On clopen sets in Cartesian products

Raushan Z. Buzyakova

Abstract. The results concern clopen sets in products of topological spaces. It is shown that a clopen subset of the product of two separable metrizable (or locally compact) spaces is not always a union of clopen boxes. It is also proved that any clopen subset of the product of two spaces, one of which is compact, can always be represented as a union of clopen boxes.

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In notation and terminology we will follow [ENG]. In particular, a set A is *clopen* in a space X if it is closed and open in X. A *clopen box* in a space $X \times Y$ is a clopen subset of the form $U \times V$, where U and V are clopen subsets of X and Y, respectively. It is known that a subset of the Cartesian product of two spaces is open if and only if it can be represented as a union of open boxes. When the same is true for clopen subsets?

§1. Example 1

The following question was first formulated by Alexander Shostak at the beginning of 90's. He posed it in his talk at a seminar on General Topology at Moscow University. Independently, Andrej Bauer asked the same question, motivated by some problems in Computer Science on which he was working on.

Question 1 (A. Bauer and A. Shostak)**.** *Is it always true that any clopen subset of the Cartesian product of two spaces can be represented as a union of clopen boxes?*

Bauer also noticed the following interpretation of Question 1.

For a topological space X let $z(X)$ be its zero-dimensional reflection, i. e., the same underlying set X but with topology generated by the collection of all clopen subsets of X. *Does zero-dimensional reflection commute with product operation?*

Motivation of Shostak for the question was purely topological. He considered the following property P of a space X: *every covering of X by clopen subsets contains a finite subcovering*. Shostak wanted to know if this property is preserved by finite products. Clearly, if the answer to Question 1 were affirmative then the answer to the question about productivity would be positive as well. Later, several counterexamples to the latter question were found (see [STE], [SHO], and [SaS]). That provides counterexamples to Question 1. However, the spaces involved in those counterexamples are of large cardinality, non-metrizable, and non-locally compact. So, it would be interesting to restrict Question 1 to separable metrizable spaces or to spaces with strong compactness-type properties.

Question 2 (A. Bauer and A. Shostak)**.** *Is it always true that any clopen subset of the Cartesian product of two separable metrizable spaces can be represented as a union of clopen boxes?*

Example 1. *There exist two spaces, X (a locally compact subspace of $\mathbb{R}^2$) and Z (a countable G_δ-subspace of $\mathbb{R}$), whose product contains a clopen set that cannot be represented as a union of clopen boxes.*

Construction of X.

Let X_1 be a subset of $\mathbb{R}^2$ consisting of the ray $\{(x,y) \in \mathbb{R}^2 : y = 0,\ x \geq 1\}$. For any $n \in \mathbb{N}$, take the sequence $S_n = \{(n, 1/k) : k \in \mathbb{N} \setminus \{1\}\}$ that converges to $(n, 0)$. Put $X = X_1 \cup (\bigcup\{S_n : n \in \mathbb{N}\})$. The topology in X is inherited from $\mathbb{R}^2$. For further reference, let O_n^k be a fixed neighborhood of $(n, 0)$ of radius $1/k$.

Construction of Z.

Let $Z = \{0\} \cup \{K_n : n \in \mathbb{N} \setminus \{1\}\}$, where $K_n = \{a_k^n : a_k^n = 1/n + (1/(n-1) - 1/n)/2^k, k \in \mathbb{N}\}$. That is, K_n is a sequence of numbers converging to $1/n$ and lying in between $1/n$ and $1/(n-1)$. So, Z is a countable metrizable non-compact space with only one non-isolated point 0.

The space $X \times Z$ is the space we are looking for.

Indeed, $X \times Z$ contains a clopen set A that cannot be represented as a union of clopen boxes.

Construction of A.

Let A_1 be the union of all copies of the connected part X_1 of X in $X \times Z$. That is,

$$A_1 = X_1 \times Z.$$

A_1 is closed in $X \times Z$ but not open since points of the form $(n, 0, z)$ are on the boundary. Let us first supply points $(n, 0, 0)$ with their neighborhoods. Let $U(n, 0, 0) = O_n^1 \times (Z \setminus \bigcup\{K_l : l < n\})$. So, the sets $U(n, 0, 0)$'s form together a staircase.

Let $A_2 = A_1 \cup (\bigcup\{U(n, 0, 0) : n \in \mathbb{N}\})$. The set A_2 is still not clopen since points $(n, 0, z)$, where $z \in K_l$ for $l < n$, are on the boundary. Let us supply these points with neighborhoods.

Consider $K_l = \{a_k^l : k \in \mathbb{N}\}$ (see the definition of Z). For each $a_k^l \in K_l$, let $U(n, 0, a_k^l) = O_n^k \times \{a_k^l\}$. That is, $U(n, 0, a_k^l)$ is the neighborhood of $(n, 0)$ of radius $1/k$ on level $X \times \{a_k^l\}$.

Put

$$A = A_2 \cup (\bigcup\{U(n, 0, a_k^l) : n, l, k \in \mathbb{N}\}).$$

A is the set we need.

The set A is open for the following reasons:

1. Every point of form (x, y, z), where x is not a natural number, is in A with an open neighborhood since $(X_1 \setminus \{(n, 0) : n \in \mathbb{N}\}) \times Z$ is open, is a subset of A, and contains all points of this form. (That is, the ray X_1 is in A on each level).

2. All other points in A are either in $U(n, 0, 0)$ or in $U(n, 0, a_k^l)$.

The set A is closed for the following reasons:

1. The closure of A is the union of the closures of the traces of A on each level $X \times \{z\}$, where z is in Z, since $X \times \{0\}$ is the only non-isolated level in our product and it is entirely in A.

2. On each discrete level $X \times \{a_k^l\}$, where a_k^l is in K_l (constructed for Z), the complement of A is a collection of isolated points, and therefore, open. So, the trace of A on each level is closed.

Thus, A is clopen. Now, let us prove that A cannot be represented as a union of clopen sets of the form $U \times V$. Indeed, any clopen neighborhood of the point $(1, 0, 0)$ of the form $U \times V$ in $X \times Z$ must contain the entire ray X_1 on 0's level (that is, set $X_1 \times \{0\}$) because of connectedness. And therefore, any clopen box of $(1, 0, 0)$ in $X \times Z$ must contain a set of the form

$$\bigcup\{O_n^k : O_n^k \text{ is one of fixed neighborhoods of } (n, 0), n \in \mathbb{N}\} \times$$
$$\times (Z \setminus \bigcup\{K_l : l < m, \text{ for some fixed } m \in \mathbb{N}\}),$$

that is, a set obtained by multiplying the union of neighborhoods of $(n, 0)$'s by the space Z without a finite number of K_l's.

(This is shown by an obvious connectedness argument).

But this is impossible, since for each n, $O_n^k \times (Z \setminus \bigcup\{K_l : l < m\})$, where m is fixed, is contained in A only when $n \leq m$.

Note. There is nothing special about $(1, 0, 0)$. You can take any point that lies in the connected part of $X \times \{0\}$.

§2. Arhangel'skii's observations

Example 1 gives us a negative answer to Question 2. However, it would be interesting to find some conditions under which a clopen set in a product is always a union of clopen boxes. Arhangel'skii noticed that the following theorem holds.

Theorem 1. *Let X and Y be topological spaces and let X be compact. Then any clopen subset of $X \times Y$ can be represented as a union of clopen boxes.*

This fact is a direct consequence of Theorem 2 proved below.

The next statement is well known and easy to prove (see Lemma 3.1.15 in [ENG]):

Lemma 1. *Suppose F is a compact subspace of a space Y, x is a point of a space X, and W is an open subset of the product space $X \times Y$ such that $\{x\} \times F \subset W$. Then there exists an open subset V in X such that $x \in V$ and $V \times F \subset W$.*

Now we apply Lemma 1 to prove the next statement:

Lemma 2. *Suppose F is a compact subspace of a space Y, X a space, and W an open and closed subset of the product space $X \times Y$. Then the set $U_F = \{x \in X : \{x\} \times F \subset W\}$ is an open and closed subset of X.*

Proof: Indeed, it is immediate from Lemma 1 that U_F is open in X. Now take any $x \in \overline{U_F}$. By the definition of U_F, we have $U_F \times F \subset W$. Therefore,

$$\{x\} \times F \subset \overline{U_F} \times F \subset \overline{U_F \times F} \subset \overline{W} = W,$$

which implies that $x \in U_F$. Hence, the set U_F is closed. □

Theorem 2. *Suppose Y is a compact space, X is a space, W an open and closed subset of the product space $X \times Y$, and (a, b) a point in W. Then there exist an open and closed subset U in X and an open and closed subset V in Y such that $(a, b) \in U \times V \subset W$.*

Proof: Put $F = \{y \in Y : (a, y) \in W\}$. Since W is closed, the set F is closed in Y and therefore, compact. Besides, F is open, since W is open. We also have $b \in F$ and $\{b\} \times F \subset W$. From Lemma 2 it follows that the set $U_F = \{x \in X : \{x\} \times F \subset W\}$ is an open and closed subset of X. Clearly, $a \in U_F$ and $(a, b) \in U_F \times F \subset W$. Thus, $U = U_F$ and $V = F$ are the sets we are looking for. □

Remark. Theorem 2 obviously generalizes to the case when Y is any space satisfying the following condition:

(lc) For each $y \in Y$ there exists an open and closed subset V of Y such that the subspace V is compact.

§3. Example 2

Since in Example 1 X is locally compact, compactness of X is important in Theorem 1. However, a natural question arises.

Question 3 (M. Reed)**.** *Let X and Y be locally compact Hausdorff spaces. Is it true that any clopen subset of $X \times Y$ can be represented as a union of clopen boxes?*

Example 2. *There exist two locally compact Hausdorff spaces X and Y whose product contains a clopen subset that cannot be represented as a union of clopen boxes.*

Let X and Z be spaces from Example 1.

Let $Y_1 = \bigcup\{N_n : N_n \text{ is a copy of } \mathbb{N},\ n \in \mathbb{N}\}$ with discrete topology. Consider $Y_2 = \beta Y_1 \setminus [\bigcup\{\beta N_n \setminus N_n : n \in \mathbb{N}\}]_{\beta Y_1}$. That is, Y_2 is obtained from βY_1 by removing the closure of the union of the remainders of N_n's.

The space Y_2 is locally compact (since it is obtained from a compactum by removing a closed subset) and satisfies the following property.

(0) *Any clopen neighborhood of closed set $Y_2 \setminus Y_1$ contains almost all N_n's except maybe finite number of them.* (Proof: any sequence of elements $\{a_n \in N_n : n \in \mathbb{N}\}$ and the set $\bigcup\{\beta N_n \setminus N_n : n \in \mathbb{N}\}$ are disjoint closed subsets of σ-compact space $\bigcup\{\beta N_n : n \in \mathbb{N}\}$. Therefore, their closures in Stone-Čech compactification (which is βY_1) do not intersect. So the remainder of any sequence $\{a_n \in N_n : n \in \mathbb{N}\}$ lies entirely in $Y_2 \setminus Y_1$.)

The space Y_2 is almost what we need, but we want $Y_2 \setminus Y_1$ to live in a connected component. To achieve this, let us again consider βY_1, take the cone over $\beta Y_1 \setminus \bigcup\{\beta N_n : n \in \mathbb{N}\}$, and denote the resulting space by Y_3. Now, the space

$$Y = Y_3 \setminus [\bigcup\{\beta N_n \setminus N_n : n \in \mathbb{N}\}]_{\beta Y_1}$$

is the space we need. The space Y has the following properties:

(1) *Y consists of closed discrete sets N_n's and a connected component $Y \setminus Y_1$ (by construction);*

(2) *Y is locally compact (since obtained from a compactum by removing a closed subset);*

(3) *any neighborhood of $Y \setminus Y_1$ contains almost all N_n's except maybe finite number of them (see property (0)).*

Let Y^* be a quotient space of Y under a partition whose only non-trivial element is the connected component $Y \setminus Y_1$. And let $p : Y \to Y^*$ be a quotient map. The map p and space Y^* have the following properties.

(4) *Y^* is homeomorphic to Z (Z is the space in Example 1). This follows from properties (1), (3), and from construction of Z.*

(5) *For any clopen subset U of Y, $p(U)$ is clopen in Y^*.* (It follows from the definition of p and the fact that any clopen set that intersect $Y \setminus Y_1$ must contain $Y \setminus Y_1$ (due to connectedness)).

Now, X and Y are both locally compact. Let us prove that $X \times Y$ contains a clopen set that cannot be represented as a union of clopen boxes. Let $f = i \times p : X \times Y \to X \times Z$ be the product of the identity map $i : X \to X$ and the quotient map $p : Y \to Z$ (see property (5)). The map f satisfies the following property.

(6) *Image of any clopen box in $X \times Y$ under f is a clopen box in $X \times Z$* (follows from the definition of product maps, property (5), and the fact that i is identity).

Let A be a clopen subset of $X \times Z$ that cannot be represented as a union of clopen boxes (we constructed such a set in Example 2). Consider $f^{-1}(A)$ in $X \times Y$. By continuity of f, $f^{-1}(A)$ is clopen in $X \times Y$ and, by property (6), $f^{-1}(A)$ cannot be represented as a union of clopen boxes.

The space Y in our example is not metrizable. Can we make it metrizable? The following theorem shows that it is impossible.

Theorem 3 (Kenneth Kunen). *Suppose X and Y are both locally compact Hausdorff and paracompact. Then any clopen subset of $X \times Y$ is a union of clopen boxes.*

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Department of Mathematics, College of Arts & Sciences, Morton Hall 321, Athens, Ohio 45701-2979, USA

E-mail: rbuzyako@bing.math.ohiou.edu

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