mu(E_N) = \lambda(B(x_0, \tfrac{\Psi(1)}{\Psi(N)}))\, \nu((z, z+1/N)) = \frac{\Psi(1)}{(\Psi(N))^2}\,.$$

Observe that

$$\bigcup_{i=1}^N J_i^N \subset \left\{(x,t) \in X \times \mathbb{R} : M_\mu(\chi_{E_N})(x,t) > \frac{C}{\Psi(N)}\right\}$$

for some constant C. In fact, if $(x,t) \in J_i^N$ then

$$\begin{aligned} M_\mu(\chi_{E_N})(x,t) &\geq \frac{1}{\mu(J_i^N)} \int_{J_i^N} \chi_{E_N}(y,s)\, d\mu(y,s) \\ &= \frac{\mu(E_N)}{\mu(J_i^N)} \\ &= \frac{\Psi(1)}{\Psi(N)}\,. \end{aligned}$$

Then

$$\mu(\{(x,t) \in X \times \mathbb{R} : M_\mu(\chi_{E_N})(x,t) > \frac{C}{\Psi(N)}\}) \geq \mu(\bigcup_{i=1}^{N} J_i^N)$$

$$= \frac{1}{\Psi(N)} + \sum_{i=2}^{N} \frac{\Psi(i) - \Psi(i-1)}{\Psi(N)} \frac{1}{\Psi(i)},$$

since

$$\bigcup_{i=1}^{N} J_i^N = \Big\{(x,t) \in X \times \mathbb{R} : z < t < z+1, x \in B(x_0, \tfrac{\Psi(1)}{\Psi(N)})\Big\}$$

$$\bigcup \bigcup_{i=2}^{N} \Big\{(x,t) \in X \times \mathbb{R} : z < t < z + \tfrac{1}{i}, x \in B(x_0, \tfrac{\Psi(i)}{\Psi(N)}) - B(x_0, \tfrac{\Psi(i-1)}{\Psi(N)})\Big\}.$$

So,

$$\Psi(N)\mu(\{(x,t) \in X \times \mathbb{R} : M_\mu(\chi_{E_N})(x,t) > \frac{C}{\Psi(N)}\}) \geq 1 + \sum_{i=2}^{N} \Big(1 - \frac{\Psi(i-1)}{\Psi(i)}\Big).$$

Observe that $\lim_{N\to+\infty} \prod_{i=2}^{N} \frac{\Psi(i-1)}{\Psi(i)} = 0$, so the above series diverges to $+\infty$. We conclude that M_μ is not of weak type $(1,1)$.

Observe that the same construction with the obvious changes works if instead of condition (4) we have

(5) there exists $z \in \mathbb{R}$ such that $\nu((z - 1/i, z)) \neq 0$ for every $i \in \mathbb{N}$.

Case 2. Suppose now that neither (4) nor (5) hold. Then ν is a discrete measure supported on an infinite set whose atoms do not accumulate; that is, there exists an infinite sequence $\{r_i\}_{i\in\mathbb{N}}$ of real numbers with no accumulation points such that $\nu(A) = \sum_{r_i \in A} a_i$ for every measurable subset $A \subset \mathbb{R}$, where $\nu(\{r_i\}) = a_i > 0$. In fact, for each $z \in \mathbb{R}$ there exists $i_z \in \mathbb{N}$ such that $\nu(z - 1/i_z, z + 1/i_z) = \nu(z)$. Now if $K \subset \mathbb{R}$ is compact, the set $\{z \in K : \nu(z) > 0\}$ is finite. So, we obtain a sequence $\{r_i\}$ with no accumulation points such that $\nu(r_i) = a_i > 0$ and $\nu(A) = \sum_{r_i \in A} a_i$ for every measurable subset $A \subset \mathbb{R}$. Observe that $\{r_i\}$ is not finite because ν is not a linear combination of Dirac measures. We assume, without loss of generality, that $0 < r_i < r_{i+1}$.

We consider two cases. (i) Assume $\sum_{i\in\mathbb{N}} a_i = +\infty$. Define for each $N \in \mathbb{N}$

$$J_i^N = \{(x,t) \in X \times \mathbb{R} : 0 < t < r_{i+1},\ x \in B(x_0, \tfrac{1}{\sum_{j=1}^{i} a_j})\} \qquad i = 1, \dots, N\,;$$

$$E_N = \bigcap_{i=1}^{N} J_i^N = \{(x,t) \in X \times \mathbb{R} : 0 < t < r_2,\ x \in B(x_0, \tfrac{1}{\sum_{j=1}^{N} a_j})\}.$$

Then, for $(x,t) \in J_i^N$, we have

$$M_\mu(\chi_{E_N})(x,t) \geq \frac{1}{\mu(J_i^N)} \int_{J_i^N} \chi_{E_N}(y,s)\,d\mu(y,s) = \frac{a_1}{\sum_{j=1}^N a_j}\,. \tag{6}$$

So, there is a constant $C > 0$ such that

$$\bigcup_{i=1}^N J_i^N \subset \Big\{(x,t) \in X \times \mathbb{R} : M_\mu(\chi_{E_N})(x,t) > \tfrac{C}{\sum_{j=1}^N a_j}\Big\}.$$

Then

$$\mu(\{(x,t) \in X \times \mathbb{R} : M_\mu(\chi_{E_N})(x,t) > \tfrac{C}{\sum_{j=1}^N a_j}\}) \geq \mu(\bigcup_{i=1}^N J_i^N) = \sum_{i=1}^N \frac{a_i}{\sum_{j=1}^i a_j}\,,$$

since

$$\bigcup_{i=1}^N J_i^N = \bigcup_{i=1}^N \Big\{(x,t) \in X \times \mathbb{R} : r_{i-1} < t \leq r_i, x \in B(x_0, \tfrac{1}{\sum_{j=1}^i a_j})\Big\}.$$

This last series diverges since $\sum_{i\in\mathbb{N}} a_i = +\infty$, see for example page 18 of [3]. Then M_μ is not of weak type $(1,1)$.

(ii) Assume now that $\sum_{i\in\mathbb{N}} a_i < +\infty$ and for each $N \in \mathbb{N}$ define

$$J_i^N = \{(x,t) \in X \times \mathbb{R} : r_{i-1} < t,\ x \in B(x_0, \frac{\sum_{j\geq N} a_j}{\sum_{j\geq i} a_j})\} \quad i = 1, \dots, N \ \ (r_0 = 0);$$

$$E_N = \bigcap_{i=1}^N J_i = \Big\{(x,t) \in X \times \mathbb{R} : r_{N-1} < t,\ x \in B(x_0, \frac{\sum_{j\geq N} a_j}{\sum_{j\geq 1} a_j})\Big\}.$$

As before, there exists a constant C such that

$$\bigcup_{i=1}^N J_i^N \subset \Big\{(x,t) \in X \times \mathbb{R} : M_\mu(\chi_{E_N})(x,t) > C \sum_{j\geq N} a_j\Big\}.$$

Then we have

$$\frac{1}{\sum_{j\geq N} a_j} \mu\Big\{(x,t) \in X \times \mathbb{R} : M_\mu(\chi_{E_N})(x,t) > C \sum_{j\geq N} a_j\Big\} \geq c\Big(1 + \sum_{i=1}^{N-1} \frac{a_i}{\sum_{j\geq i} a_j}\Big)$$

for some constant c. The series above diverges to $+\infty$ so we conclude that M_μ is not of weak type $(1,1)$. $\square$

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