# On inverses of $\delta$-convex mappings

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*Abstract.* In the first part of this paper, we prove that in a sense the class of bi-Lipschitz $\delta$-convex mappings, whose inverses are locally $\delta$-convex, is stable under finite-dimensional $\delta$-convex perturbations. In the second part, we construct two $\delta$-convex mappings from $\ell_1$ onto $\ell_1$, which are both bi-Lipschitz and their inverses are nowhere locally $\delta$-convex. The second mapping, whose construction is more complicated, has an invertible strict derivative at 0. These mappings show that for (locally) $\delta$-convex mappings an infinite-dimensional analogue of the finite-dimensional theorem about $\delta$-convexity of inverse mappings (proved in [7]) cannot hold in general (the case of $\ell_2$ is still open) and answer three questions posed in [7].

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## 1. Introduction

Let $X, Y$ be normed linear spaces, $A \subset X$ be an open convex set. A mapping $F: A \to Y$ is called *$\delta$-convex on $A$*, if there exists a continuous function $f: A \to \mathbb{R}$ such that $y^* \circ F + f$ is a continuous convex function on $A$ for each $y^* \in Y^*$, $\|y^*\| = 1$. If this is the case, we say that $f$ is a *control function of $F$*. A mapping $G: B \to Y$ defined on an open set $B \subset X$ is said to be *locally $\delta$-convex*, if for each point $b \in B$ there exists an open convex neighborhood $V$ of $b$ so that $G|_V$ is $\delta$-convex.

This definition of (local) $\delta$-convexity for Banach space-valued mappings is due to L. Veselý and L. Zajíček and was introduced in [7]. Much about properties of (locally) $\delta$-convex mappings can be found in that article. The history of the notion of a $\delta$-convex function goes back to A.D. Alexandrov ([1], [2]). P. Hartman [5] defined and investigated the notion of delta-convex mappings between Euclidean spaces. For the history of notions of $\delta$-convex functions and mappings, we refer the interested reader to [7]. They have applications in many areas of mathematics, for example in the non-smooth optimization theory. For a recent application of $\delta$-convex functions in the theory of Banach spaces, see articles of M. Cepedello Boiso [3], [4].

In the first part of this paper, we prove a theorem about $\delta$-convexity of inverses of $\delta$-convex mappings (an analogue of the finite-dimensional Theorem 5.2 in [7])

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for a special class of (infinite-dimensional) $\delta$-convex mappings. This class contains bi-Lipschitz $\delta$-convex mappings, that arose as a sum of a bi-Lipschitz $\delta$-convex mapping with a locally $\delta$-convex inverse and a finite-dimensional $\delta$-convex mapping. Our theorem is also a strengthening of Theorem 4.5 in [7] for the considered special class of mappings. So we obtain that a counterexample to Problem 1 in [7] cannot be found in that class.

L. Veselý and L. Zajíček ask in [7] (Problem 1) whether the inverse of a locally $\delta$-convex bi-Lipschitz mapping is also locally $\delta$-convex. They prove that it is so when we consider the finite dimensional case (see Theorem 4.5 in [7]) and that the answers is yes "almost everywhere" (on an open dense set), when the source space is an Asplund-Banach space and we consider bi-Lipschitz locally $\delta$-convex bijections between open convex sets (see Theorem 4.6 in [7]). In the second part of this paper we construct two $\delta$-convex mappings from $\ell_1$ onto $\ell_1$, which are both bi-Lipschitz and whose inverses are nowhere locally $\delta$-convex. This gives a negative answer to the question asked in Problem 1 ([7]). The second mapping also has an invertible strict derivative at 0 (however, we pay for this property by substantial technical complications). This gives a (negative) solution to Problem 2 from [7].

The authors of [7] also ask (Problem 3) whether a $\delta$-convex mapping, which is strictly differentiable at a point, admits a control function, which is strictly differentiable at that point. In [6] the authors gave an answer to that question by constructing a $\delta$-convex function $\mathbb{R}^2 \to \mathbb{R}$, which is strictly differentiable at 0, but which does not admit a control function having this property. It is possible to prove (using a part of proof of Theorem 4.6 from [7]) that our second mapping neither admits a control function, which is strictly differentiable at 0, so we give another solution to this problem.

Let $F\colon X \to Y$ be a mapping between two normed linear spaces and $K > 0$. By $\operatorname{Lip} F$ we shall denote the smallest Lipschitz constant of $F$. We shall say, that $F$ is *$K$-bi-Lipschitz* if for all $x, y \in X$ it holds that $\frac{1}{K}\|x-y\| \le \|F(x)-F(y)\| \le K\|x-y\|$. We say that $F\colon X \to Y$ is *bi-Lipschitz*, if there is a constant $L > 0$ such that $F$ is $L$-bi-Lipschitz.

Let $X, Y$ be normed linear spaces, $D \subset X$ and $F\colon D \to Y$ a mapping. We say that $A \in L(X,Y)$ is a *strict derivative of $F$ at a point $a \in D$* (see [7]), if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|F(y)-F(x)-A(y-x)\| \le \varepsilon\|y-x\|$, whenever $x, y \in B(a,\delta)$, where we take $B(a,\delta) = \{x \in X; \|x-a\| < \delta\}$.

Let us recall some facts about $\delta$-convex mappings:

**Lemma 1.1** ([7, Lemma 1.5]). *Let $X, Y, Z, T$ be normed linear spaces, let $A \subset X$ and $B \subset Z$ be open convex sets. Suppose that $F\colon A \to Y$ is a $\delta$-convex mapping with a control function $f$ on $A$ and let $G\colon Z \to X$, $H\colon Y \to T$ be continuous affine mappings. Then the following assertions hold.*

(a) *The mapping $H \circ F$ is $\delta$-convex with the control function* $\operatorname{Lip}(H) \cdot f$ *on $A$.*
(b) *If $G(B) \subset A$, then $F \circ G$ is $\delta$-convex with the control function $f \circ G$ on $B$.*

**Proposition 1.2** ([7, Proposition 1.10]). *Every $\delta$-convex mapping is locally Lipschitz.*

**Corollary 1.3** ([7, Corollary 1.18]). *Let $X, Y$ be normed linear spaces, $A \subset X$ be an open convex set and let both $F\colon A \to Y$, $f\colon A \to \mathbb{R}$ be continuous. Then the following assertions are equivalent:*

(i) *$F$ is $\delta$-convex on $A$ with a control function $f$;*
(ii) $\left\| \frac{F(x)+F(y)}{2} - F\left(\frac{x+y}{2}\right) \right\|_Y \le \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)$ *whenever $x, y \in A$.*

**Proposition 1.4** ([7, Proposition 4.1]). *Let $X, Y, Z$ be normed linear spaces and let $A \subset X$, $B \subset Y$ be open convex sets. Let $F\colon A \to B$ be $\delta$-convex on $A$ with a control function $f$ and let $G\colon B \to Z$ be $\delta$-convex on $B$ with a control function $g$. Suppose further that $G, g$ are Lipschitz on $B$ with constants $L_G$, $L_g$.*

*Then the composite mapping $G \circ F$ is $\delta$-convex on $A$ with a control function $h = g \circ F + (L_G + L_g)f$.*

**Theorem 1.5** ([7, Theorem 5.1]). *Let $X, Z$ be normed linear spaces and let $Y$ be a finite dimensional normed linear space. Let $A \subset X$, $B \subset Y$ be open convex sets, $c > 0$ and let $G\colon A \times B \to Z$ be a $\delta$-convex mapping such that $\|G(x,y) - G(x,\widetilde{y})\| \ge c\|y - \widetilde{y}\|$ whenever $x \in A$, $y, \widetilde{y} \in B$. Let $\varphi\colon A \to B$ be a mapping satisfying $G(x, \varphi(x)) = 0$ on $A$.*

*Then $\varphi$ is locally $\delta$-convex on $A$.*

## 2. Inverse theorem

**Theorem 2.1.** *Let $X, Z$ be Banach spaces, $A \subset X$, $B, G \subset Z$ be nonempty open sets, let further $A$ be convex, and let $F\colon A \to B$ be a bi-Lipschitz $\delta$-convex mapping onto $B$, such that $F^{-1}$ is locally $\delta$-convex on $B$. Let $\xi\colon A \to Z$ be $\delta$-convex and such that* $\dim \operatorname{span} \xi(A) < \infty$. *Further let $H = F + \xi$ be a bi-Lipschitz mapping onto $G$.*

*Then the mapping $H^{-1}\colon G \to A$ is locally $\delta$-convex.*

*Remark* 1. The mapping $H$ from Theorem 2.1 is $\delta$-convex because it is a sum of two such mappings.

Proof: We want to prove that $H^{-1}$ is locally $\delta$-convex. Let us denote $Y = \operatorname{span} \xi(A)$. Choose $z_0 \in G$. Denote $x_0 = H^{-1}(z_0)$ and choose $\varepsilon > 0$ so, that $B(F(x_0), \varepsilon) \subset B$ and so that $F^{-1}$ is $\delta$-convex on $B(F(x_0), \varepsilon)$. Put $V = B_Y(\xi(x_0), \varepsilon/2)$ and choose an open convex neighborhood $U$ of $z_0$ so that

$$\xi\left(H^{-1}(U)\right) \subset V \quad \text{and} \quad U \subset B(z_0, \varepsilon/2). \tag{2.1}$$

This is possible since $H$ is bi-Lipschitz and $\xi$ is locally Lipschitz (see Proposition 1.2). Then $U - V \subset B(F(x_0), \varepsilon)$ holds, as for $x \in U, y \in V$ we have the following inequality

$$\|x - y - F(x_0)\| = \|x - F(x_0) - \xi(x_0) + \xi(x_0) - y\| < 2\,\frac{\varepsilon}{2} = \varepsilon.$$

Let us define

$$L\colon U \times V \to Y, L(x,y) = H\left(F^{-1}(x-y)\right) - x.$$

It follows from Proposition 1.4 that the mapping $L$ is $\delta$-convex. Take arbitrary $x \in U, y, \overline{y} \in V$. Then the following holds for $L$:

$$\begin{aligned}\|L(x,y) - L(x,\overline{y})\| &= \left\|H\left(F^{-1}(x-y)\right) - H\left(F^{-1}(x-\overline{y})\right)\right\| \\ &\geq K^{-1}\left\|F^{-1}(x-y) - F^{-1}(x-\overline{y})\right\| \\ &\geq K^{-1}C^{-1}\|\overline{y} - y\|,\end{aligned}$$

where $K > 0$ ($C > 0$, respectively) is a bi-Lipschitz constant of the mapping $H$ (of the mapping $F$, respectively). To be able to apply Theorem 5.1 from [7], it remains to show that for each $x \in U$ it holds for $\varphi(x) = \xi \circ H^{-1}(x)$ that $L(x,\varphi(x)) = 0$ and $\varphi(x) \in V$. We put $z = H^{-1}(x)$ and then the following holds:

$$\begin{aligned}L(x,\varphi(x)) &= H\left(F^{-1}(x - \varphi(x))\right) - x \\ &= H\left(F^{-1}\left(F(z) + \xi(z) - \xi(z)\right)\right) - H(z) \\ &= H(z) - H(z) = 0.\end{aligned}$$

From the first formula in (2.1) it is easy to see that $\varphi(x) \in V$. Thus we obtained a mapping $\varphi\colon U \to V$. Now all the assumptions of Theorem 1.5 are fulfilled (following the notation of [7] we take $X = Y, Z, Y$, $A = U$, $B = V$, $c = K^{-1}C^{-1}$, $G = L, \varphi$). So, we get that $\varphi$ is locally $\delta$-convex in $U$.

Pick a neighborhood $U_0$ of $z_0$ so that $\varphi$ is $\delta$-convex on $U_0$. Then in $W = U \cap U_0$ we have $H^{-1}(x) = F^{-1}(x - \varphi(x))$ and it follows from Proposition 1.4 that $H^{-1}$ is $\delta$-convex on $W$. □

## 3. Two examples

The following theorem gives answers to questions asked in Problems 1, 2, and 3 in [7].

**Theorem 3.1.** *There is a mapping $N\colon \ell_1 \to \ell_1$, which is bi-Lipschitz, maps $\ell_1$ onto $\ell_1$, is $\delta$-convex, and such that the inverse mapping $N^{-1}$ is nowhere locally $\delta$-convex.*

*There even exists a mapping $\widetilde{N}: \ell_1 \to \ell_1$, which is bi-Lipschitz, $\delta$-convex, onto $\ell_1$, strictly differentiable at 0, $\widetilde{N}'(0) = Id_{\ell_1}$, and such that the inverse $\widetilde{N}^{-1}$ is nowhere locally $\delta$-convex.*

*Remark* 2. 1. A mapping is nowhere locally $\delta$-convex, when it is not locally $\delta$-convex at any point.

2. The mapping $N$ only gives answer to question in Problem 1, but it is the most interesting one. The construction of $N$ is substantially simpler than that of $\widetilde{N}$, regardless of the fact, that they both use a similar idea.
3. Let us also note, that the mapping $\widetilde{N}$ is a counterexample to Problem 3, because it does not admit a control function, which is strictly differentiable at 0. Suppose such a function exists. Then it follows from the proof of Theorem 4.6 in [7] that the mapping $\widetilde{N}^{-1}$ is $\delta$-convex in a neighbourhood of 0 and that is a contradiction with the fact that $\widetilde{N}^{-1}$ is nowhere locally $\delta$-convex.
4. In the proof of Theorem 3.1 we always consider $\mathbb{R}^n$ endowed with the $\ell_1$-norm (i.e. $\|x\| = \sum_{i=1}^n |x_i|$ for $x \in \mathbb{R}^n$).

Let us first prove some auxiliary lemmas. The "building blocks" for our mappings will be mappings between $\mathbb{R}^n$ with some suitable properties.

**Lemma 3.2.** *Let $c \in (0,1)$, $L > 0$, and let $\xi_i: \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n-1$, be c-Lipschitz $\delta$-convex functions and let $\varphi_i: \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n-1$, be their L-Lipschitz control functions satisfying $\varphi_i(0) = 0$. Then the mapping $\Psi: \mathbb{R}^n \to \mathbb{R}^n$ (defined as $(\Psi(x))_i = \xi_i(x_{i+1})$ for $i < n$ and $(\Psi(x))_n = 0$) is c-Lipschitz and $\delta$-convex with control function $\varphi: \mathbb{R}^n \to \mathbb{R}$ defined as $\varphi(x) = \sum_{i=1}^{n-1} \varphi_i(x_{i+1})$ (note that $\varphi(0) = 0$). If we further define a mapping $F: \mathbb{R}^n \to \mathbb{R}^n$ as $F(x) = x - \Psi(x)$, then $F$ and $F^{-1}$ are Lipschitz with the constant $\max\left\{\frac{1}{1-c}, 1+c\right\}$, $F$ is $\delta$-convex with the control function $\varphi$, and $F$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. It also holds that* $\operatorname{Lip} \varphi \le L$*.*

*Let $\varepsilon > 0$ and $M \ge 0$. If there exists an $M$-Lipschitz function $\theta: B(0, \varepsilon) \to \mathbb{R}$, which is a control function for $F^{-1}|_{B(0,\varepsilon)}$, then there exists an $M$-Lipschitz control function for $\xi_1 \circ \dots \circ \xi_{n-1}$ on $(-\varepsilon, \varepsilon)$.*

Proof of Lemma 3.2: Let us first prove, that $\Psi$ is Lipschitz. For the rest of the proof choose $x, y \in \mathbb{R}^n$. Then

$$\|\Psi(x) - \Psi(y)\|_1 = \sum_{i=1}^{n-1} |\xi_i(x_{i+1}) - \xi_i(y_{i+1})| \le \sum_{i=1}^{n-1} c\,|x_{i+1} - y_{i+1}| \le c\,\|x - y\|_1.$$

Considering $\varphi$, we get

$$|\varphi(x) - \varphi(y)| = \left|\sum_{i=1}^{n-1} \varphi_i(x_{i+1}) - \varphi_i(y_{i+1})\right| \le L \sum_{i=1}^{n-1} |x_{i+1} - y_{i+1}| \le L\|x - y\|_1.$$

Let us see why $\Psi$ is $\delta$-convex:

$$
\begin{aligned}
\left\|\frac{\Psi(x)+\Psi(y)}{2}-\Psi\left(\frac{x+y}{2}\right)\right\|_1 &= \sum_{i=1}^{n-1}\left|\frac{\xi_i(x_{i+1})+\xi_i(y_{i+1})}{2}-\xi_i\left(\frac{x_{i+1}+y_{i+1}}{2}\right)\right| \\
&\le \sum_{i=1}^{n-1}\frac{\varphi_i(x_{i+1})+\varphi_i(y_{i+1})}{2}-\varphi_i\left(\frac{x_{i+1}+y_{i+1}}{2}\right) \\
&= \frac{\varphi(x)+\varphi(y)}{2}-\varphi\left(\frac{x+y}{2}\right).
\end{aligned}
\tag{3.2}
$$

It follows from Corollary 1.3 that $\Psi$ is $\delta$-convex with the control function $\varphi$.

Let us now look at $F$ — it is certainly a $\delta$-convex mapping as a sum of such maps. To see that $F$ is bi-Lipschitz, let us look at the following estimates:

$$
\begin{aligned}
(1-\operatorname{Lip}\Psi)\|x-y\|_1 &\le \|x-y\|_1-\|\Psi(x)-\Psi(y)\|_1 \\
&\le \|F(x)-F(y)\|_1 \\
&\le (1+\operatorname{Lip}\Psi)\|x-y\|_1.
\end{aligned}
$$

So $(1-c)\|x-y\|_1 \le \|F(x)-F(y)\|_1 \le (1+c)\|x-y\|_1$.

Let us show that a convex function is a control function of $F$ iff it is a control function of $\Psi$. It follows from Corollary 1.3 and from the following equality:

$$
\left\|\frac{F(x)+F(y)}{2}-F\left(\frac{x+y}{2}\right)\right\|_1 = \left\|\frac{\Psi(x)+\Psi(y)}{2}-\Psi\left(\frac{x+y}{2}\right)\right\|_1.
$$

So we get (see (3.2)), that $\varphi$ is a control function of $F$.

Now we show that $F$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Suppose we have $y=F(x)$. Then $y_1=x_1-\xi_1(x_2),\dots,y_{n-1}=x_{n-1}-\xi_{n-1}(x_n), y_n=x_n$. We see that we can express $x_i$ using $y_j$, $j=1,\dots,n$. We can also use a different argument, which is based on the Banach fixed point theorem.

Let $\theta$ be according to the assumptions. For $y=(y_1,\dots,y_n)\in B(0,\varepsilon)$ such that $y_i=0$ for $i<n$ it holds, that $F^{-1}(y)=(\xi_1\circ\dots\circ\xi_{n-1}(y_n),\dots,\xi_{n-1}(y_n),y_n)$, what is shown by direct computation. Let us define a function $t\colon\mathbb{R}\to\mathbb{R}^n$ as $t(x)=(\underbrace{0,\dots,0}_{(n-1)-\text{times}},x)$ and denote $\pi\colon\mathbb{R}^n\to\mathbb{R}$ the projection onto the first coordinate (i.e. $\pi((x_1,\dots,x_n))=x_1$). Then for $x\in(-\varepsilon,\varepsilon)$ it clearly holds, that $\xi_1\circ\dots\circ\xi_{n-1}(x)=\pi\circ F^{-1}\circ t(x)$. According to Lemma 1.1 it is true, that $F^{-1}\circ t$ is on $(-\varepsilon,\varepsilon)$ $\delta$-convex with the control function $\theta\circ t$. Applying the same lemma, we get that $\pi\circ F^{-1}\circ t$ is $\delta$-convex with the control function $\operatorname{Lip}\pi\cdot(\theta\circ t)$. Note that $\operatorname{Lip}\pi=\operatorname{Lip}t=1$. As

$$
\operatorname{Lip}(\operatorname{Lip}\pi\cdot(\theta\circ t)) \le \operatorname{Lip}\pi\cdot\operatorname{Lip}\theta\cdot\operatorname{Lip}t=\operatorname{Lip}\theta=M,
$$

the function $\xi_1 \circ \cdots \circ \xi_{n-1}$ is $\delta$-convex on $(-\varepsilon, \varepsilon)$ with the control function $\theta \circ t$, which is $M$-Lipschitz. This concludes the proof. $\square$

*Remark* 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Then for $x \in \mathbb{R}$ we denote by $f'_+(x)$ ($f'_-(x)$, respectively) the right derivative (the left derivative, respectively) of the function $f$ at $x$, if it exists.

**Lemma 3.3.** *Let $U \subset \mathbb{R}$ be an open interval, $f\colon U \to \mathbb{R}$ be a $\delta$-convex function and $\varphi\colon U \to \mathbb{R}$ be its control function. Then the following holds:*

$$\varphi'_+(x) - \varphi'_-(x) \geq \left| f'_+(x) - f'_-(x) \right| \quad \text{for all} \ \ x \in U.$$

*Let $x_1, \dots, x_k \in U$ be an increasing sequence of distinct real numbers, $k \in \mathbb{N}$. Then*

$$\operatorname{Lip} \varphi \geq \frac{1}{2} \sum_{i=1}^{k} \left| f'_+(x_i) - f'_-(x_i) \right|.$$

Proof of Lemma 3.3: Concerning the first part of the lemma: since $\varphi$ is a control function for $f$, the functions $f + \varphi$ and $-f + \varphi$ are convex in $U$. Take an arbitrary $x \in U$. Then

$$(f + \varphi)'_+(x) \geq (f + \varphi)'_-(x) \text{ and } (-f + \varphi)'_+(x) \geq (-f + \varphi)'_-(x).$$

It is easy to see that for a $\delta$-convex function unilateral derivatives exist. We get that

$$\varphi'_+(x) - \varphi'_-(x) \geq \left| f'_+(x) - f'_-(x) \right|.$$

Concerning the second part of the lemma: it is easy to see that

$$\begin{aligned} \varphi'_+(x_k) - \varphi'_-(x_1) &= \sum_{i=1}^{k} \left( \varphi'_+(x_i) - \varphi'_-(x_i) \right) + \sum_{i=2}^{k} \left( \varphi'_-(x_i) - \varphi'_+(x_{i-1}) \right) \\ &\geq \sum_{i=1}^{k} \left( \varphi'_+(x_i) - \varphi'_-(x_i) \right). \end{aligned}$$

We only used the fact that $\varphi$ is convex. Now we have

$$\begin{aligned} 2 \operatorname{Lip} \varphi &\geq \left| \varphi'_+(x_k) \right| + \left| \varphi'_-(x_1) \right| \quad \geq \varphi'_+(x_k) - \varphi'_-(x_1) \\ &\geq \sum_{i=1}^{k} \left( \varphi'_+(x_i) - \varphi'_-(x_i) \right) \geq \sum_{i=1}^{k} \left| f'_+(x_i) - f'_-(x_i) \right|. \end{aligned}$$

We again used the fact that $\varphi$ is a convex function. $\square$

**Definition 3.4.** In the sequel we shall use the following notation: let $\varepsilon > 0$ and $k > 0$ be given. Then we define $f_\varepsilon^k : \mathbb{R} \to \mathbb{R}$ as

$$f_\varepsilon^k(x) = \begin{cases} 0 & \text{for } \ x \le 0, \\ k\,x & \text{for } \ x \in (0, \varepsilon], \\ 2\,k\,\varepsilon - k\,x & \text{for } \ x \in (\varepsilon, 2\,\varepsilon], \\ 0 & \text{for } \ x > 2\,\varepsilon. \end{cases}$$

We see, that this function is $k$-Lipschitz. Let us define $g_\varepsilon^k : \mathbb{R} \to \mathbb{R}$ as

$$g_\varepsilon^k(x) = \begin{cases} 0 & \text{for } \ x \le 0, \\ k\,x & \text{for } \ x \in (0, \varepsilon], \\ 3\,k\,x - 2\,k\,\varepsilon & \text{for } \ x \in (\varepsilon, 2\,\varepsilon], \\ 4\,k\,x - 4\,k\,\varepsilon & \text{for } \ x > 2\,\varepsilon. \end{cases}$$

Again, it is easy to see that $g_\varepsilon^k$ is $4k$-Lipschitz, convex and further, that $(f_\varepsilon^k + g_\varepsilon^k)$, $(-f_\varepsilon^k + g_\varepsilon^k)$ are convex, so $f_\varepsilon^k$ is $\delta$-convex with the control function $g_\varepsilon^k$.

The following two lemmas will allow us to construct a sequence of functions with suitable properties. We shall use them for the construction of our mappings.

**Definition 3.5.** Let $U \subset \mathbb{R}$ be open, $I \subset U$ be an interval, $c \in \mathbb{R}$ and $f : U \to \mathbb{R}$ a function. Then we say, that $f$ *is affine in the interval $I$ with tangent $c$*, if there exists $d \in \mathbb{R}$ so that for all $x \in I$ the equality $f(x) = c\,x + d$ holds. Further we define $\operatorname{supp} f = \{x \in U\,;\ f(x) \neq 0\}$.

**Lemma 3.6.** *Suppose we are given $\delta > 0$ and $c > 0$. Then there exists a sequence of functions $\{h_n : \mathbb{R} \to \mathbb{R}\}_{n \in \mathbb{N}}$ such that the following conditions are fulfilled for all $n \in \mathbb{N}$:*

1. *$h_n(0) = 0$, $h_n$ is $c$-Lipschitz, $\delta$-convex and there exists $\nu_n$ convex control function for $h_n$ satisfying* $\operatorname{Lip} \nu_n \le 4c$, $\nu_n(0) = 0$,
2. *if $\phi_n$ is a control function for $h_n \circ \cdots \circ h_1$ in $(0, \delta)$, then* $\operatorname{Lip} \phi_n \ge c\,(2c)^{n-1}$.

Proof of Lemma 3.6: We shall construct functions $h_n$ by induction so that conditions 1 and 2 of the lemma are satisfied and also that the following conditions hold for all $n \in \mathbb{N}$:

3. $h_n(x) \ge 0$ for all $x \in \mathbb{R}$, $\operatorname{supp} h_n \subset [0, \delta)$,
4. there exist $2^n$ disjoint intervals $(a_i, b_i)$, where $i = 1, \dots, 2^n$, so that $h_n \circ \cdots \circ h_1$ is in $[a_i, b_i]$ affine with tangent $\pm\, c^n$ and $h_n \circ \cdots \circ h_1$ is equal to 0 in one of the boundary points of each of these intervals,
5. for the function $\beta = h_n \circ \cdots \circ h_1$ there exist $2^{n-1}$ points in $(0, \delta)$, where the following condition is fulfilled:

$$\left|\beta'_+(x) - \beta'_-(x)\right| \ge 2\,c^n.$$

We take $h_1 = f^c_{\delta/4}$ and $\nu_1 = g^c_{\delta/4}$. Everything holds, if we take $(a_1, b_1) = (0, \delta/4)$ and $(a_2, b_2) = (\delta/4, \delta/2)$. Suppose that $n > 1$ and we have constructed $h_i$ for $i < n$. Now it suffices to prove, that there exists $h_n$, so that the required conditions are satisfied. Let us define

$$\widetilde{d} = \min\left\{\max\left\{h_{n-1}\circ\cdots\circ h_1([a_i, b_i])\right\}; i = 1, \ldots, 2^{n-1}\right\},$$

where $a_i$, $b_i$ are as in condition 4 for $(n-1)$ and finally

$$d = \min\left\{\widetilde{d}, \delta/2\right\}. \tag{3.3}$$

Then obviously $d > 0$. We take $h_n = f^c_{d/2}$ and $\nu_n = g^c_{d/2}$. Conditions 1 and 3 are clearly satisfied. It remains to show that the rest of the conditions holds.

**Ad 4.** Let $(a_i, b_i)$, $i = 1, \ldots, 2^{n-1}$, be as in condition 4 for $(n-1)$. Take $1 \le i \le 2^{n-1}$. Suppose that $h_{n-1}\circ\cdots\circ h_1(a_i) = 0$. The case when $h_{n-1}\circ\cdots\circ h_1(b_i) = 0$ is analogous. Then the function $h_{n-1}\circ\cdots\circ h_1$ is $[a_i, b_i]$ increasing and equal to $c^{n-1}(x - a_i)$. It follows from (3.3) that there exists $t_i \in (a_i, b_i]$ so that $h_{n-1}\circ\cdots\circ h_1(t_i) = d$. In $\left[a_i, \frac{a_i+t_i}{2}\right]$ the function $h_n\circ\cdots\circ h_1$ is affine with tangent $c^n$, it is equal to 0 in $a_i$, in $\left[\frac{a_i+t_i}{2}, t_i\right]$ the function $h_n\circ\cdots\circ h_1$ is affine with tangent $-c^n$ and it is equal to 0 in $t_i$.

Intervals of kind either $\left(a_i, \frac{a_i+t_i}{2}\right)$, $\left(\frac{a_i+t_i}{2}, t_i\right)$ or $\left(t_i, \frac{t_i+b_i}{2}\right)$, $\left(\frac{t_i+b_i}{2}, b_i\right)$ (in case that $h_{n-1}\circ\cdots\circ h_1(b_i) = 0$) form for $i = 1, \ldots, 2^{n-1}$ a family of $2^n$ intervals, where condition 4 for $n$ is fulfilled.

**Ad 5.** It is enough to realize that at points of kind $y_i = \frac{a_i+t_i}{2}$ (or $y_i = \frac{t_i+b_i}{2}$) for $i = 1, \ldots, 2^{n-1}$, $t_i$ is taken as in the last two paragraphs, the equality $\left|\beta'_+(y_i) - \beta'_-(y_i)\right| = 2c^n$ holds, where $\beta = h_n\circ\cdots\circ h_1$. It follows from the selection of $h_n$ and points $a_i$, $b_i$. But then also condition 5 from the construction is fulfilled.

**Ad 2.** Let $\phi\colon (0, \delta) \to \mathbb{R}$ be a convex function and a control function for $\beta = h_n\circ\cdots\circ h_1$. We select points $z_i$ for $i = 1, \ldots, 2^{n-1}$. These are taken to be the $2^{n-1}$ points of condition 5 for $n$. Then according to Lemma 3.3 the following holds:

$$\operatorname{Lip}\phi \ge \frac{1}{2}\sum_{i=1}^{2^{n-1}}\left|\beta'_+(z_i) - \beta'_-(z_i)\right| = \frac{1}{2}\cdot 2^{n-1}\cdot(2c^n) = c\,(2c)^{n-1}.$$

□

The more complicated version is the following:

**Lemma 3.7.** *Suppose we are given $\delta > 0$ and $M \in (0,1)$. Then there exist $m \in \mathbb{N}$, such that $\frac{1}{2^m} < M$, and a sequence of functions $\left\{\widetilde{h}_n : \mathbb{R} \to \mathbb{R}\right\}_{n\in\mathbb{N}}$ satisfying the following conditions for all $n \in \mathbb{N}$:*

1. *$\widetilde{h}_n(0) = 0$, $\widetilde{h}_n$ is $\left(\frac{1}{2^m}\right)$-Lipschitz, $\delta$-convex and there exists $\widetilde{\nu}_n$, a convex control function for $\widetilde{h}_n$ satisfying $\operatorname{Lip} \widetilde{\nu}_n \leq 4$ and $\widetilde{\nu}_n(0) = 0$,*
2. *let $\psi : (0,\delta) \to \mathbb{R}$ be a control function for $\widetilde{h}_n \circ \cdots \circ \widetilde{h}_1$ in $(0,\delta)$. Then*
$$\operatorname{Lip} \psi \geq 2^{n-1},$$
3. *there exists $\lambda_n > 0$ such that $\widetilde{h}_i([0,\lambda_n]) = \{0\}$ for $i \leq n$.*

**Definition 3.8.** Suppose we are given $a < b, a, b \in \mathbb{R}$, $l \in \mathbb{R}$ and $n \in \mathbb{N}$. Let us put $\varepsilon = (b-a)/n$. We divide the interval $[a,b]$ into $n$ subintervals of the same length, with boundary points $c_1 = a, \dots, c_{n+1} = b$ (thus $c_i = a + (i-1)\cdot\varepsilon$, where $i = 1, \dots, n+1$). We define a function $f(a,b,n,l) : \mathbb{R} \to \mathbb{R}$ as

$$f(a,b,n,l)(x) = \sum_{i=1}^{n} f^l_{\varepsilon/2}(x - c_i).$$

It is easy to see that $f(a,b,n,l)$ is $l$-Lipschitz. Further we define a function $g(a,b,n,l) : \mathbb{R} \to \mathbb{R}$ as

$$g(a,b,n,l)(x) = \sum_{i=1}^{n} g^l_{\varepsilon/2}(x - a_i).$$

Then $g(a,b,n,l)$ is a convex, $4nl$-Lipschitz function, which is a control function for $f(a,b,n,l)$. So $f(a,b,n,l)$ is $\delta$-convex on $\mathbb{R}$. Also note that $f(a,b,n,l)$ is equal to 0 outside of $(a,b)$.

It simply follows that for $f(a,b,n,l)$ there exist $2n$ intervals, in which $f(a,b,n,l)$ is affine with tangent $\pm l$, so that it is also equal to 0 in one of the boundary points and the interiors of these intervals are disjoint. Note that there exist $n$ points in $(a,b)$, where $\left|f'_+(x) - f'_-(x)\right| = 2l$.

Proof of Lemma 3.7: Take $m \in \mathbb{N}$, so that $2^{-m} < M$ and we shall define functions $\widetilde{h}_n$, again by induction, to satisfy conditions 1, 2, 3 and further for all $n \in \mathbb{N}$:

4. it is true that $\widetilde{h}_n(x) \geq 0$ for all $x \in \mathbb{R}$ and $\operatorname{supp} \widetilde{h}_n \subset [0,\delta)$,
5. there exist $2^{(m+1)n}$ disjoint intervals $(a_i, b_i)$, where $i = 1, \dots, 2^{(m+1)n}$, so that $\widetilde{h}_n \circ \cdots \circ \widetilde{h}_1$ is affine in $[a_i, b_i]$ with tangent $\pm(1/2^m)^n$ and in one of the boundary points of each interval the function $\widetilde{h}_n \circ \cdots \circ \widetilde{h}_1$ is equal to 0,

6. for the function $\widetilde{\beta} = \widetilde{h}_n \circ \dots \circ \widetilde{h}_1$ there exist $2^{n-1+mn}$ points in $(0, \delta)$, where the following inequality holds:

$$\left|\widetilde{\beta}'_+(x) - \widetilde{\beta}'_-(x)\right| \geq 2\left(\frac{1}{2^m}\right)^n.$$

We define $\widetilde{h}_1$, $\widetilde{\nu}_1$ as $\widetilde{h}_1 = f(\delta/2, \delta, 2^m, 1/2^m)$ and $\widetilde{\nu}_1 = g(\delta/2, \delta, 2^m, 1/2^m)$. Further we put $\lambda_1 = \frac{\delta}{2}$ and $\varepsilon = \frac{\delta}{2^{m+2}}$. If we take for $i = 1, \dots, 2^{m+1}$, the points $a_i$, $b_i$ to be $a_i = \frac{\delta}{2} + (i-1)\varepsilon$, $b_i = \frac{\delta}{2} + i\varepsilon$, then the intervals $(a_i, b_i)$ satisfy condition 5. For $j = 1, \dots, 2^m$, we take $t_j = a_{2j}$. Then in points $t_j$ the condition 6 is fulfilled and the validity condition 2 is clear by the choice of $\lambda_1$. Now suppose that $n > 1$ and we have constructed $\widetilde{h}_i$ for $i < n$. It suffices to show that there exists $\widetilde{h}_n$ so that all the conditions hold. Define

$$\widetilde{d} = \min\left\{\max\left\{\widetilde{h}_{n-1} \circ \dots \circ \widetilde{h}_1([a_i, b_i])\right\}; i = 1, \dots, 2^{(m+1)(n-1)}\right\},$$

where $a_i$, $b_i$ are taken as in condition 5 for $(n-1)$ and finally

$$d = \min\left\{\widetilde{d}, \delta\right\}. \tag{3.4}$$

Then clearly $d > 0$. Take $\widetilde{h}_n = f\left(\frac{d}{2}, d, 2^m, \frac{1}{2^m}\right)$ and $\widetilde{\nu}_n = g\left(\frac{d}{2}, d, 2^m, \frac{1}{2^m}\right)$. Conditions 1 and 4 are clearly satisfied. It remains to prove that the remaining conditions hold.

**Ad 5.** Let $(a_i, b_i)$, $i = 1, \dots, 2^{(m+1)(n-1)}$, be taken as in condition 5 for $(n-1)$. Take $1 \leq i \leq 2^{(m+1)(n-1)}$. Suppose that $\widetilde{h}_{n-1} \circ \dots \circ \widetilde{h}_1(a_i) = 0$. The other case when $\widetilde{h}_{n-1} \circ \dots \circ \widetilde{h}_1(b_i) = 0$ is analogous. Then the function $\widetilde{h}_{n-1} \circ \dots \circ \widetilde{h}_1$ is increasing in $[a_i, b_i]$ and equal to $(1/2^m)^{n-1}(x - a_i)$. The choice of $d$ in (3.4) implies, that there exists $t_i \in (a_i, b_i]$ such that $\widetilde{h}_{n-1} \circ \dots \circ \widetilde{h}_1(t_i) = d$. Then in $[a_i, b_i]$ the following equality holds:

$$\widetilde{h}_n \circ \dots \circ \widetilde{h}_1 = f\left(\frac{a_i + t_i}{2}, t_i, 2^m, (1/2^m)^n\right), \tag{3.5}$$

what follows from the special form of $\widetilde{h}_n$ and of $\widetilde{h}_{n-1} \circ \dots \circ \widetilde{h}_1$ on $[a_i, b_i]$.

It follows from the properties of $f(\cdot, \cdot, \cdot, \cdot)$ which were mentioned in Definition 3.8 that there exist $2^{m+1}$ intervals, with disjoint interiors, contained in $[a_i, b_i]$, where the function $\widetilde{h}_n \circ \dots \circ \widetilde{h}_1$ is affine with tangent $\pm(1/2^m)^n$ and in one of the boundary points of each interval it is equal to 0.

Thus for each interval $[a_i, b_i]$, where $i = 1, \dots, 2^{(m+1)(n-1)}$, we found $2^{m+1}$ subintervals, whose interiors are disjoint and for each of these (sub)intervals the condition 5 for $n$ holds. So we get $2^{(m+1)(n-1)} \cdot 2^{m+1} = 2^{(m+1)n}$ intervals.

**Ad 6.** It follows from above that in each interval $[a_i, b_i]$, which are taken as in (Ad 5.), there exist $2^m$ distinct points, where $\left|\widetilde{\beta}'_+(x) - \widetilde{\beta}'_-(x)\right| = 2\left(\frac{1}{2^m}\right)^n$. It is a consequence of the equality (3.5) and of properties of $f(\cdot,\cdot,\cdot,\cdot)$ mentioned in Definition 3.8. Altogether we obtain $2^{(m+1)(n-1)} \cdot 2^m = 2^{mn+n-1}$ points with the desired property.

**Ad 3.** Take $\lambda_n$ to be $\min\{\lambda_1, \dots, \lambda_{n-1}, d/2\} > 0$. Then for $i < n$ condition 3 is fulfilled thanks to the fact, that $\lambda_n \le \lambda_i$. It is enough to prove that $\widetilde{h}_n \equiv 0$ on $[0, \lambda_n]$. But we have $\widetilde{h}_n = f(d/2, d, 2^m, (1/2^m))$ and from the definition of $f(a,b,n,l)$ this function is equal to 0 outside of $(a,b)$. As we have $\lambda_n \le \frac{d}{2}$, the desired property of $h_n$ simply follows.

**Ad 2.** We define $z_i$ for $i = 1, \dots, 2^{n-1+mn}$, as the points of condition 6 for $n$. Let $\psi$ be a control function for $\widetilde{\beta} = \widetilde{h}_n \circ \dots \circ \widetilde{h}_1$ on $(0, \delta)$. Lemma 3.3 implies

$$\operatorname{Lip}\psi \ge \frac{1}{2}\sum_{i=1}^{2^{n-1+mn}} \left|\widetilde{\beta}'_+(z_i) - \widetilde{\beta}'_-(z_i)\right| = \frac{1}{2}\cdot 2^{n-1+mn}\cdot 2\left(\frac{1}{2^m}\right)^n = 2^{n-1},$$

which was to be proved. □

Proof of Theorem 3.1: We shall simultaneously construct mappings $N$ and $\widetilde{N}$. We shall write

$$Y = \sum_{n=2}^{\infty} \oplus_{\ell_1} (\mathbb{R}^n, \|\cdot\|_1)$$

and find mappings $N, \widetilde{N}: Y \to Y$ in form

$$N(x_2, x_3, \dots) = (F_2(x_2), F_3(x_3), \dots),$$
$$\widetilde{N}(x_2, x_3, \dots) = \left(\widetilde{F}_2(x_2), \widetilde{F}_3(x_3), \dots\right),$$

where $F_n, \widetilde{F}_n: \mathbb{R}^n \to \mathbb{R}^n$. Note that $Y$ is obviously isometrically isomorphic to $\ell_1$. In the sequel we shall use the symbol $\|x\|_Y = \sum_{n=2}^{\infty} \|x_n\|_{1,\mathbb{R}^n}$ even for points $x = (x_2, x_3, \dots) \in \prod_{n=2}^{\infty} \mathbb{R}^n$ which might not belong to $Y$. It makes proofs shorter.

First we define $F_n$. Choose $c \in \left(\frac{1}{2}, 1\right)$, fix $K$ such that $K > \max\left\{\frac{1}{1-c}, c+1\right\}$, and $L = 4c$. We shall find $F_n$, $n > 1$, so that they will satisfy the following conditions for all $n > 1$:

1. $F_n(0) = 0$, $F_n$ is $K$-bi-Lipschitz, $F_n$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$, and is $\delta$-convex on $\mathbb{R}^n$,
2. there exists a convex function $\varphi_n: \mathbb{R}^n \to \mathbb{R}$, which is $L$-Lipschitz, $\varphi_n(0) = 0$ and $\varphi_n$ is a control function for $F_n$ on $\mathbb{R}^n$,
3. suppose that $\varepsilon > \frac{1}{n}$ and the function $\theta: B(0,\varepsilon) \to \mathbb{R}$ is a control function for $F_n^{-1}|_{B(0,\varepsilon)}$, then $\operatorname{Lip}\theta \ge c \cdot (2c)^{n-2}$.

Choose $n \in \mathbb{N}$, $n > 1$. Put $\delta = \frac{1}{n}$ and apply Lemma 3.6 with chosen $\delta, c$. We obtain a sequence of functions $\{h_j\}_{j\in\mathbb{N}}$. We shall use only the first $(n-1)$ functions. For $j = 1, \dots, n-1$, we define $\xi_{n-j}: \mathbb{R} \to \mathbb{R}$ as $\xi_{n-j}(x) = h_j(x)$ and $\psi_{n-j}: \mathbb{R} \to \mathbb{R}$ as $\psi_{n-j}(x) = \nu_j(x)$.

Such $\xi_i$ and $\psi_i$ satisfy the assumptions of Lemma 3.2. Denote by $F_n$ the mapping $F$ obtained by the application of Lemma 3.2 with $\xi_i$, $\psi_i$, $i = 1, \dots, n-1$. Then the mapping $F_n$ is $\delta$-convex, $K$-bi-Lipschitz, there exists a control function $\varphi_n$ for $F_n$, which is $L$-Lipschitz and $\varphi_n(0) = 0$. It further holds that $F_n(0) = 0$ (because $\xi_i(0) = 0$ for $i \le n$) and $F_n$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Now we define $\widetilde{F}_n$. Choose $\widetilde{K} \ge 2$ and $\widetilde{L} = 2$. We shall find $\widetilde{F}_n$, $n > 1$, so that they will satisfy the following conditions for all $n > 1$:

1. $\widetilde{F}_n(0) = 0$, $\widetilde{F}_n$ is $\widetilde{K}$-bi-Lipschitz, $\widetilde{F}_n$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$ and is $\delta$-convex on $\mathbb{R}^n$,
2. there exists a convex function $\widetilde{\varphi}_n: \mathbb{R}^n \to \mathbb{R}$, which is $\widetilde{L}$-Lipschitz, $\widetilde{\varphi}_n(0) = 0$ and $\widetilde{\varphi}_n$ is a control function for $\widetilde{F}_n$ on $\mathbb{R}^n$,
3. suppose that $\varepsilon > \frac{1}{n}$ and the function $\theta: B(0,\varepsilon) \to \mathbb{R}$ is a control function for $\widetilde{F}_n^{-1}|_{B(0,\varepsilon)}$, then $\operatorname{Lip}\theta \ge 2^{n-2}$,
4. there exists $\Lambda_n > 0$ such that for all $x \in B(0,\Lambda_n)$ it holds that $\widetilde{\Psi}_n(x) = \widetilde{F}_n(x) - x = 0$ and $\widetilde{\Psi}_n$ is $\frac{1}{n}$-Lipschitz.

Choose $n \in \mathbb{N}$, $n > 1$. Put $\delta = M = \frac{1}{n}$ and we apply Lemma 3.7. We obtain a sequence $\widetilde{h}_i$ and denote $m_n = m$. Put $\Lambda_n = \lambda_{n-1}$, where $\lambda_{n-1}$ is taken as in condition 3 in Lemma 3.7. Again we shall use the first $(n-1)$ functions. For $j = 1, \dots, n-1$, we define $\widetilde{\xi}_{n-j}: \mathbb{R} \to \mathbb{R}$ as $\widetilde{\xi}_{n-j}(x) = \widetilde{h}_j(x)$ and $\widetilde{\psi}_{n-j}: \mathbb{R} \to \mathbb{R}$ as $\widetilde{\psi}_{n-j}(x) = \widetilde{\nu}_j(x)$.

Such $\widetilde{\xi}_i$ and $\widetilde{\psi}_i$ satisfy the assumptions of Lemma 3.2 if we take $c = \frac{1}{n}$, $K = \widetilde{K}$, $L = \widetilde{L}$, $\xi_i = \widetilde{\xi}_i$, $\psi_i = \widetilde{\psi}_i$. Denote $\widetilde{F}_n$ the mapping $F$ from Lemma 3.2 used on $\widetilde{\xi}_i$, $\widetilde{\psi}_i$, $i = 1, \dots, n-1$. Then the mapping $\widetilde{F}_n$ is $\delta$-convex, $\widetilde{K}$-bi-Lipschitz, there exists a control function $\widetilde{\varphi}_n$ for $\widetilde{F}_n$, which is $\widetilde{L}$-Lipschitz and $\widetilde{\varphi}_n(0) = 0$. Note that the mapping $\widetilde{\Psi}_n(x) = \widetilde{F}_n(x) - x$ from Lemma 3.7 is $\frac{1}{n}$-Lipschitz. Further $\widetilde{F}_n(0) = 0$ and $\widetilde{F}_n$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Because it holds for $i \le n-1$ that $\widetilde{h}_i([0,\Lambda_n]) = \{0\}$, then for $x \in \mathbb{R}$, $\|x\| \le \Lambda_n$, it is true, that $\widetilde{\Psi}_n(x) = 0$, what is an easy consequence of the definition of $\widetilde{\Psi}_n$.

It remains to show that conditions 3 hold both for $F_n$ and $\widetilde{F}_n$. It follows from the next proposition. Choose $n \in \mathbb{N}$.

**Proposition 3.9.** *Let $\varepsilon > \frac{1}{n}$ and let $\psi$ be a control function of $F_n^{-1}|_{B(0,\varepsilon)}$ ($\widetilde{F}_n^{-1}|_{B(0,\varepsilon)}$, respectively). Then $\operatorname{Lip}(\psi) \ge c \cdot (2c)^{n-2}$ ($\operatorname{Lip}(\psi) \ge 2^{n-2}$, respectively).*

Proof of Proposition 3.9: Let us suppose first, that $\psi$ is a control function of $F_n^{-1}|_{B(0,\varepsilon)}$. Further, we might suppose, that $\operatorname{Lip}\psi < \infty$. Then it follows from Lemma 3.2 that there exists a control function for $\xi_1 \circ \cdots \circ \xi_{n-1}$ on $(-\varepsilon, \varepsilon)$, which is $(\operatorname{Lip}\psi)$-Lipschitz; we denote the function $\alpha$. Because $\xi_{n-j}(x) = h_j(x)$, it holds that $\xi_1 \circ \cdots \circ \xi_{n-1} = h_{n-1} \circ \cdots \circ h_1$. The function $\alpha$ is certainly a control function for $h_{n-1} \circ \cdots \circ h_1$ on $(-\varepsilon, \varepsilon)$. From Lemma 3.6, condition 2, it follows that $\operatorname{Lip}\alpha \geq c \cdot (2c)^{n-2}$. As $\operatorname{Lip}\alpha \leq \operatorname{Lip}\psi$, we have proved the first part of the proposition.

Now suppose, that $\psi$ is a control function for $\widetilde{F}_n^{-1}|_{B(0,\varepsilon)}$. Then everything is analogous to the case of $F_n$, the only difference being that we are working with $\widetilde{\xi}_i$, $\widetilde{h}_i$, $i = 1, \ldots, n-1$, and the estimate follows from Lemma 3.7, condition 2. This concludes the proof. $\square$

Let us now look closer at the properties of mappings $N$ and $\widetilde{N}$, that were defined above.

We show first that $N$ maps $Y$ into $Y$ and that it is bi-Lipschitz. Choose $x, y \in Y$. Remember, that $x = (x_2, x_3, \ldots)$, where $x_n \in \mathbb{R}^n$ (the same holds for $y$). Then

$$\|N(x) - N(y)\|_Y = \sum_{n>1} \|F_n(x_n) - F_n(y_n)\|_{1,\mathbb{R}^n} \leq K \sum_{n>1} \|x - y\|_{1,\mathbb{R}^n}.$$

So we get, that $\|N(x) - N(y)\|_Y \leq K\|x - y\|_Y$. Because $N(0) = 0$, then if we take $y = 0$, we get that $N(x) \in Y$. Similar argument gives, that $\|N(x) - N(y)\|_Y \geq \frac{1}{K}\|x - y\|_Y$. For $\widetilde{N}$ we use an analogous computation with $\widetilde{K}$.

For the proof of $\delta$-convexity of $N$ we define a function $\varphi\colon Y \to \mathbb{R}$ as $\varphi(x) = \sum_{n>1} \varphi_n(x_n)$, where $\varphi_n$ are control functions of $F_n$, $\varphi_n(0) = 0$ and $\varphi_n$ is $L$-Lipschitz. The function $\varphi$ is well defined, because for $x \in Y$, we obtain

$$|\varphi(x)| = \left|\sum_{n>1} \varphi_n(x_n)\right| = \left|\sum_{n>1} (\varphi_n(x_n) - \varphi_n(0))\right| \leq L \sum_{n>1} \|x_n\|_{1,\mathbb{R}^n}. \tag{3.6}$$

By similar estimates as in (3.6) we get, that $\varphi$ is $L$-Lipschitz (and thus continuous). Convexity of $\varphi$ follows from that fact that it is a limit of finite partial sums of convex functions, which are obviously convex.

Note that $\varphi$ is a control function of $N$. It follows from Corollary 1.3 and from the following estimate:

$$
\begin{aligned}
&\left\|\frac{1}{2}(N(x)+N(y))-N\left(\frac{x+y}{2}\right)\right\|_Y \\
&=\sum_{n>1}\left\|\frac{F_n(x_n)+F_n(y_n)}{2}-F_n\left(\frac{x_n+y_n}{2}\right)\right\|_{1,\mathbb{R}^n} \\
&\leq \sum_{n>1}\frac{\varphi_n(x_n)+\varphi_n(y_n)}{2}-\varphi_n\left(\frac{x_n+y_n}{2}\right) \\
&=\frac{\varphi(x)+\varphi(y)}{2}-\varphi\left(\frac{x+y}{2}\right),
\end{aligned}
$$

for $x, y \in Y$. The proof of $\delta$-convexity of $\widetilde{N}$ follows by an analogous argument using $\widetilde{\varphi}_n$, $n \in \mathbb{N}$.

It is easy to show that $N$ is onto $Y$. It follows from the fact that $F_n$'s are uniformly bi-Lipschitz and onto. Suppose we are given $y \in Y$. Then $y = (y_2, y_3, \dots)$, where $y_i \in \mathbb{R}^i$. Define $x_i \in \mathbb{R}^i$ as $x_i = F_i^{-1}(y_i)$ for $i \in \mathbb{N}$. Then $x = (x_2, x_3, \dots) \in Y$, as

$$
\|x\|_Y = \sum_{i>1}\|x_i - 0\| = \sum_{i>1}\left\|F_i^{-1}(y_i) - F_i^{-1}(0)\right\| \leq K\sum_{i>1}\|y_i\| = K\|y\|_Y.
$$

Thus $N(x) = y$. That $\widetilde{N}$ is onto $Y$ follows by a similar argument.

Let us show that $N^{-1}$ is nowhere locally $\delta$-convex. For a contradiction let us suppose that we have a point $z \in Y$ and there exists $\varepsilon > 0$ and a continuous convex function $\theta\colon B_Y(z,\varepsilon) \to \mathbb{R}$ so that $\theta$ is a control function of $N^{-1}|_{B(0,\varepsilon)}$. By possibly making the $\varepsilon > 0$ smaller, we can suppose that $\operatorname{Lip}\theta < \infty$ (as continuous convex functions are locally Lipschitz).

First, there exists $n_0 \in \mathbb{N}$ so that

1. $\frac{1}{n} < \frac{\varepsilon}{4}$ for $n \geq n_0$;
2. $\sum_{n \geq n_0}\|z_n\| \leq \frac{\varepsilon}{4}$.

Fix $n > n_0$. For $x \in B_{\mathbb{R}^n}(0, \varepsilon/4)$ we define $E^n(x) \in Y$ as

$$
E^n(x)_i = \begin{cases} z_i & \text{for } \ i \leq n_0; \\ x & \text{for } \ i = n; \\ 0 & \text{elsewhere.} \end{cases}
$$

Then $E^n(x) \in B_Y(z,\varepsilon)$, because

$$
\begin{aligned}
\|z - E^n(x)\| &= \sum_{i>n_0}\|z_i - E^n(x)_i\| = \sum_{\substack{i>n_0\\ i\neq n}}\|z_i\| + \|z_n - x\| \\
&\leq \sum_{\substack{i>n_0\\ i\neq n}}\|z_i\| + \|z_n\| + \|x\| \quad \leq \quad \frac{3\varepsilon}{4} \quad < \varepsilon.
\end{aligned}
$$

Let us denote $\pi_n\colon Y \to \mathbb{R}^n$ the projection onto the $n$-th coordinate (that is $\pi_n((x_2, x_3, \dots)) = x_n$ for $x \in Y$). Then it follows from Lemma 1.1, part (b), that $N^{-1} \circ E^n$ is $\delta$-convex with the control function $\theta \circ E^n$ on $B(0, \varepsilon/4)$. Another application of Lemma 1.1, now part (a), yields that $\pi_n \circ N^{-1} \circ E^n$ is $\delta$-convex with the control function $\mathrm{Lip}(\pi_n) \cdot (\theta \circ E^n)$. As $\mathrm{Lip}(\pi_n) = \mathrm{Lip}(E^n) = 1$, we get

$$\mathrm{Lip}(\mathrm{Lip}(\pi_n) \cdot (\theta \circ E^n)) \leq \mathrm{Lip}(\pi_n) \cdot \mathrm{Lip}\,\theta \cdot \mathrm{Lip}(E^n) = \mathrm{Lip}(\theta). \tag{3.7}$$

Note that for $x \in B_{\mathbb{R}^n}(0, \varepsilon/4)$ it is true, that $F_n^{-1}(x) = \pi_n \circ N^{-1} \circ E^n$. So we obtain, that $\theta \circ E^n$ is a control function for $F_n^{-1}$ on $B(0, \varepsilon/4)$. Condition 3 in definition of $F_n$ implies, that $\mathrm{Lip}(\theta \circ E^n) \geq c \cdot (2c)^{n-2}$, and this, together with (3.7), implies that $\mathrm{Lip}\,\theta \geq \mathrm{Lip}(\theta \circ E^n)$. So we obtained that $\mathrm{Lip}\,\theta \geq c \cdot (2c)^{n-2}$ for all $n > n_0$ and that is a contradiction with the fact that $\mathrm{Lip}\,\theta < \infty$, because $\lim_{n\to\infty} c \cdot (2c)^{n-2} = \infty$ thank to the choice of $c > \frac{1}{2}$.

The proof of the fact that $\widetilde{N}^{-1}$ is nowhere locally $\delta$-convex follows the same lines; the only difference is in the estimates following from Proposition 3.9.

Now we show that $\widetilde{N}$ is strictly differentiable at 0. Choose $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$, so that $1/n < \varepsilon$ for all $n \geq n_0$. Take $\delta > 0$ such that $\delta < \min\{\Lambda_i; i \leq n_0\}$ (see definition of $\widetilde{F}_n$, condition 4). Then $\widetilde{\Psi}_j(x) = 0$ for $x \in \mathbb{R}^j$, $\|x\| \leq \delta$ and $j \leq n_0$. Pick $x, y \in B_Y(0, \delta)$. Then

$$\begin{aligned}
&\left\|\widetilde{N}(x) - \widetilde{N}(y) - Id_Y(x-y)\right\|_Y \\
&\quad = \sum_{n>1} \left\|\widetilde{F}_n(x_n) - \widetilde{F}(y_n) - (x_n - y_n)\right\|_{1,\mathbb{R}^n} = \sum_{n>1} \left\|\widetilde{\Psi}_n(x_n) - \widetilde{\Psi}_n(y_n)\right\|_{1,\mathbb{R}^n} \\
&\quad = \sum_{n=2}^{n_0} \left\|\widetilde{\Psi}_n(x_n) - \widetilde{\Psi}_n(y_n)\right\|_{1,\mathbb{R}^n} + \sum_{n>n_0} \left\|\widetilde{\Psi}_n(x_n) - \widetilde{\Psi}_n(y_n)\right\|_{1,\mathbb{R}^n} \\
&\quad \leq \sum_{n>n_0} \frac{1}{n} \|x_n - y_n\| \leq \sum_{n>n_0} \frac{1}{n_0} \|x_n - y_n\| \\
&\quad \leq \frac{1}{n_0} \|x - y\|_Y \leq \varepsilon \|x - y\|_Y.
\end{aligned}$$

Thus $Id_Y$ is the strict derivative of $\widetilde{N}$ at 0. The mapping $Id_Y$ is obviously invertible.

□

*Remark* 4. The case $X = Y = \ell_2$ remains open for Problems 1 and 2 from [7].

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