# Maximal nowhere dense P-sets in basically disconnected spaces and F-spaces

A.V. Koldunov, A.I. Veksler

*Abstract.* In [5] the following question was put: are there any maximal n.d. sets in $\omega^*$? Already in [9] the negative answer (under **MA**) to this question was obtained. Moreover, in [9] it was shown that no $P$-set can be maximal n.d. In the present paper the notion of a maximal n.d. $P$-set is introduced and it is proved that under **CH** there is no such a set in $\omega^*$. The main results are Theorem 1.10 and especially Theorem 2.7(ii) (with Example in Section 3) in which the problem of the existence of maximal n.d. $P$-sets in basically disconnected compact spaces with rich families of n.d. $P$-sets is actually solved.

*Keywords:* maximal n.d. set, $P$-set, maximal n.d. $P$-set, compact space, basically disconnected space, $F$-space

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## 0. Introduction

In [9], the notion of a maximal (closed) nowhere dense set in a topological space was studied and some results were obtained. These studies were continued in [4], [12] and others.

Let $\mathfrak{N}(X)$ denote the family of all (closed) nowhere dense (*n.d.*) sets in a topological space $X$. In $\mathfrak{N}(X)$ the order is introduced in the following way: $F_1 \prec F_2$ if $F_1 \in \mathfrak{N}(F_2)$ (in the induced topology on $F_2$). In [9] it was proved that in some spaces, particularly without isolated points, there are n.d. sets which are maximal with respect to this order (but, for example, there is no such a set in a segment). Such sets are called *maximal nowhere dense* (*m.n.d.*), and the family of all such sets is denoted by $\mathfrak{M}(X)$. It should be pointed that the notion of the m.n.d. set appeared in studies of spaces of measurable and summable functions ([9, Section 2]; [12, Theorem 14]).

It seems natural to study $\mathfrak{M}(X)$ only for compact $X$. Now we shall deliver some results on m.n.d. sets from [4], [9], [12] combining them in the following theorem. We recall that a $\Theta$-*set* is any n.d. zero-set (the letter $\Theta$ will be used for $\Theta$-sets only). An $F$-*space* (*quasi-$F$-space*) is a space in which any cozero-set (correspondingly, any dense cozero-set) is $C^*$-embedded. A space is called *basically disconnected* (*b.d.*), if the closure of any cozero-set is a clopen set.

**0.1 Theorem.** *For a compact space $X$ the following statements are true.*

(i) *If $\Theta$ is a non-empty $C^*$-coembedded $\Theta$-set, then $\Theta \in \mathfrak{M}(X)$.*
(ii) *Therefore, if $X$ is an $F$-space or a quasi-$F$-space and $\Theta \neq \emptyset$ is a $\Theta$-set in $X$, then $\Theta \in \mathfrak{M}(X)$.*
(iii) *For b.d. $X$ and $F \in \mathfrak{N}(X)$ it is true that $F \in \mathfrak{M}(X)$ if and only if $F$ contains a non-empty $\Theta$-set in $X$. In particular, $\mathfrak{M}(X) \neq \emptyset$ for every infinite b.d. compact space $X$.*
(iv) (**MA**). *$\mathfrak{M}(\omega^*) = \emptyset$.*
(v) *A $P$-set is not a m.n.d. set.*
(vi) *If $X$ contains only a finite number of isolated points, then any m.n.d. set in $X$ cannot have a countable $\pi$-base.*
(vii) *This implies that, if $X$ is a metric compact space with only a finite number of isolated points, then $\mathfrak{M}(X) = \emptyset$.*
(viii) *$\mathfrak{M}(2^\tau) = \mathfrak{M}(I^\tau) = \emptyset$.*

Let us pay some more attention to the statement (iv). It is connected with Hechler's conjecture which appeared in [1] published in 1978 (i.e., three years after [9]). S.H. Hechler formulated his conjecture in the following way. Let $F \in \mathfrak{N}(\omega^*)$; is it true that there exists a family $\{G_\alpha : \alpha < 2^\omega\}$ of disjoint open non-empty sets in $\omega^*$ such that $F \subset \operatorname{cl} G_\alpha \backslash G_\alpha$. He showed that it is true under **MA**. In [7] (1990) P. Simon proved that Hechler's conjecture is equivalent to the conjecture "$(\mathfrak{M}(\omega^*)) = \emptyset$". The latter conjecture was also introduced in [5] in the form of Q222. But it is still unknown whether Hechler's conjecture is true, though it is proved in other set-theoretical models (see [8]).

Now, let $\mathfrak{K}$ be some class of (closed) n.d. sets. Let $\mathfrak{K}(X)$ denote the family of all sets from this class $\mathfrak{K}$ in $X$. We may consider elements of $\mathfrak{K}(X)$ which are maximal in $\mathfrak{K}(X)$ with respect to the order introduced before. For example, the class $\mathfrak{P}$ of all non-empty n.d. $P$-sets and the class $\mathfrak{P}'$ of all non-empty n.d. $P'$-sets ([10]) may be considered. We recall that a closed set $F \subset X$ is called a *$P$-set* (a *$P'$-set*), if $F \cap \operatorname{cl} E = \emptyset$ (correspondingly $F \cap \operatorname{int} \operatorname{cl} E = \emptyset$) for any $F_\sigma$-set $E$ which is disjoint with $F$.

Let $\mathfrak{K}_{\mathfrak{M}}(X)$ denote the family of all maximal elements in $\mathfrak{K}(X)$. Note that $\mathfrak{N}_{\mathfrak{M}}(X) = \mathfrak{M}(X)$. In this paper some results on $\mathfrak{P}_{\mathfrak{M}}(X)$ will be obtained. Elements of this family will be called *maximal n.d. $P$-sets* (*m.n.d. $P$-sets*). Compact b.d. spaces will mainly be considered since $c(X) > \omega$ for a b.d. compact space $X$ implies $\mathfrak{P}(X) \neq \emptyset$. Any other natural large classes of compact spaces with $\mathfrak{P}(X) \neq \emptyset$ for any spaces $X$ from this class are unknown for us. Besides for a b.d. compact space $\mathfrak{P}(X) = \mathfrak{P}'(X)$, and therefore $\mathfrak{P}'_{\mathfrak{M}}(X) = \mathfrak{P}_{\mathfrak{M}}(X)$. But in the case of $X = \omega^*$ the situation is different since $\mathfrak{P}'(\omega^*) = \mathfrak{N}(\omega^*)$. So the conjectures "$\mathfrak{M}(\omega^*) = \emptyset$" and "$\mathfrak{P}'_{\mathfrak{M}}(\omega^*) = \emptyset$" are equivalent. Note also that in the general case the equality $\mathfrak{M}(X) \cap \mathfrak{P}'(X) = \emptyset$ is not a theorem in ZFC (see [10]).

In Section 1 we consider classes of compact spaces (usually b.d., but not always) having m.n.d. $P$-sets; besides conditions for a $P$-point to be a m.n.d. $P$-set are discussed. In Section 2 classes of compact spaces with $\mathfrak{P}(X) \neq \emptyset$ and $\mathfrak{P}_{\mathfrak{M}}(X) = \emptyset$ are studied. The existence of such b.d. compact spaces is due to Theorem 2.7(ii). It should be noted that it is much more complicated to find compact spaces of this kind than to find compact spaces with non-empty $\mathfrak{P}_{\mathfrak{M}}(X)$; in Section 3 the desired example is constructed. In Section 2 compact $F$-spaces are considered, which form a wider class than b.d. compact spaces. In particular, in Corollary 2.3 it is proved (**CH**) that $\mathfrak{P}_{\mathfrak{M}}(\omega^*) = \emptyset$. Maximal n.d. weak $P'$-sets could be studied here, but we put them off until somewhere later. Section 4 is devoted to unsolved problems.

We recall that in any b.d. compact space (compact $F$-space) the closure of the union of any family (correspondingly, of a sequence) of $P$-sets is a $P$-set.

Let $M \subset X$ and $E \subset M$ be a closed (open, n.d. ... ) set in $M$ in the induced topology on $M$; then $E$ is called a *closed* (*open, n.d.* ...) *subset* of $M$. If $M$ is a clopen set in $X$, then $E \in \mathfrak{P}(M)$ ($E \in \mathfrak{P}_{\mathfrak{M}}(M)$) if and only if $E \in \mathfrak{P}(X)$ (correspondingly, $E \in \mathfrak{P}_{\mathfrak{M}}(X)$). The letter $X$ will be used only to denote compact spaces which are b.d., if the converse is not stated. The letter $F$ will be used for closed sets, and $G$ will be used for open ones. The family of all cozero-sets in $X$ is denoted by $\mathbf{C}z(X)$. The notation of the type of $\cup E_\alpha$ will be used sometimes for $\cup\{E_\alpha : \alpha \in A\}$.

In Section 2 we shall consider the notion of the *sequential absolute* $a_sX$ of a compact space $X$, which was introduced in [3] using a different term (other terms: the smallest quasi-$F$-preimage; quasi-$F$-cover, sequential cover; see [2], [6], [14]). The space $a_sX$ may be defined as the smallest irreducible compact quasi-$F$-preimage of $X$. We shall present those results on the space $a_sX$, which will be used in our arguments. Let $\tau$ denote a canonical surjection: $a_sX \to X$.

**0.2 Theorem.** *Let $X$ be any compact space and $Y = a_sX$ be its sequential absolute. The following statements are true.*

(i) *For any $V \in \mathbf{C}z(Y)$ there is $W \in \mathbf{C}z(X)$ such that $\tau^{-1}W \subset V \subset \operatorname{cl} \tau^{-1}W$.*
(ii) *For any $h \in C(Y)$ and any $\varepsilon > 0$ there are a $\Theta$-set $\Theta$ in $X$ and $g \in C^*(X\backslash\Theta)$ such that $|g \circ \tau - h| < \varepsilon$ on $Y\backslash\tau^{-1}\Theta$.*
(iii) *A quasi-$F$-space $Y$ is an $F$-space if and only if for any disjoint $V_1, V_2 \in \mathbf{C}z(X)$ there is a $\Theta$-set $\Theta$ such that $\operatorname{cl} V_1 \cap \operatorname{cl} V_2 \subset \Theta$.*
(iv) *A quasi-$F$-space $Y$ is b.d., i.e., $Y$ is the smallest irreducible compact b.d. preimage of $X$ (or $\sigma$-absolute of $X$; see* [3], [13], [15]*) if and only if for any $V \in \mathbf{C}z(X)$ there is $V' \in \mathbf{C}z(X)$ such that $V \cap V' = \emptyset$ and $V \cup V'$ is dense in $X$.*

## 1. Spaces with maximal nowhere dense $P$-sets

First we find conditions on a (closed) n.d. set in a b.d. compact space to be a m.n.d. $P$-set.

**1.1 Definition.** An ideal $I$ of clopen sets in a b.d. compact space $X$ is called a *$\sigma$-ideal* if $U_n \in I$ $(n < \omega)$ implies $\operatorname{cl} \cup U_n \in I$.

**1.2 Lemma.** *If $I$ is a $\sigma$-ideal of clopen sets in a b.d. compact space $X$, then $F_I = X \backslash \cup I$ is a $P$-set. Conversely, if $F$ is a $P$-set, then the family $I$ of all clopen sets disjoint with $F$ is a $\sigma$-ideal and $F = F_I$.*

PROOF: Suppose that an $F_\sigma$-set $E$ is disjoint with $F$. Then $E = \cup F_n$, where $F_n$ are closed in $X$. Find clopen $U_n \supset F_n$ such that $U_n \cap F = \emptyset$. So $E \subset \operatorname{cl} \cup F_n \subset \operatorname{cl} \cup U_n \in I$ since $I$ is a $\sigma$-ideal. The remaining part is proved likewise. □

Note that a nonempty $F_I$ belongs to $\mathfrak{P}(X)$ if and only if $\cup I$ is dense in $X$.

**1.3 Definition.** Any family of disjoint non-empty clopen sets in an arbitrary topological space $X$, whose union is dense, is called a *sieve*. If each member of a sieve is disjoint with a given set $F \in \mathfrak{N}(X)$, then this sieve is called an *$F$-sieve*.

**1.4 Definition.** Let $\Delta$ be an uncountable sieve in a b.d. compact space $X$. Let $I = I_\Delta$ be the smallest $\sigma$-ideal containing $\Delta$. In this case a n.d. $P$-set $F_I$ (see Lemma 1.2) is called a *standard* n.d. $P$-set and is denoted by $F_\Delta$.

Evidently, if $c(X) > \omega$, then there exist standard n.d. $P$-sets in $X$.

Note that if $\Delta'$ is a $F_\Delta$-sieve, then $\Delta'$ is uncountable (since a non-empty $P$-set $F_\Delta$ cannot be contained in a $\Theta$-set) and $F_{\Delta'} \supset F_\Delta$. Besides, if non-empty clopen $U \subset X$ and $\Delta'$ is a trace of the sieve $\Delta$ on $U$, then $F_{\Delta'} = F_\Delta \cap U$.

**1.5 Lemma.** *The following statements hold.*

(i) *If $F \in \mathfrak{P}(X)$, then there is a sieve $\Delta$ such that $F \subset F_\Delta$.*
(ii) *If $F \in \mathfrak{P}_{\mathfrak{M}}(X)$, then there is a sieve $\Delta$ such that $\emptyset \neq F_\Delta \subset F$.*

PROOF: (i) Let $F \in \mathfrak{P}(X)$. By Lemma 1.2, we may take any $F$-sieve $\Delta$ for the given set $F$.

(ii) Now let $F \in \mathfrak{P}_{\mathfrak{M}}(X)$. Take $F_\Delta$ from (i). Then $F \overline{\in} \mathfrak{N}(F_\Delta)$. It means that there is a clopen set $U$ such that $\emptyset \neq U \cap F = U \cap F_\Delta$. Let $\Delta = \{U_\alpha : \alpha \in A\}$. Put $\Delta' = \{U_\alpha \cap U : \alpha \in A,\ U_\alpha \cap U \neq \emptyset\} \cup \{X \backslash U\}$. Evidently, $F_{\Delta'} = F_\Delta \cap U = F \cap U \subset F$. □

Lemma 1.5 shows that for b.d. $X$, $\mathfrak{P}_{\mathfrak{M}}(X) \neq \emptyset$ if and only if there is $F_\Delta \in \mathfrak{P}_{\mathfrak{M}}(X)$. That is why we shall pay some more attention to standard n.d. $P$-sets.

**1.6 Lemma.** *Let $\Delta = \{U_\alpha : \alpha \in A\}$ be an uncountable sieve in a b.d. compact space $X$. The following statements are true.*

(i) *There is no non-empty $P$-set $F \subset (X \backslash \cup \Delta) \backslash F_\Delta$.*
(ii) *If $F_\Delta \subset F \in \mathfrak{P}(X)$, then $F = \operatorname{cl} \cup (F \cap U_\alpha) \cup F_\Delta$.*
(iii) *$F_\Delta$ is the largest $P$-set in $X$ disjoint with each member of $\Delta$.*

Proof: (i) Suppose that $\emptyset \neq F \subset (X \setminus \cup \Delta) \setminus F_\Delta$. Then there are $\alpha_n \in A$ such that $F \cap \mathrm{cl} \cup \{U_{\alpha_n} : n < \omega\} \neq \emptyset$. So $F$ is not a $P$-set.

(ii) Suppose there is $x \in F$ such that $x \overline{\in} \mathrm{cl} \cup (F \cap U_\alpha) \cup F_\Delta$. Find a clopen set $U$ containing $x$ and disjoint with $\mathrm{cl} \cup (F \cap U_\alpha) \cup F_\Delta$. Then $x \in F \cap U \in \mathfrak{P}(X)$ and $F \cap U \subset (X \setminus \cup \Delta) \setminus F_\Delta$. But this contradicts (i).

(iii) follows from (ii). □

Now some conditions for the given set $F_\Delta$ to be a m.n.d. $P$-set will be obtained.

**1.7 Proposition.** *Let $\Delta = \{U_\alpha : \alpha \in A\}$ be an uncountable sieve in a b.d. compact space $X$. The following statements hold.*

(i) *$F_\Delta \in \mathfrak{P}_{\mathfrak{M}}(X)$ if and only if for any family $\{F_\alpha : \alpha \in A\}$ of n.d. $P$-sets with $F_\alpha \subset U_\alpha$ (some $F_\alpha$ can be empty, but the set of others is uncountable) there are a uncountable set $A_0 \subset A$ and points $x_\alpha \in U_\alpha (\alpha \in A_0)$ such that $\mathrm{cl}\{x_\alpha : \alpha \in A_0\} \cap \mathrm{cl} \cup \{F_\alpha : \alpha \in A\} = \emptyset$.*

(ii) *If $\mathrm{cl} \cup U_\alpha$ is $C^*$-embedded in $X$, then $F_\Delta \in \mathfrak{P}_{\mathfrak{M}}(X)$.*

Proof: (i) *Necessity.* Let $F_\Delta \in \mathfrak{P}_{\mathfrak{M}}(X)$, $F_\alpha \in \mathfrak{P}(X)$ and $F_\alpha \subset U_\alpha$ $(\alpha \in A)$. Then the set $F = (\mathrm{cl} \cup F_\alpha) \cup F_\Delta \in \mathfrak{P}(X)$. Since $F_\Delta \in \mathfrak{P}_{\mathfrak{M}}(X)$, then there is $x \in F_\Delta$ such that $x \overline{\in} \mathrm{cl}(F \setminus F_\Delta) = \mathrm{cl} \cup F_\alpha$. Find clopen $U$ with $x \in U$ and $U \cap \mathrm{cl} \cup F_\alpha = \emptyset$. Obviously, $U = \mathrm{cl} \cup \{U \cap U_\alpha : \alpha \in A\}$. Thus, the set $A_0 = \{\alpha : U \cap U_\alpha \neq \emptyset\}$ is uncountable. Indeed, otherwise $U \cap F_\Delta = \emptyset$ because $F_\Delta$ is a $P$-set. Take any $x_\alpha \in U \cap U_\alpha$ $(\alpha \in A_0)$. Since $\mathrm{cl}\{x_\alpha : \alpha \in A_0\} \subset U$ we get $\mathrm{cl}\{x_\alpha : \alpha \in A_0\} \cap \mathrm{cl} \cup \{F_\alpha : \alpha \in A\} = \emptyset$.

*Sufficiency.* Let $F_\Delta \subset F \in \mathfrak{P}(X)$ and $F_\alpha = F \cap U_\alpha$. Evidently, each $F_\alpha$ is an n.d. $P$-set in $X$. Let uncountable $A_0 \subset A$ and $\mathrm{cl}\{x_\alpha : \alpha \in A_0\} \cap \mathrm{cl} \cup \{F_\alpha : \alpha \in A\} = \emptyset$. Since $A_0$ is uncountable, there is $x \in (\mathrm{cl}\{x_\alpha : \alpha \in A_0\}) \cap F_\Delta$. By Lemma 1.6(ii), we get $\mathrm{cl}(F \setminus F_\Delta) \subset \mathrm{cl} \cup \{F_\alpha : \alpha \in A\}$. So $x \overline{\in} \mathrm{cl}(F \setminus F_\Delta)$. This implies $F_\Delta \overline{\in} \mathfrak{N}(F)$. So $F_\Delta \in \mathfrak{P}_{\mathfrak{M}}(X)$.

(ii) If $F_\alpha \subset U_\alpha$ and $F_\alpha$ is a n.d. $P$-set, then there is $x_\alpha \in U_\alpha \setminus F_\alpha$. We can separate $x_\alpha$ and $F_\alpha$ in $U_\alpha$ by zero-sets. Then the sets $\cup \{x_\alpha : \alpha \in A\}$ and $\cup \{F_\alpha : \alpha \in A\}$ are separated in $\cup U_\alpha$ by zero-sets. Since $\cup U_\alpha$ is $C^*$-embedded in $X$, we get $\mathrm{cl}\{x_\alpha : \alpha \in A_0\} \cap (\mathrm{cl} \cup F_\alpha) = \emptyset$. Now use (i). □

**1.8 Remark.** Let $X$ be an arbitrary compact space with a uncountable sieve $\Delta = \{U_\alpha : \alpha \in A\}$ of clopen sets and let $\cup U_\alpha$ be $C^*$-embedded; let $F_\Delta = X \setminus \cup \{\mathrm{cl} \cup_{n<\omega} U_{\alpha_n} : (\alpha_n) \subset A\}$. In this case it also may be proved that $F_\Delta \in \mathfrak{P}_{\mathfrak{M}}(X)$.

Now we shall find b.d. compact spaces having m.n.d. $P$-sets.

**1.9 Definition.** Let $X$ be a b.d. compact space. This space is called $\omega_1$-*extremally disconnected* ($\omega_1$-*e.d.*), if a closure of the union of any family of $\omega_1$ clopen sets is clopen. If in a b.d. space $Y$ the closure of the union of any family of $\omega_1$ disjoint clopen non-empty sets is not open, then we call $Y$ a *anti-$\omega_1$-b.d.* space.

Evidently, the class of all $\omega_1$-b.d. compact spaces coincides with the class of Stone spaces of all $\omega_2$-complete Boolean algebras.

**1.10 Theorem.** *Let $X$ be a $\omega_1$-b.d. compact space. The following statements are true.*

(i) *Any standard n.d. $P$-set $F_\Delta$ is a maximal n.d. $P$-set.*
(ii) *A n.d. $P$-set is a maximal n.d. $P$-set if and only if it contains some standard n.d. $P$-set. Any maximal n.d. $P$-set is contained in some standard n.d. $P$-set.*
(iii) *Any non-empty clopen subset $F'$ of a standard set $F_\Delta$ is a maximal n.d. $P$-set.*

Proof: (i) Let $\Delta = \{U_\alpha : \alpha \in A\}$ be an uncountable sieve. In general $\cup U_\alpha$ need not be $C^*$-embedded in $X$. But still this obstacle can be avoided. Let $\{U'_\gamma : \gamma < \omega_1\} \subset \Delta$ and $F_\Delta \cap U' \neq \emptyset$, where $U' = \mathrm{cl}\,\cup U'_\gamma$ is clopen (since $X$ is $\omega_1$-e.d.). Then $\Delta' = \{U'_\gamma : \gamma < \omega_1\} \cup \{X \backslash U'\}$ is an uncountable sieve and $F_{\Delta'} = F_\Delta \cap U' \neq \emptyset$.

The sieve $\Delta'$ has the $C^*$-embedded union in $X$. Indeed, if $F_1$ and $F_2$ are disjoint closed subsets in $\cup\Delta'$, then their intersections with each member of the sieve may be separated by clopen subsets which are clopen in $X$ as well. The unions of these clopen sets have the clopen closures since $X$ is $\omega_1$-e.d. Therefore these closures are disjoint, i.e., $\mathrm{cl}\, F_1 \cap \mathrm{cl}\, F_2 = \emptyset$. Thus, $\cup\Delta'$ is $C^*$-embedded in $X$. By Proposition 1.7(ii), $F_{\Delta'} \in \mathfrak{P}_{\mathfrak{M}}(X)$. Hence $F_\Delta \in \mathfrak{P}_{\mathfrak{M}}(X)$.

(ii) The first part follows from Lemma 1.5(ii), and the second one follows from Lemma 1.5(i).

(iii) As it was obtained in the proof of Lemma 1.5(ii), there is a clopen set $U$ in $X$ such that $F' = F \cap U = F_\Delta \cap U = F_{\Delta'}$ for some sieve $\Delta'$. Now use (i). $\square$

Now we consider conditions on b.d. compact spaces, under which there exists the largest (by inclusion) n.d. $P$-set. Obviously, spaces with such a property exist. The space $X = \beta\omega_1$ with a n.d. $P$-set $F = u\omega_1$ presents a simple example of this kind. Now, we identify this $P$-set $F$ at a point $y_0$. This point is the unique (and largest) n.d. $P$-set in a new compact space $Y$. This simple consideration shows that there may be some relation between the question which we are discussing and the following question: under what conditions a $P$-point is a maximal n.d. set or the largest one. The remaining part of Section 1 will be devoted to these two questions in b.d. compact spaces. To study them we need one definition more. But before introducing it we note that the image of $u\omega_1 \subset X$ remains to be the largest n.d. $P$-set under any identification. In some cases the image of $X$ may not be an $F$-space, in other cases it may be an $F$-space but not a b.d. space. However, if the point $y_0 \in Y$ is identified with a nonisolated point of $Y$, then we obtain a compact $F$-space without n.d. $P$-sets, although this space does not have the Suslin property.

**1.11 Definition.** An arbitrary space $X$ is said to have a local Suslin property if in $X$ there is a $\pi$-base of open sets each having the Suslin property. If $X$ is b.d., then the local Suslin property is equivalent to the existence of a sieve $\Delta$ of clopen sets each with the Suslin property. If $X$ does not have the Suslin property, then the non-empty set $F_\Delta$ will be denoted by $F_{ccc}(X)$ or simply $F_{ccc}$.

**1.12 Lemma.** *The set $F_{ccc}(X)$ does not depend on the choice of the sieve consisting of clopen sets having the Suslin property.*

Proof: Let $\Delta_1 = \{U_\alpha : \alpha \in A\}$ and $\Delta_2$ be two such sieves and $U \in \Delta_2$. Then $\cup\{U_\alpha \cap U : U_\alpha \cap U \neq \emptyset\}$ is dense in $U$. As $U$ has the Suslin property, this union can be presented as $\cup\{U_{\alpha_n} \cap U : n < \omega\}$, where the sequence $(\alpha_n) \subset A$. Since $I_{\Delta_1}$ is a $\sigma$-ideal, then $U = \operatorname{cl} \cup(U_{\alpha_n} \cap U) \in I_{\Delta_1}$. Thus, $F_{\Delta_1} \subset F_{\Delta_2}$. Analogously, $F_{\Delta_2} \subset F_{\Delta_1}$. □

**1.13 Proposition.** *Let $X$ be a b.d. compact space without the Suslin property (this provides that $\mathfrak{P}(X) \neq \emptyset$). The following statements are true.*

(i) *There is the largest (by inclusion) n.d. $P$-set in $X$ if and only if $X$ has the local Suslin property. In this case the required set is $F_{ccc}$.*
(ii) *A $P$-point $x$ is a maximal n.d. $P$-set if and only if in $X$ there is a clopen set $U$ with the local Suslin property and such that $F_{ccc}(U) = \{x\}$.*
(iii) *There is the unique maximal n.d. $P$-set in $X$ if and only if $X$ has the local Suslin property and $F_{ccc}$ is a singleton (this point is the unique non-empty $P$-set in $X$).*
(iv) *If $X$ is an $\omega_1$-e.d. space, then $X$ does not contain maximal n.d. one-point $P$-sets.*

Proof: (i) *Sufficiency.* Let us take a $F_{ccc}$-sieve $\Delta = \{U_\alpha : \alpha \in A\}$. Obviously, $F_{ccc} = F_\Delta$. There are no non-empty $P$-sets in $U_\alpha$ since $c(U_\alpha) \leq \omega$. By Proposition 1.7(i) and Lemma 1.6(iii), $F_\Delta$ is the largest n.d. $P$-set in $X$.

*Necessity.* Let $F \in \mathfrak{P}(X)$ and $\Delta = \{U_\alpha : \alpha \in A\}$ be some $F$-sieve. If $X$ does not have the local Suslin property, then $c(U_{\alpha_0}) > \omega$ for some $\alpha_0 \in A$. In this case there exists $F' \in \mathfrak{P}(X)$ in $U_{\alpha_0}$. Then $F \cup F' \neq F$.

(ii) Let $\{x\} \in \mathfrak{P}_{\mathfrak{M}}(X)$ and let $\Delta = \{U_\alpha : \alpha \in A\}$ be some $x$-sieve. Since $F_\Delta$ is a n.d. $P$-set, $x \in F_\Delta$, and $\{x\} \in \mathfrak{P}_{\mathfrak{M}}(X)$, then $x$ is an isolated point in $F_\Delta$. There is a clopen set $U_0$ in $X$ separating $x$ from closed $F_\Delta \backslash \{x\}$. Let the sieve $\Delta'$ contain all non-empty sets $U_\alpha \cap U_0$ and $X \backslash U_0$. Denote the members of this sieve by $U'_\gamma$ $(\gamma \in \Gamma)$.

Let $\Gamma_1 = \{\gamma \in \Gamma : c(U'_\gamma) > \omega\}$ and $\Gamma_2 = \Gamma \backslash \Gamma_1$. We shall check that $\Gamma_1$ is at most countable. Suppose that the opposite statement is true. In this case $x \in \operatorname{cl}\{U'_\gamma : \gamma \in \Gamma_1\}$. Since $c(U'_\gamma) > \omega$ for any $\gamma \in \Gamma_1$, there is $F_\gamma \subset U'_\gamma$ such that $F_\gamma \in \mathfrak{P}(X)$. Then $F = \operatorname{cl} \cup F_\gamma$ is a n.d. $P$-set in a b.d. compact space $X$. Finally, $x \in F$ and $\{x\} \in \mathfrak{N}(F)$, a contradiction.

Thus, $\Gamma_1$ is at most countable. So $x\overline{\in}W = \mathrm{cl} \cup\{U'_\gamma : \gamma \in \Gamma_1\}$. Let $\Delta'' = \{U'_\gamma : \gamma \in \Gamma_2\} \cup \{X\backslash(U_0\backslash W)\}$. Obviously, $\{x\} = F_{\Delta''} = F_{ccc}(U_0\backslash W)$. Finally, put $U = U_0\backslash W$. Necessity is proved.

Sufficiency follows from (i). Of course, $\{x\} = F_{ccc}(U)$.

(iii) follows from (i) and (ii) and Theorem 1.10(iii).

(iv) Let $\{x\} \in \mathfrak{P}_{\mathfrak{M}}(X)$. As it was established in the proof of (ii), $\{x\} = F_{ccc}(U)$ for some clopen $U$. So without a loss of generality we may assume that $\{x\} = F_{ccc}(X) = F_\Delta$, where $\Delta = \{U_\alpha : \alpha \in A\}$. It is clear that card $A \geq \omega_1$. Let $A_1 \subset A$, $A_2 = A\backslash A_1$, card $A_1 = \omega_1$, and card $A_2 \geq \omega_1$. Since $X$ is $\omega_1$-e.d., the set $W = \mathrm{cl} \cup\{U_\alpha : \alpha \in A_1\}$ is clopen. Therefore, $X\backslash W = \mathrm{cl} \cup\{U_\alpha : \alpha \in A_2\}$ is clopen too. But the sets $\cup\{U_\alpha : \alpha \in A_i\}$ $(i = 1, 2)$ are not closed. So $x \in W \cap (X\backslash W) = \emptyset$, a contradiction. □

**1.14 Remark.** A compact space $X$ with the following properties can be constructed: $X$ has a dense set of nonisolated $P$-points, and finite and only finite $P$-sets are maximal n.d. $P$-sets. Evidently, this compact space cannot be b.d. This example will be presented in Section 3.

## 2. On spaces with non-empty nowhere dense $P$-sets and without maximal nowhere dense $P$-sets

It appears to be more difficult to find compact spaces $X$ with $\mathfrak{P}(X) \neq \emptyset$ and $\mathfrak{P}_{\mathfrak{M}}(X) = \emptyset$ than to find compact spaces with non-empty $\mathfrak{P}_{\mathfrak{M}}(X)$.

**2.1 Definition.** An arbitrary compact space $X$ without isolated points is said to have a *rich family of n.d. $P$-sets*, if for every open $G \neq \emptyset$ there is non-empty $F \in \mathfrak{P}(X)$ in $G$. If $X$ is b.d., then $X$ has a rich family of n.d. $P$-sets if and only if $c(G) > \omega$ for any open $G \neq \emptyset$.

**2.2 Proposition.** *Let $X$ be a compact $F$-space without isolated points, and $X$ has a rich family of n.d. $P$-sets. Let two following conditions hold for some $F \in \mathfrak{P}(X)$.*

- (a) *There is a family $\{H_\alpha : \alpha < \omega_1\}$ of closed neighborhoods of $F$ such that $H_{\alpha_2} \subset \mathit{int}\ H_{\alpha_1}$ for $\alpha_2 > \alpha_1$, and $H = \cap\{H_\alpha : \alpha < \omega_1\}$ is n.d. in $X$.*
- (b) *There is a family $\{G_\alpha : \alpha < \omega_1\}$ of open sets such that $F \cap G_\alpha \neq \emptyset$ and if $G \cap F \neq \emptyset$ for some open $G$, then $G_\alpha \subset G$ for some $\alpha < \omega_1$.*

*In this case there is $F' \in \mathfrak{P}(X)$ with $F' \succ F$.*

Proof: Evidently, $H \in \mathfrak{P}(X)$. By induction, we choose non-empty n.d. $P$-sets $F_\alpha \subset ((\ \mathrm{int}\ H_\alpha \cap G_\alpha)\backslash \cup\{F_{\alpha'} : \alpha' < \alpha\})\backslash H$. This can be done because $\mathrm{cl} \cup\{F_{\alpha'} : \alpha' < \alpha\}$ is a $P$-set in an $F$-space $X$ (since card$\{\alpha' : \alpha' < \alpha\} \leq \omega$). Besides $\mathrm{cl} \cup\{F_{\alpha'} : \alpha' < \alpha\}$ is n.d. as all n.d. $P$-sets $F_{\alpha'}$ are disjoint. Denote $F' = H \cup \mathrm{cl} \cup\{F_\alpha : \alpha < \omega_1\}$. First, we check that $F' \in \mathfrak{N}(X)$.

Suppose that it is not so. Then there are open sets $G$ and $G'$ such that $\emptyset \neq G' \subset \operatorname{cl} G' \subset G \backslash H \subset G \subset F'$. The open set $G'$ is contained in $X \backslash H_\alpha$ for some $\alpha < \omega_1$. Therefore, $\emptyset \neq G' \subset \operatorname{cl} \cup \{F_{\alpha'} : \alpha' < \alpha\} \in \mathfrak{N}(X)$. The contradiction shows that $F'$ is n.d.

Now we are to prove that $F'$ is a $P$-set. Let $V \in \mathbf{C}z(X)$ and $V \cap F' = \emptyset$. We are to check that $\operatorname{cl} V \cap F' = \emptyset$. We have $\operatorname{cl} V \cap H = \emptyset$ as $H$ is a $P$-set. Then $\operatorname{cl} V \cap H_{\alpha_0} = \emptyset$ for some $\alpha_0 < \omega_1$. This implies $\operatorname{cl} V \cap \operatorname{cl} \cup \{F_\alpha : \alpha \geq \alpha_0\} = \emptyset$. Finally,

$$\emptyset = (\operatorname{cl} V \cap H) \cup (\operatorname{cl} V \cap \operatorname{cl} \cup \{F_\alpha : \alpha \geq \alpha_0\}) \cup (\operatorname{cl} V \cap \operatorname{cl} \cup \{F_\alpha : \alpha < \alpha_0\}) = \operatorname{cl} V \cap F'.$$

At last, we obtain that $F' \succ F$. Since $(\cup F_\alpha) \cap F = \emptyset$, it is sufficient to prove $F \subset \operatorname{cl} \cup F_\alpha$. Suppose that it is not so. Then find an open set $G$ such that $G \cap F \neq \emptyset$ and $\operatorname{cl} G \cap \operatorname{cl} \cup F_\alpha = \emptyset$. By condition (b), we get $G_{\alpha_0} \subset G$ for some $\alpha_0 < \omega_1$. Thus, $\emptyset \neq F_{\alpha_0} \subset G_{\alpha_0} \subset G$, which contradicts the choice of $G$. □

**2.3 Corollary.** (i) (**CH**) *$\mathfrak{P}_{\mathfrak{M}}(\omega^*) = \emptyset$ though $\omega^*$ has a rich family of n.d. $P$-sets.*

(ii) (**CH**) *Also $\mathfrak{P}_{\mathfrak{M}}(a_\sigma \omega^*) = \emptyset$ for the $\sigma$-absolute of the compact $\omega^*$ though $a_\sigma \omega^*$ has a rich family of n.d. $P$-sets.*

Proof: In the both cases we have $w(X) = c(X) = 2^\omega = \omega_1$. It allows to apply Proposition 2.2 and to construct families $\{H_\alpha : \alpha < \omega_1\}$ and $\{G_\alpha : \alpha < \omega_1\}$. In particular, we may take all sets from the base of open sets intersecting with $F$ as a second family. □

**2.4 Remark.** Note that for any infinite compact $F$-space $X$ we have $w(X) \geq c$. It means that it is difficult (if possible at all) to use Proposition 2.2 to prove Corollary 2.3 without some set-theoretical conjectures.

The existence of compact $F$-spaces and b.d. compact spaces with a rich family of n.d. $P$-sets but without maximal n.d. $P$-sets will follow from Theorem 2.7 in which Proposition 2.2 is applied not to the given space but to its sequential absolute.

**2.5 Lemma.** *Let $X$ be an arbitrary compact space without isolated points. Let for any non-empty open $G$ there exists $F \in \mathfrak{P}'(X)$ such that $F \subset G$, and for any $V \in \mathbf{C}z(X)$ there exists a $\Theta$-set $\Theta$ such that $(V \cap F = \emptyset) \Longrightarrow \operatorname{cl} V \cap F \subset \Theta$. In this case the sequential absolute $Y = a_s X$ has a rich family of n.d. $P$-sets.*

Proof: Let $\tau : Y \to X$. For a set $F$ from the condition of Lemma, we denote

$$F' = \tau^{-1} F \backslash \cup \{V \in \mathbf{C}z(Y) : V \cap \tau^{-1} F \subset \Theta' \text{ for some } \Theta\text{-set in } Y\}.$$

We are to prove that $\emptyset \neq F' \in \mathfrak{P}(Y)$. First, we check that $\tau F' = F$ (therefore $F' \neq \emptyset$).

Suppose $F\backslash\tau F' \neq \emptyset$. There is open $G \subset X$ such that $G \cap F \neq \emptyset$ and $\mathrm{cl}\, G \cap \tau F' = \emptyset$. Then $\mathrm{cl}\, \tau^{-1}G \cap F' = \emptyset$. Hence, if $y \in \mathrm{cl}\, \tau^{-1}G \cap \tau^{-1}F$, then $y \overline{\in} F'$. By definition of the set $F'$, we find a cozero-set $V_y$ and a $\Theta$-set $\Theta_y$ such that $y \in V_y$ and $V_y \cap \tau^{-1}F \subset \Theta_y$. We choose a finite subcovering $\{V_{y_1}, \dots V_{y_n}\}$ of a compact set $\mathrm{cl}\, \tau^{-1}G \cap \tau^{-1}F$. In this case

$$\mathrm{cl}\, \tau^{-1}G \cap \tau^{-1}F \subset \cup\{V_{y_i} \cap \tau^{-1}F \cap \mathrm{cl}\, \tau^{-1}G : i = 1, 2, \dots n\} \subset \cup \Theta_{y_i} = \Theta'.$$

By Theorem 0.2, there is a $\Theta$-set $\Theta$ in $X$ such that $\Theta' \subset \tau^{-1}\Theta$. It implies $\emptyset \neq F \cap G \subset \Theta$. So $\Theta$ contains a non-empty open subset $F \cap G$ of the n.d. $P$-set $F$. But this contradicts properties of $P'$-sets([10]). The contradiction shows that $\tau F' = F$.

Now it remains to prove that $F'$ is a $P$-set. Let $V \in \mathbf{C}z(Y)$ and $V \cap F' = \emptyset$. We must obtain that $\mathrm{cl}\, V \cap F' = \emptyset$. By Theorem 0.2(i), there is $W \in \mathbf{C}z(X)$ such that $\tau^{-1}W \subset V \subset \mathrm{cl}\, \tau^{-1}W$. Since $\tau^{-1}W \cap F' \subset V \cap F' = \emptyset$, we get $W \cap F = \emptyset$. Indeed, we have $W \cap \tau F' = \emptyset$; but we have already proved that $\tau F' = F$.

Further, by the assumption of Lemma, there is a $\Theta$-set $\Theta \supset \mathrm{cl}\, W \cap F$. It implies $\mathrm{cl}\, V \cap \tau^{-1}F \subset \tau^{-1}\Theta$. So $(\mathrm{cl}\, V\backslash\tau^{-1}\Theta) \cap (\tau^{-1}F\backslash\tau^{-1}\Theta) = \emptyset$. Now we shall prove that $\mathrm{cl}\, V \cap F' \subset \mathrm{cl}(\tau^{-1}F\backslash\tau^{-1}\Theta)$. Suppose it is not so. Then there is $y \in \mathrm{cl}\, V \cap F'$ such that $y \overline{\in} \mathrm{cl}(\tau^{-1}F\backslash\tau^{-1}\Theta)$. We find $V' \in \mathbf{C}z(Y)$ such that $y \in V'$ and $V' \cap (\tau^{-1}F\backslash\tau^{-1}\Theta) = \emptyset$. Then $y \in V' \cap F' \subset V' \cap \tau^{-1}F \subset \tau^{-1}\Theta$. By definition of the set $F'$, we have $y \overline{\in} F'$. A contradiction proves the desired result. Thus, we obtain a new inclusion $\mathrm{cl}\, V \cap F' \subset \mathrm{cl}(\mathrm{cl}\, V\backslash\tau^{-1}\Theta) \cap \mathrm{cl}(\tau^{-1}F\backslash\tau^{-1}\Theta)$.

To prove that $F'$ is a $P$-set we shall check that the right part of this inclusion is empty. We recall that $Y = a_s X$ is a quasi-$F$-space. So the $\Theta$-set $\tau^{-1}\Theta$ is $C^*$-coembedded. As the sets $\mathrm{cl}\, V\backslash\tau^{-1}\Theta$ and $\tau^{-1}F\backslash\tau^{-1}\Theta$ are disjoint closed subsets of the normal space $Y\backslash\tau^{-1}\Theta$, these closures in $X$ are disjoint too. $\square$

**2.6 Remark.** Obviously, if $X$ has a rich family of $P$-sets, then the conditions of Lemma 2.5 are satisfied. But it is easy to prove that any irreducible preimage of a compact space $X$ with a rich family of n.d. $P$-sets also has a rich family of n.d. $P$-sets. Indeed, a preimage of a $P$-set is a $P$-set.

**2.7 Theorem.** *Let $X$ be a compact space with a weight $\omega_1$ and without isolated points. Then the following statements are true.*

(i) *Let conditions of Lemma 2.5 hold for $X$, and besides for disjoint cozero-sets $V_1$, $V_2$ there exists a $\Theta$-set $\Theta \supset \mathrm{cl}\, V_1 \cap \mathrm{cl}\, V_2$. Then $Y = a_s X$ is a compact $F$-space with a rich family of n.d. $P$-sets, but $\mathfrak{P}_{\mathfrak{M}}(Y) = \emptyset$.*

(ii) *Let any open set in $X$ have not the Suslin property, and for any cozero-set $V$ there exists a cozero-set $V'$ such that $V \cap V' = \emptyset$ and $V \cup V'$ is dense in $X$. In this case $Y = a_s X$ is a b.d. compact space (and therefore is the $\sigma$-absolute of $X$) with a rich family of n.d. $P$-sets, but $\mathfrak{P}_{\mathfrak{M}}(X) = \emptyset$.*

Proof: (i) By Theorem 0.2(iii), $Y$ is an $F$-space. By Lemma 2.5, $Y$ has a rich family of n.d. $P$-sets. Let $F \in \mathfrak{P}(Y)$. We shall check that conditions (a) and (b) from Proposition 2.2 are true for $F$.

Let $\{W_\gamma : \gamma < \omega_1\}$ be a base of open sets in $X$, which we may assume to be additive and multiplicative. Then it is easy to see that the base of closed neighborhoods of the set $\tau F$ is contained in the family of all sets $E_\gamma = X \backslash W_\gamma$. Since $\tau F$ is a $P'$-set, then this base is uncountable and may be presented as $\{E_\alpha : \alpha < \omega_1\}$. Obviously, $\cap\{\tau^{-1}E_\alpha : \alpha < \omega_1\} = \tau^{-1}\tau F \supset F$. Then (in a standard way) we construct a family $\{H_\alpha : \alpha < \omega_1\}$ of closed neighborhoods of a $P$-set $F$ such that $H_\alpha \subset \cap\{\text{ int } H_{\alpha'} : \alpha' < \alpha\}$ and $H_\alpha \subset \cap\{\tau^{-1}E_{\alpha'} : \alpha' \leq \alpha\}$. So $F \subset H = \cap\{H_\alpha : \alpha < \omega_1\} \subset \tau^{-1}\tau F$. Thus, condition (a) holds.

Now we consider the family of all open sets $\tau^{-1}W_\gamma$ intersecting $F$ and present it as $\{G_\alpha : \alpha < \omega_1\}$. We are to prove that this family satisfies condition (b).

Let $G$ be open in $Y$ and $G \cap F \neq \emptyset$. We shall find $G_\alpha \subset G$. Let $y \in G \cap F$. There is a function $h \in C^*(Y)$ such that $0 \leq h \leq 1$, $h(y) = 1$, and $h(Y \backslash G) = 0$. Let $\varepsilon < 1/3$. By Theorem 0.2(ii), there exist a $\Theta$-set $\Theta$ in $X$ and $g \in C^*(X \backslash \Theta)$ such that $0 \leq g \leq 1$, $|g \circ \tau - h| < \varepsilon$ on $Y \backslash \tau^{-1}\Theta$. Since $\tau^{-1}\Theta$ is $C^*$-coembedded in an $F$-space $Y$, there is a continuous extension $f = \overline{g \circ \tau}$ of the function $g \circ \tau$ on $Y$ such that $|f - h| \leq \varepsilon$. It implies $f(Y) \geq 1 - \varepsilon$ and $f(Y \backslash G) \leq \varepsilon$. Since $F \cap \tau^{-1}\Theta$ is a closed n.d. subset of a $P$-set $F$, there is a point $y' \in (F \backslash \tau^{-1}\Theta) \cap f^{-1}$ $(1/2, 1]$. It means that $f(y') = g(\tau y') > 1/2$. The function $g$ is continuous in the point $\tau y'$, and therefore there is $W_\gamma$ such that $\tau y' \in W_\gamma$, $\operatorname{cl} W_\gamma \cap \Theta = \emptyset$, and $g(W_\gamma) > 1/2$ (here we use the multiplicativity of the base $\{W_\gamma : \gamma < \omega_1\}$). Hence $y' \in \tau^{-1}W_\gamma \cap F$, and $\tau^{-1}W_\gamma \cap F \neq \emptyset$. It remains to prove that $\tau^{-1}W_\gamma \subset G$ because this inclusion implies that $\tau^{-1}W_\gamma = G_\alpha$ for some $\alpha < \omega_1$.

Suppose that there is $y_0 \in \tau^{-1}W_\gamma \backslash G$. Then $y_0 \in (Y \backslash \tau^{-1}\Theta) \backslash G$, and thus $(g \circ \tau)(y_0) = f(y_0) \leq \varepsilon$. But on the other hand, $\tau y_0 \in W_\gamma$ and $g(W_\gamma) \geq 1/2$. So $(g \circ \tau)(y_0) \geq 1/2$. The contradiction shows that condition (b) from Proposition 2.2 holds. So we may use this Proposition.

(ii) By Theorem 0.2(iv), the compact space $Y = a_s X$ is b.d. In (i) it was proved that $\mathfrak{P}_{\mathfrak{M}}(Y) = \emptyset$. Since any non-empty open set in $X$ does not have the Suslin property, then the same is true for $Y$. So $Y$ has a rich family of n.d. $P$-sets. $\square$

There is another way to get b.d. compact spaces with a rich family of n.d. $P$-sets but without m.n.d. $P$-sets.

**2.8 Remark.** In Theorem 2.7(ii) one cannot weaken the condition "every non-empty set in $X$ does not have the Suslin property" to the condition "$X$ does not have the Suslin property". At least, under **CH** it is easy to present a corresponding b.d. compact space with a unique non-empty n.d. $P$-set (which is, of course, $P$-point).

**2.9 Proposition.** *Let $X$ be a b.d. anti-$\omega_1$-e.d. compact space with $w(X) = \omega_2$ and without isolated points. Besides, we assume that any non-empty open set $G \subset X$ does not have the Suslin property. Then $\mathfrak{P}_{\mathfrak{M}}(X) = \emptyset$ though $X$ has a rich family of n.d. $P$-sets.*

Proof: Note that $c(X) = \omega_2$. Let $\{G_\gamma : \gamma < \omega_2\}$ be a base of clopen sets in $X$ and $F \in \mathfrak{P}(X)$. We are to check that $F \overline{\in} \mathfrak{P}_{\mathfrak{M}}(X)$. By Lemma 1.5(i), we may assume that $F = F_\Delta$ is a standard $P$-set. The sieve $\Delta$ cannot be countable since $F_\Delta$ is a $P$-set. $\Delta$ cannot have cardinality of $\omega_1$ since $X$ is anti-$\omega_1$-e.d. and so no $\omega_1$-family of disjoint sets can be dense in $X$. Thus, the sieve $\Delta$ has cardinality of $\omega_2$. Let $\Delta = \{U_\alpha : \alpha < \omega_2\}$.

Denote $\Gamma = \{\gamma < \omega_2 : G_\gamma \cap F_\Delta \neq \emptyset\}$. For any $\gamma \in \Gamma$ let $A_\gamma = \{\alpha < \omega_2 : U_\alpha \cap G_\gamma \neq \emptyset\}$. Obviously, card $A_\gamma = \omega_2$ (since $F_\Delta \in \mathfrak{P}(X)$ and $X$ is anti-$\omega_1$-e.d.). Denote $\alpha_\gamma = \min[A_\gamma \setminus \{\alpha_{\gamma'} : \gamma' < \gamma, \gamma' \in \Gamma\}]$ for any $\gamma \in \Gamma$. By the construction, we have $\alpha_{\gamma_1} \neq \alpha_{\gamma_2}$ for $\gamma_1 \neq \gamma_2$.

The clopen set $G_\gamma \cap U_{\alpha_\gamma}$ does not have the Suslin property. Then there is $F_\gamma \in \mathfrak{P}(X)$ in $G_\gamma \cap U_{\alpha_\gamma}$. Let $F' = \operatorname{cl} \cup \{F_\gamma : \gamma \in \Gamma\}$. Then $F'$ is a n.d. $P$-set since $X$ is b.d. To prove that $F_\Delta \overline{\in} \mathfrak{P}_{\mathfrak{M}}(X)$ we shall obtain $\operatorname{cl} \cup F_\gamma \supset F_\Delta$, which implies $F_\Delta \prec F'$.

Let $G_\gamma \cap F_\Delta \neq \emptyset$, i.e., $\gamma \in \Gamma$. By the construction, $F_\gamma \subset (G_\gamma \cap F') \setminus F_\Delta$. Moreover $(F' \setminus F_\Delta) \cap G_\gamma \neq \emptyset$. Thus, $\operatorname{cl} \cup F_\gamma \supset F_\Delta$. □

**2.10 Corollary** ($2^{\omega_1} = \omega_2$). *The space $u\omega_1$ of all uniform ultrafilters in $\omega_1$ contains no maximal n.d. $P$-sets though it has a rich family of n.d. $P$-sets.*

## 3. Example

In this section a compact space $X$ without isolated points and with the following properties will be constructed.

(1) $w(X) = \omega_1$.
(2) In $X$ no non-empty open set has the Suslin property.
(3) For any $G \in \mathbf{C}z(X)$ there exists $G' \in \mathbf{C}z(X)$ such that $G \cap G' = \emptyset$ and $G \cup G'$ is dense in $X$.
(4) Any point in $X$ is either a $P$-point or a $G_\delta$-point.
(5) The set of all $P$-points is dense in $X$.
(6) A n.d. set in $X$ is a $P$-set if and only if it is the union of a finite family of $P$-points. In particular, $\mathfrak{P}_{\mathfrak{M}}(X) = \mathfrak{P}(X)$.

By Theorem 2.7(ii), the $\sigma$-absolute of this space is a b.d. compact space with a rich family of n.d. $P$-sets but without maximal n.d. $P$-sets.

The desired compact space $X$ is constructed as a limit of the inverse spectrum $\varprojlim\{Y_\alpha, \pi^\alpha_\delta, \omega_1\}$ of metric compact spaces (here $\omega_1$ is considered as a linearly ordered set of smaller ordinals).

**3.1.** In this subsection a special functor $T$ will be presented. We consider a class of all metric compact spaces without isolated points and with a fixed (for every such a compact space $Y$) dense sequence $E = \{e_n : n < \omega\}$ with a fixed order on $E$. In this class a functor $T$ is presented in the following way. A compact metric space $T(Y)$ is a closed subspace of the product $Y \times [0,1]$, which is the union of a *horizontal* $H = Y \times \{0\}$ and all *verticals* $V_n = \{e_n\} \times (0, 1/n]$. A *closed vertical* $\{e_n\} \times [0, 1/n]$ is denoted by $\overline{V}_n$. Any point of the horizontal is called a *lower point*, and any point $(e_n, 1/n)$ is called an *upper point* in $T(Y)$. Let $\pi$ denote a natural projection of $T(Y)$ on the horizontal $H$. So $\pi$ is a continuous surjection on $Y : \pi(y, r) = y$. Finally, we choose a dense sequence in $T(Y)$ in the following way: $E_{T(Y)} = \{(e_n, r_m) : r_m \in \mathbb{Q} \cap (0, 1/n], n < \omega\}$. Some rule of numeration is fixed in $E_{T(Y)}$.

Here some properties of the given construction are presented.

(a) Obviously, $H$ is n.d. in $T(Y)$, and each vertical $V_n$ is open in $T(Y)$.
(b) If $y \overline{\in} E$, then $\pi^{-1}y$ is a lower point in $T(Y)$.
(c) If $y \in E$, then $\pi^{-1}y = \overline{V}_n$ for the corresponding number $n < \omega$.
(d) If $M \subset Y$, then $\pi^{-1}M$ is the union of some family of closed verticals $\overline{V}_n$ and some subset of n.d. $H$.

**3.2.** In this subsection we construct a family of compact metric spaces $\{Y_\alpha : \alpha < \omega_1\}$ with dense sequences $E_\alpha \subset Y_\alpha$ and surjective (continuous) mappings $\pi^\alpha_\delta : Y_\alpha \to Y_\delta$ such that $\pi^\alpha_\delta = \pi^\lambda_\delta \circ \pi^\alpha_\lambda$ $(0 \le \delta < \lambda < \alpha < \omega_1)$ and also $\pi^\alpha_\delta E_\alpha = E_\delta$.

Let $Y_0 = [0,1]$, $E_0 = \mathbb{Q} \cap [0,1]$, $Y_1 = T(Y_0)$, $E_1 = E_{T(Y_0)}$. Then $E_1$ is dense in $Y_1$. Now we suppose that $Y_\delta$, $\pi^\alpha_\delta$, $\pi^\delta_\lambda$ and countable dense sets $E_\delta$ are constructed for all $\delta \le \alpha$. In this case we put $Y_{\alpha+1} = T(Y_\alpha)$, $\pi^{\alpha+1}_\delta = \pi^\alpha_\delta \circ \pi^{\alpha+1}_\alpha$, where $\pi^{\alpha+1}_\alpha = \pi$ is a natural projection on the horizontal $H = H_{\alpha+1} \approx Y_\alpha$. We take $E_{T(Y_\alpha)}$ as a set $E_{\alpha+1}$.

Now let $\mu < \omega_1$ be some limit ordinal, and assume that for any $\delta < \mu$ compact metric spaces $Y_\delta$ with countable dense sets $E_\delta$ and with corresponding mappings $\pi^\delta_\lambda$ are constructed. In this case we put $Y_\mu = \varprojlim(Y_\delta, \pi^\delta_\lambda, \mu)$, where $\mu$ is considered as a linearly ordered set of all smaller ordinals. Evidently, $w(Y_\mu) = \omega$ since $Y_\mu$ is a compact metric space (without isolated points). Any element $y^{(\mu)} \in Y_\mu$ is a *$\mu$-thread* $(y^{(\alpha)} : \alpha < \mu)$, where $y^{(\alpha)} \in Y_\alpha$ and $y^{(\delta)} = \pi^\alpha_\delta y^{(\alpha)}$. The canonical mapping $\pi^\mu_\alpha : Y_\mu \to Y_\alpha$ is generated by the corresponding mapping $y^{(\mu)} \to y^{(\alpha)}$.

To introduce the set $E_\mu$ we take the family of all the $\mu$-threads $(y^{(\alpha)} : \alpha < \mu)$ such that for each of them there exists non-limit $\delta < \mu$ with the following property: $y^{(\alpha)}$ is an upper point in $Y_\alpha$ for any non-limit ordinal $\alpha \ge \delta$. Now we shall prove that any $\mu$-thread $(y^{(\alpha)} : \alpha < \mu)$ from $E_\mu$ may contain only a finite number of non-upper members $y^{(\alpha)}$ for non-limit $\alpha$. Indeed, if it is not so for some $\mu$-thread, we find a increasing sequence of non-limit ordinals $(\alpha_n)$ such that $\alpha_n < \mu$ and

$y^{(\alpha_n)}$ is not an upper point. Let $\lambda = \sup \alpha_n \leq \mu$. If $\lambda = \mu$, then by definition this $\mu$-thread does not belong to $E_\mu$. If $\lambda < \mu$, then $\lambda + 1 < \mu$ and $y^{(\lambda+1)}$ is a lower point in $Y_{\lambda+1}$. Therefore by property (b) of the functor $T$, $y^{(\delta)}$ is a lower point for any non-limit $\delta$ such that $\lambda + 1 \leq \delta < \mu$. So again this $\mu$-thread does not belong to $E_\mu$. Using this property and the fact that both $\{\delta : \delta < \mu\}$ and $E_\delta$ are countable we get that $E_\mu$ is countable too.

Now we shall prove that $E_\mu$ is dense in $Y_\mu$. It is easy to see that in this case the base of open sets in $Y_\mu$ may be presented by all sets of the type of $(\pi^\mu_\alpha)^{-1} G_\alpha$, where $\alpha < \mu$ and $G_\alpha$ is a base open set in $Y_\alpha$ (if open bases are chosen so that, if $G_\alpha$ is a base set, then $(\pi^\lambda_\alpha)^{-1} G_\alpha$ is a base set too).

In each $G_\alpha$ there is some $y^{(\alpha)} \in E_\alpha$. By property (c) of the functor $T$, there exists a unique $\mu$-thread $y^{(\mu)}$ such that $y^{(\delta)}$ is an upper point in $Y_\delta$ for every non-limit $\delta \geq \alpha + 1$. Thus $y^{(\mu)} \in (\pi^\mu_\alpha)^{-1} G_\alpha$.

**3.3.** In this subsection we construct the required compact space $X$. We assume that each countable dense $E_\mu$ is ordered in some manner. Then we put

$$X = \varprojlim (Y_\alpha, \pi^\alpha_\delta, \omega_1).$$

By the definition, the space $X$ consists of $\omega_1$-*threads* $(y^{(\alpha)} : \alpha < \omega_1)$. We note that if $\alpha$ is a limit ordinal, then $y^{(\alpha)}$ is defined by the $\alpha$-thread $(y^{(\delta)} : \delta < \alpha)$. So any $\omega_1$-thread is well defined by all $y^{(\delta)}$, where $\delta$ is non-limit ordinal less than $\omega_1$. So one can assume that any point $x$ of the space $X$ is a *thread* $(y^{(\alpha)} : \alpha \in \omega_1')$, where $\omega_1'$ is the set of all non-limit $\alpha < \omega_1$. Now we define $\tau_\alpha : X \to Y_\alpha$, where $\tau_\alpha(x) = y^{(\alpha)} (\alpha \in \omega_1')$.

Show that for any thread $(y^{(\alpha)} : \alpha < \omega_1')$ there can be only two cases. In the first case all $y^{(\alpha)} \in E_\alpha$ and all but a finite numbers $y^{(\alpha)}$ are upper points in $Y_\alpha$. In the second case there is $\delta \in \omega_1'$ such that $y^{(\alpha)}$ is a lower point in $Y_\alpha$ for any $\alpha \geq \delta$, $\alpha \in \omega_1'$. One can prove this by repeating the arguments which were used when introducing the set $E_\mu$ (see 3.2). The points described in the first case will be called *upper*, and the family of all such points will be denoted by $X_U$. Likewise, the points from the second case will be called *lower*, and the set of all these points will be denoted by $X_L$.

**3.4.** In this subsection we check the properties (1)–(6) for $X$.

Let $x \in X_L$. Then $y^{(\alpha)}$ is a lower point for some $\alpha \in \omega_1'$, and $y^{(\alpha)}$ is a $G_\delta$-point in a metric space $Y_\alpha$, i.e., $y^{(\alpha)} = \cap\{G_n : n < \omega\}$, where $G_n$ are open in $Y_\alpha$. By property (b), we have $\tau_\alpha^{-1} y^{(\alpha)} = \cap(\tau_\alpha^{-1} G_n) = x$. Therefore, $x$ is a $G_\delta$-point in $X$.

For the given $M \subset X$ we denote $M^{(\alpha)} = \tau_\alpha M$. Then the set $\tau_\alpha^{-1} M^{(\alpha)}$ decreases as the index $\alpha$ increases, and $M = \cap \tau_\alpha^{-1} M^{(\alpha)}$. Now if $F$ is closed in $X$ and $x = \{y^{(\alpha)} : \alpha \in \omega_1'\} \overline{\in} F$, then $y^{(\alpha)} \overline{\in} F^{(\alpha)}$ for some $\alpha \in \omega_1'$ ( here we used a compactness of $X$). Now we can prove that, if $x \in X_U$, then $x$ is a $P$-point in $X$.

Let $F_k$ be closed in $X$ and $x\overline{\in}F_k$ $(k<\omega)$. We must prove that $x\overline{\in}\operatorname{cl}\cup F_k$. For each $k$ by the result of the preceding paragraph, there exists $\alpha_k\in\omega_1'$ such that $y^{(\alpha_k)}\overline{\in}F_k^{(\alpha_k)}$. Let $\alpha\in\omega_1'$ and $\alpha>\sup\{\alpha_k : k<\omega\}$. Since $F_k^{(\alpha+1)}\subset(\pi_\alpha^{\alpha+1})^{-1}F_k^\alpha$ we use property (d) of the functor $T$ and get that $F_k^{(\alpha+1)}$ is contained in the union of some verticals in $Y_{\alpha+1}=T(Y_\alpha)$ and of the horizontal $H=H_{\alpha+1}$. Besides, $y^{(\alpha+1)}\in V_{n_0}$ (see property (c) of the functor $T$). Hence $\cup\{F_k^{(\alpha+1)} : k<\omega\}\cap\overline{V}_{n_0}=\emptyset$. By property (a) of the functor $T$, we have $\operatorname{cl}\cup F_k^{(\alpha+1)}\cap V_{n_0}=\emptyset$. It means that $x\overline{\in}\operatorname{cl}\cup F_k$. So property (4) is proved.

As $X$ has nonisolated $P$-points, $w(X)\geq\omega_1$. The base of open sets in $X=\varprojlim(Y_\alpha,\pi_\delta^\alpha,\omega_1)$ consists of all sets $\tau_\alpha^{-1}G_\alpha$, where $G_\alpha$ belongs to the base of open sets in the space $Y_\alpha(\alpha\in\omega_1')$. Thus, $w(X)\leq\omega_1$. Finally, $w(X)=\omega_1$, i.e., property (1) holds.

We can apply the arguments used in the proof of the density of $E_\mu$ in $Y_\mu$ (see 3.2) to the set $X_U$ and get that this set is dense in $X$. So properties (5) and (2) hold in $X$. Since no $P$-set can contain $G_\delta$-points we get that each $P$-set $F$ in $X$ is a compact $P$-space in induced topology. Thus, $F$ is the union of a finite set of $P$-points. Therefore, each such a set is a maximal n.d. $P$-set. So property (6) holds too.

It remains to check property (3). Let $G$ be some cozero-set in $X$. Then $G$ is an $F_\sigma$-set; besides $X$ is a compact space. Hence $G=\cup\{\tau_{\alpha_k}^{-1}G_k : k<\omega\}$, where each $G_k$ is a base open set in $Y_{\alpha_k}$. Fix $\alpha>\sup\{\alpha_k : k<\omega\}$ and replace each $G_k$ by $(\pi_{\alpha_k}^\alpha)^{-1}G_k$ with the same preimage in $X$. Thus, $G=\cup\{\tau_\alpha^{-1}G_k : k<\omega\}=\tau_\alpha^{-1}(\cup\{G_k : k<\omega\})$, where all $G_k$ are open sets in one space $Y_\alpha$. Let $W_k=(\pi_\alpha^{\alpha+1})^{-1}G_k$. Then $G$ can be represented as $\tau_{\alpha+1}^{-1}\cup\{W_k : k<\omega\}$. By property (d), the set $\cup W_k$ is the union of some set of verticals $V_n\subset Y_{\alpha+1}$ and a set from the horizontal $H=H_{\alpha+1}$. Let a set $W'$ be the union of all other verticals from $Y_{\alpha+1}$. Evidently, $W'$ is a cozero-set and $W'\cap(\cup W_k)=\emptyset$. Put $G'=\tau_{\alpha+1}^{-1}(W')$. Then $G'\in\mathbf{C}z(X)$ and $G\cap G'=\emptyset$. Moreover, $X\backslash(G\cup G')\subset\tau_{\alpha+1}^{-1}H\subset X_L$. Thus, the closed set $X\backslash(G\cup G')$ does not intersect the set $X_U$ which is dense in $X$. Hence, $G\cup G'$ is dense in $X$. □

## 4. Unsolved problems

**4.1** *Question.* Is it true in ZFC that $\mathfrak{P}_{\mathfrak{M}}(a_s\omega^*)=\mathfrak{P}_{\mathfrak{M}}(u\omega_1)=\emptyset$?

**4.2** *Question.* Under what set-theoretical conjectures which are weaker than **CH** does the identity $\mathfrak{P}_{\mathfrak{M}}(\omega^*)=\emptyset$ hold?

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Chair of Math. Analysis, St-Petersburg State Pedagogical University, Mojka 48, St-Petersburg 191186, Russia

Chair of High Math. St-Petersburg State University of Technology and Design, Bolshaya Morskaya 18, St-Petersburg 191065, Russia

*E-mail*: artifex@increatum.spb.ru


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