Criteria for $k_M < \infty$ in Musielak-Orlicz spaces

Lianying Cao, Tingfu Wang

Abstract. In this paper, some necessary and sufficient conditions for $\sup\{k_x : \|x\|^0 = 1\} < \infty$ in Musielak-Orlicz function spaces as well as in Musielak-Orlicz sequence spaces are given.

Keywords: Musielak-Orlicz space, Orlicz norm

Classification: 46B20, 46B30

In Orlicz spaces endowed with the Orlicz norm, denote

$$k_M = \sup_{\|x\|_M^0=1} \Big\{k > 0 : \|x\|^0 = \frac{1}{k}(1+\rho_M(kx))\Big\}.$$

Since the study of many geometric properties in Orlicz spaces is related to whether $k_M < \infty$ is true, the criterion for $k_M < \infty$ has been discussed extensively. In 1986 S. Chen obtained a concise result in classical Orlicz spaces:

$$k_M < \infty \Longleftrightarrow M \in \nabla_2 \text{ (i.e. } N \in \Delta_2).$$

But because of the fact that the Musielak-Orlicz functions and condition Δ are more complicated, the corresponding problem in Musielak-Orlicz spaces has not been solved. And it has become to be an obstacle for the study of many geometric properties in these spaces. In this paper, we shall generalize the result of S. Chen to Musielak-Orlicz function and sequence spaces.

The triple (T, Σ, μ) stands for a finite nonatomic measurable space. A mapping $M : T \times [0, \infty) \to [0, \infty]$ is said to be Musielak-Orlicz function if it satisfies:

(*) for each $u \in [0, \infty)$, $M(t, u)$ is a μ-measurable function of t on T;
(**) for $t \in T$ (a.e.), $M(t, u)$ is convex and left-continuous with respect to u;
(***) for $t \in T$ (a.e.), $M(t, 0) = 0$, $\lim_{u\to\infty} M(t, u) = \infty$ and $M(t, u') < \infty$ for some $u' > 0$.

The subject supported by NSFC (19871020).

We denote by $N(t,v)$ the complementary function of $M(t,u)$, where

$$N(t,v) = \sup_{u\geq 0}\{uv - M(t,u)\} \qquad (t\in T,\ v\geq 0).$$

It is easy to see that N is also a Musielak-Orlicz function.

Let $x(t): T\to(-\infty,\infty)$ be a μ-measurable function. The linear set

$$\{x(t); \exists\,\lambda>0 \ \text{ such that } \ \rho_M(\lambda x) = \int_T M(t,\lambda x(t))\,d\mu < \infty\}$$

equipped with Orlicz norm

$$\|x\|^0 = \sup_{\rho_X(y)\leq 1}\int_T x(t)y(t)\,d\mu = \inf_{k>0}\frac{1}{k}(1+\rho_M(kx))$$

forms a Banach space denoted by L^0_M. It is called the Musielak-Orlicz function space. For $0\neq x\in L^0_M$, $\|x\|^0 = \frac{1}{k}(1+\rho_M(kx))$ iff $k\in k(x) = [k^*_x, k^{**}_x]$, where

$$k^*_x = \inf\Big\{k>0 : \rho_N(\rho(k|x|)) = \int_T N(t,p(t,k|x(t)|))\,d\mu \geq 1\Big\},$$
$$k^{**}_x = \sup\Big\{k>0 : \rho_N(\rho(k|x|)) = \int_T N(t,p(t,k|x(t)|))\,d\mu \leq 1\Big\}.$$

We say that $M(t,u)$ satisfies condition Δ ($M\in\Delta$ for short) if there exist $\lambda>1$ and a measurable nonnegative function δ defined on T with $\int_T\delta(t)\,d\mu<\infty$ such that

$$M(t,2u)\leq \lambda M(t,u)+\delta(t) \qquad (t\in T \text{ a.e.}, \quad -\infty<u<+\infty).$$

The right derivative of $M(t,u)$ ($N(t,v)$) at u (v) is denoted by $p(t,u)$ ($q(t,v)$, respectively).

We start with the following lemmas.

Lemma 1. *The following statements are equivalent:*

(1) *$N\in\Delta$, i.e. there exist $\lambda>1$ and $0\leq\delta(t)\in L^1$ such that*

$$N(t,2v)\leq\lambda N(t,v)+\delta(t) \qquad (t\in T \text{ a.e.},\ v\in R);$$

(2) *for any $\varepsilon>0$ there exist $\lambda>1$ and $0\leq\delta(t)\in L^1$ such that*

$$N(t,\frac{v}{\varepsilon})\leq\lambda N(t,v)+\delta(t) \qquad (t\in T \text{ a.e.},\ v\in R);$$

(3) *for any $\varepsilon\in(0,1)$ there exist $\theta\in(0,1)$ and $0\leq\delta(t)\in L^1$ such that*

$$M(t,\varepsilon u)\leq\theta\varepsilon M(t,u)+\delta(t) \qquad (t\in T \text{ a.e.},\ v\in R);$$

(4) *there exist $\varepsilon,\theta\in(0,1)$ and $0\leq\delta(t)\in L^1$ such that*

$$M(t,\varepsilon u)\leq\theta\varepsilon M(t,u)+\delta(t) \qquad (t\in T \text{ a.e.},\ v\in R).$$

Proof: See Theorem 1.13 in [2]. □

Lemma 2. *For any $0 \neq x \in L_M^0$, if $\int_{\{t\in T: x(t)\neq 0\}} N(t, B(t))\, d\mu > 1$ then $K(x) \neq \emptyset$, where $B(t) = \sup\{v \geq 0 : N(t, v) < \infty\}$.*

PROOF: See Theorem 1.35 in [2]. $\square$

In the following, we always denote

$$k_M = \sup_{\|x\|^0=1} \Big\{k > 0 : \|x\|^0 = \frac{1}{k}(1 + \rho_M(kx))\Big\}.$$

Theorem 1. *The necessary and sufficient condition for $k_M < \infty$ is $N \in \Delta$.*

PROOF: Necessity. Suppose that $N \notin \Delta$. For any $\varepsilon > 0$, define

$$\delta(t) = \sup\Big\{u \geq 0 : M(t, \varepsilon u) > \frac{\varepsilon}{1+\varepsilon} M(t, u)\Big\}.$$

Then

$$\int_T M(t, \delta(t))\, d\mu = \infty.$$

(Otherwise

$$M(t, \varepsilon u) \leq \frac{\varepsilon}{1+\varepsilon} M(t, u) + M(t, \delta(t)) \qquad (t \in T \text{ a.e., } \ u \in R).$$

This shows that $N \in \Delta$.)

Thus we can take $u(t) \geq 0$ satisfying

$$M(t, \varepsilon u(t)) > \frac{\varepsilon}{1+\varepsilon} M(t, u(t))$$

and

$$\int_T M(t, u(t))\, d\mu > \frac{1+\varepsilon}{\varepsilon}\,.$$

Then

$$\int_T M(t, \varepsilon u(t))\, d\mu > 1.$$

So $\|\varepsilon u\|^0 > 1$. And recalling $M(t, \varepsilon u(t)) < \infty$ ($t \in T$ a.e.), it is easy to check that there exists $\Omega \subset T$ such that $\|\varepsilon u\,|_{\Omega}\|^0 = 1$. Take $k \in K(\varepsilon u\,|_{\Omega})$, i.e.

$$1 = \|\varepsilon u\,|_{\Omega}\|^0 = \frac{1}{k}(1 + \rho_M(k\varepsilon u\,|_{\Omega})).$$

Then

$$\begin{aligned}\frac{1}{k}+\rho_M(\varepsilon u\,|\,_\Omega)\le \frac{1}{k}(1+\rho_M(k\varepsilon u\,|\,_\Omega)) &= \|\varepsilon u\,|\,_\Omega\|^0\\ &\le \varepsilon(1+\rho_M(\frac{1}{\varepsilon}\cdot \varepsilon u\,|\,_\Omega)) = \varepsilon(1+\rho_M(u\,|\,_\Omega))\\ &\le \varepsilon(1+\frac{1+\varepsilon}{\varepsilon}\int_\Omega M(t,\varepsilon u(t))\,d\mu)\\ &= \varepsilon+(1+\varepsilon)\rho_M(\varepsilon u\,|\,_\Omega).\end{aligned}$$

So $\frac{1}{k}\le \varepsilon+\varepsilon\rho_M(\varepsilon u\,|\,_\Omega)\le 2\varepsilon$. By the arbitrariness of $\varepsilon>0$, we obtain the contradiction $k_M=\infty$.

Sufficiency. Since $N\in\Delta$, according to Lemma 1, there exist $\eta>0$ and $0\le\delta(t)\in L^1$ such that

$$M(t,2u)\ge 2(1+2\eta)M(t,u)-\delta(t)\qquad (t\in T \text{ a.e., } v\in R).$$

So for u satisfying $M(t,u)>\frac{\delta(t)}{2\eta}$, we have

$$M(t,2u)\ge 2(1+\eta)M(t,u). \tag{1}$$

Take $D>0$ such that $D-1-\frac{1}{2\eta}\int_T\delta(t)\,d\mu\ge 1$. For any $x\in L^0_M$ with $\|x\|^0=1$, denote

$$H_x=\Big\{t\in T: M(t,D|x(t)|)>\frac{\delta(t)}{2\eta}\Big\}.$$

Since $1=\|x\|^0\le\frac{1}{D}(1+\rho_M(Dx))$, we get

$$\rho_M(Dx)\ge D-1.$$

It follows that

$$\begin{aligned}\int_{H_x}M(t,Dx(t))\,d\mu &= \rho_M(Dx)-\int_{T\setminus H_x}M(t,Dx(t))\,d\mu\\ &\ge D-1-\frac{1}{2\eta}\int_{T\setminus H_x}\delta(t)\,d\mu\\ &\ge D-1-\frac{1}{2\eta}\int_T\delta(t)\,d\mu\ge 1.\end{aligned}\tag{2}$$

By $N\in\Delta$, we get $B(t)=\infty$ (a.e.). Consequently, by virtue of Lemma 2, for any $x\in L^0_M$ with $\|x\|^0=1$, $K(x)\ne\emptyset$. If $k\in K(x)$, then $k\le D$ or $k>D$. If $k>D$ there exists $j\ge 1$ such that

$$2^{j-1}D<k\le 2^jD.$$

From (1) and (2), we have

$$\begin{aligned} 2^j D \geq k = 1 + \rho_M(kx) &> \int_{H_x} M(t, kx(t))\, d\mu \\ &\geq \int_{H_x} M(t, 2^{j-1} D|x(t)|)\, d\mu \\ &\geq 2^{j-1}(1+\eta)^{j-1} \int_{H_x} M(t, D|x(t)|)\, d\mu \\ &\geq (1+\eta)^{j-1} 2^{j-1}. \end{aligned}$$

It implies that $j - 1 \leq \log_{1+\eta} 2D$. Thus $k \leq D \cdot 2^{\log_{1+\eta} 2D+1}$. This shows that $k_M < \infty$.

Next we present the criterion for $k_M < \infty$ in the Musielak-Orlicz sequence space l_M^0. Let $M = \{M_i\}_{i=1}^\infty$ be a sequence of functions. For each i, $M_i(u)$ is convex and left continuous with respect to u, and satisfies $M_i(0) = 0$, $\lim_{u\to\infty} M_i(u) = \infty$ and $M_i(u') < \infty$ for some $u' > 0$. We denote by $N_i(v)$ the complementary function of $M_i(u)$, where $N_i(v) = \mathrm{Sup}_{u\geq 0}\{uv - M_i(u)\}$. $p_i(u)$ and $q_i(v)$ denote their right derivatives, respectively. $p_i^-(u)$ denotes the left derivative of $M_i(u)$. We say that M satisfies condition δ if there exist $\lambda > 1$, $i_0, c_i \geq 0$ $(i > i_0)$ with $\sum_{i>i_0} c_i < \infty$, and $a > 0$ such that

$$M_i(2u) \leq \lambda M_i(u) + c_i \qquad (i \geq i_0,\ M_i(u) \leq a).$$

□

In what follows, denote by $x = (x(i))_{i=1}^\infty$ a real sequence, and define a modular of x with respect to M by

$$\rho_M(x) = \sum_{i=1}^\infty M_i(|x(i)|).$$

The linear set

$$\left\{x : \exists\, c > 0,\ \rho_M\left(\frac{x}{c}\right) < \infty\right\}$$

equipped with the Orlicz norm

$$\|x\|^0 = \inf_{k>0} \frac{1}{k}(1 + \rho_M(kx)) = \sup_{\rho_N(y)\leq 1} \sum_{i=1}^\infty x(i)y(i)$$

forms a Banach space denoted by l_M^0 and called the Musielak-Orlicz sequence space. For any $0 \neq x \in l_M^0$ we define $K(x)$, k_x^*, k_x^{**} and k_M in the same way as in the function case. For each i, denote

$$b_i = \sup\{v \geq 0 : N_i(v) < \infty\}.$$

We obtain the following theorem.

Theorem 2. *$k_M < \infty$ if and only if*

(1) *$N \in \delta$;*

(2) *for any $a > 0$ there exists $\lambda > 1$ such that if $N_i(p_i(u)) \geq a$ then*

$$N_i(p_i(\lambda u)) > 1 \qquad (i = 1, 2, \dots).$$

Proof: Necessity. The necessity of (1) can be verified analogously to Theorem 1.

If (2) does not hold, there exist $a > 0$, $u_n > 0$ and i_n such that

$$N_{i_n}(p_{i_n}(u_n)) \geq a, \; N_{i_n}(p_{i_n}(nu_n)) \leq 1, \qquad (n = 1, 2, \dots).$$

Let $x_n : x_n(i_n) = u_n$, $x_n(i) = 0 \;\; (i \neq i_n) \;\; (n = 1, 2, \dots)$. Then $x_n \in l_M^0$ and

$$\|x\|^0 \geq u_n p_{i_n}(u_n) \geq N_{i_n}(p_{i_n}(u_n)) \geq a.$$

Since

$$\rho_N(p(nx_n)) = N_{i_n}(p_{i_n}(nu_n)) \leq 1,$$

by the definition of $k_{x_n}^{**}$ we have $k_{x_n}^{**} \geq n$. Consequently

$$k^{**}_{\frac{x_n}{\|x_n\|^0}} = \|x_n\|^0 k_{x_n}^{**} \geq na.$$

This shows that $k_M = \infty$.

Sufficiency. Since $N \in \delta$, there exist $\lambda' > 1$, i_0', $c_i \geq 0$ $(i > i_0')$ with $\sum_{i > i_0'} c_i < \infty$, and $a' > 0$ such that

$$N_i(2v) \leq \lambda' N_i(v) + c_i \qquad (i > i_0', \; N_i(v) \leq a').$$

Take $i_0 > i_0'$ satisfying $\sum_{i > i_0} c_i < 1$ and $a < a'$ satisfying

$$M_i(q_i(N_i^{-1}(a))) \leq \frac{1}{i_0} \qquad (i = 1, 2, \dots i_0).$$

Next take $\lambda > \lambda'$ such that

$$N_i(p_i(u)) \geq a \Longrightarrow N_i(p_i(\lambda u)) > 1 \qquad (i - 1, 2, \dots). \tag{3}$$

It is easy to prove that

$$N_i(2v) \leq \lambda N_i(v) + c_i \qquad (i > i_0, \; N_i(v) \leq a). \tag{4}$$

Notice that $\frac{1}{\lambda}N_i(2v)$ and $\frac{1}{\lambda}M_i(\frac{\lambda}{2}u)$ are complementary to each other. So for $i > i_0$, $N_i(p_i(u)) \le a$, we have

$$\begin{aligned} M_i(u) + N_i(p(u)) = up_i(u) &\le \frac{1}{\lambda}M_i\left(\frac{\lambda u}{2}\right) + \frac{1}{\lambda}N_i(2p_i(u)) \\ &\le \frac{1}{\lambda}M_i\left(\frac{\lambda u}{2}\right) + N_i(p_i(u)) + \frac{c_i}{\lambda} \\ &\le \frac{1}{2\lambda}M_i(\lambda u) + N_i(p_i(u)) + \frac{c_i}{\lambda}\,. \end{aligned}$$

Then

$$M_i(\lambda u) \ge 2\lambda M_i(u) - 2c_i \qquad (i > i_0,\ N_i(p_i(u)) \le a). \tag{5}$$

Condition (2) implies $N_i(b_i) > 1$ $(i = 1, 2, \ldots)$. Applying Theorem 1 in [3], for any $x \in l_M^0$ with $\|x\|^0 = 1$, we get $K(x) \ne \emptyset$. Then for any given $k \in K(x)$, we have $k \le 5\lambda$ or $k > 5\lambda$. If $k > 5\lambda$, then

$$N_i(p_i(5\lambda|x(i)|)) \le \rho_N(p(5\lambda|x|)) \le \rho_N(p^-(kx)) \le 1 \qquad (i = 1, 2, \ldots).$$

Applying (3) we have

$$N_i(p_i(5|x(i)|)) < a \qquad (i = 1, 2, \ldots),$$

i.e.

$$5|x(i)| \le q_i(N_i^{-1}(a)) \qquad (i = 1, 2, \ldots).$$

Consequently

$$\sum_{i=1}^{i_0} M_i(5|x(i)|) \le \sum_{i=1}^{i_0} M_i(q_i(N_i^{-1}(a))) \le 1.$$

From $1 = \|x\|^0 \le \frac{1}{5}(1 + \rho_M(5x))$, we deduce that $\rho_M(5x) \ge 4$. So

$$\sum_{i>i_0} M_i(5|x(i)|) \ge 3. \tag{6}$$

Take $j \ge 1$ such that $5\lambda^j < k < 5\lambda^{j+1}$. Combining (3) with

$$N_i(p_i(\lambda 5\lambda^{j-1}|x(i)|)) \le \rho_N(p(5\lambda^j|x|)) \le 1,$$

we obtain

$$N_i(p_i(5\lambda^{j-1}|x(i)|)) < a \qquad (i = 1, 2, \ldots).$$

From (5) and (6), we conclude that

$$\begin{aligned}
5\lambda^{j+1} > k = 1 + \rho_M(kx) &> \sum_{i>i_0} M_i(k|x(i)|) \\
&\geq \sum_{i>i_0} M_i(5\lambda^j|x(i)|) \\
&\geq \sum_{i>i_0} \Big\{(2\lambda)^j M_i(5|x(i)|) - (2\lambda)^{j-1}(2c_i) - (2\lambda)^{j-2}(2c_i) - \cdots - 2c_i\Big\} \\
&= \sum_{i>i_0} (2\lambda)^j \Big\{M_i(5|x(i)|) - \Big(\frac{1}{2\lambda} + \frac{1}{(2\lambda)^2} + \cdots + \frac{1}{(2\lambda)^j}\Big) \cdot 2c_i\Big\} \\
&\geq (2\lambda)^j \sum_{i>i_0} \Big\{M_i(5|x(i)|) - 2c_i\Big\} \geq (2\lambda)^j.
\end{aligned}$$

Thus $j \leq \log_2 5\lambda$, i.e. $k \leq 5\lambda^{\log_2 5\lambda + 1}$. Hence $k_M < \infty$. □

References

[1] Chen S., *Some rotundities of Orlicz space with Orlicz norm*, Bull. Polish Acad. Sci. Math. **34** (1986), 585–596.

[2] Chen S., *Geometry of Orlicz space*, Dissertationes Math. **356** (1996), 1-204.

[3] Wu C., Sun H., *Norm calculation and complex convexity of the Musielak-Orlicz sequence space*, Chinese Ann. Math. **12A** (1991), suppl., 98–102.

Harbin University of Science and Technology, P.O. Box 123, 150080 Harbin, China

E-mail: wangtf@ems.hrbmu.edu.cn

(*Received* July 3, 2000, *revised* January 2, 2001)