Almost* realcompactness

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Abstract. We provide a new generalization of realcompactness based on ultrafilters of cozero sets and contrast it with almost realcompactness.

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All the spaces mentioned in this paper are assumed to be Tychonoff. For the definitions of zero-sets, cozero-sets, and ultrafilters, see [8]. We say that an ultrafilter $\mathcal{F}$ is *fixed* if and only if $\bigcap \bar{\mathcal{F}} \neq \emptyset$. The *zero-set community of a set* B (or *z-community* of B) is the collection $\mathcal{Z}_B = \{Z : Z$ is a zero-set of X and $B \subseteq Z\}$. The *z-community of a family* $\mathcal{B}$ is the collection $\mathcal{Z} = \bigcup_{B \in \mathcal{B}} \mathcal{Z}_B$. Since the zero-sets of X are a basis for the closed sets in a Tychonoff space, note that $\bigcap \mathcal{Z}_B = \mathrm{cl}_X B$ when $\mathcal{Z}_B$ is the z-community of B. Consequently $\bigcap \mathcal{Z} = \bigcap \bar{\mathcal{F}}$, whenever $\mathcal{Z}$ is the z-community of $\mathcal{F}$.

A space is said to be *realcompact* if every ultrafilter of zero-sets with the countable intersection property (cip) is fixed. A space is said to be *almost realcompact* if every ultrafilter of open sets with the closed countable intersection property (ccip) is fixed.

A result of Frolík [6, 1] states that a topological space X is almost realcompact if and only if the collection of countable open covers of X is complete. Moreover it can be inferred from Frolík's work (cf. [7]) that X is realcompact if and only if the collection of countable cozero covers of X is complete. The fact that cozero-sets produce a stronger result in this particular context leads one to wonder whether they might also strengthen the original definition of almost realcompact. We shall see that this is not the case.

Definition 1. A space X is said to be *almost* realcompact* if whenever $\mathcal{F}$ is an ultrafilter of cozero-sets such that $\bar{\mathcal{F}}$ has the countable intersection property, then $\bigcap \bar{\mathcal{F}} \neq \emptyset$.

We call a topological property $\mathcal{P}$ a *weak realcompactness condition* if every realcompact space has $\mathcal{P}$ and if every pseudocompact space with $\mathcal{P}$ is compact. We shall see that almost* realcompact is a weak realcompactness condition. We shall also see that the almost* realcompact property neither implies nor is implied by almost realcompactness.

Lemma 2 ([13, 12]). *X is realcompact if and only if every ultrafilter of cozero subsets of X whose z-community has the countable intersection property is fixed.*

Proposition 3. *If X is realcompact, then X is almost* realcompact.*

PROOF: Let $\mathcal{F}$ be a cozero ultrafilter on X with $\bigcap \bar{\mathcal{F}} = \emptyset$. Then since X is realcompact, the z-community $\mathcal{Z}$ of $\mathcal{F}$ does not have the countable intersection property. Now if $\bar{\mathcal{F}}$ had the countable intersection property, then since $\mathrm{cl}_X F \subseteq Z$ for every $F \in \mathcal{F}$ and for every $Z \in \mathcal{Z}_F$, $\mathcal{Z}$ would have the countable intersection property as well. Thus $\bar{\mathcal{F}}$ does not have the countable intersection property and so X is almost* realcompact. □

The converse is false; the following lemma will assist us with the counterexample.

Lemma 4. *Let $\mathcal{U}$ be an open (resp. cozero-set) ultrafilter on X with ccip, and let $U \in \mathcal{U}$. If $\mathrm{cl}_X U$ is almost (resp. almost*) realcompact, then $\bigcap \overline{\mathcal{U}} \neq \emptyset$.*

PROOF: We give the "cozero-set" version. The "open" version is similar.

Let $\mathcal{U}$ be a cozero-set ultrafilter with ccip and let U be a cozero-set of X with $U \in \mathcal{U}$. Assume also that $\mathrm{cl}_X U$ is almost* realcompact. We will show

$$(*) \qquad \text{If } H \in \mathcal{U} \restriction \mathrm{cl}_X U, \text{ then } H \cap U \in \mathcal{U}.$$

Let $H \in \mathcal{U} \restriction \mathrm{cl}_X U$ and note that $H \cap U \neq \emptyset$. Further note that $H \cap U$ is a cozero-set of U, hence of X. Now if $P \in \mathcal{U}$, then $P \cap U \in \mathcal{U}$ and so $P \cap U \in \mathcal{U} \restriction \mathrm{cl}_X U$. Thus $H \cap U \cap P \neq \emptyset$. We conclude that $H \cap U \in \mathcal{U}$.

We show next that $\mathcal{U} \restriction \mathrm{cl}_X U$ is a cozero-set ultrafilter base on $\mathrm{cl}_X U$. Let W be a cozero-set of $\mathrm{cl}_X U$ and assume that $W \cap G \neq \emptyset$ for all $G \in \mathcal{U} \restriction \mathrm{cl}_X U$. In particular, $W \cap U \neq \emptyset$ and is a cozero-set of X. Let $G \in \mathcal{U}$. Then $G \cap U \in \mathcal{U}$ and so $G \cap U \in \mathcal{U} \restriction \mathrm{cl}_X U$ which implies that $W \cap U \cap G \neq \emptyset$. We conclude that $W \cap U \in \mathcal{U}$ and so $W \cap U \in \mathcal{U} \restriction \mathrm{cl}_X U$. Since $W \cap U \subset W$ our claim is proved.

Let $\mathcal{P}$ be a cozero-set ultrafilter on $\mathrm{cl}_X U$ with $\mathcal{U} \restriction \mathrm{cl}_X U \subseteq \mathcal{P}$. We claim that $\mathcal{P}$ has ccip. Let $\{V_n : n \in \omega\} \subset \mathcal{P}$. Each $V_n \supset H_n$ where $H_n \in \mathcal{U} \restriction \mathrm{cl}_X U$. By $(*)$ $\{H_n \cap U : n \in \omega\} \subset \mathcal{U}$. Thus there is $x \in \bigcap_{n\in\omega} \mathrm{cl}_X(H_n \cap U)$. Clearly $x \in \bigcap_{n\in\omega} \mathrm{cl}_{\mathrm{cl}_X U} V_n$.

Since $\mathrm{cl}_X U$ is almost* realcompact, there is $x \in \bigcap \overline{\mathcal{P}}$. We claim that $x \in \bigcap \overline{\mathcal{U}}$. Let $P \in \mathcal{U}$. Then $P \cap U \in \mathcal{U}$ and so $P \cap U \in \mathcal{U} \restriction \mathrm{cl}_X U \subseteq \mathcal{P}$ and so $x \in \mathrm{cl}_X(P \cap U) \subset \mathrm{cl}_X P$. This completes the proof. □

Example 5. *The Mysior plane is almost* realcompact, but not realcompact.*

In [11], Mysior provides an example of an almost realcompact space that is not realcompact. He defines a topology on $\mathbb{R}^2$ by first isolating the points not on the x-axis. For every point $(x, 0)$ on the x-axis, he defines a base of neighborhoods to be the family $\{U_n(x, 0) : n \in \mathbb{N}\}$ where each $U_n(x, 0)$ is the union of three segments: $\{(x, y) : -\frac{1}{n} < y < \frac{1}{n}\}$, $\{(x + 1 + y, y) : 0 < y < \frac{1}{n}\}$ and $\{(x + \sqrt{2} + y, -y) :$

$0 < y < \frac{1}{n}\}$. Mysior demonstrates that the half-planes $X_+ = \{(x,y) : y \geq 0\}$ and $X_- = \{(x,y) : y \leq 0\}$ are both closed in X and realcompact.

To see that the Mysior plane is almost* realcompact, let $\mathcal{P}$ be an ultrafilter of cozero sets of X with the closed countable intersection property. Note that the open half-planes $U = \{(x,y) : y > 0\}$ and $L = \{(x,y) : y < 0\}$ are both cozero-sets in X whose union is dense in X. Therefore either U or L must be a member of $\mathcal{P}$. Without loss of generality, assume $U \in \mathcal{P}$. But $X_+ = \mathrm{cl}_X U$ is realcompact and so $\mathcal{P}$ must be fixed by Lemma 4. Consequently X is almost* realcompact.

In a similar way it follows from Lemma 4 that the more well known Mrówka example (see e.g. [15, 16.4]) — the example traditionally used to show that the union of two realcompact spaces need not be realcompact — is also almost* realcompact.

We now wish to know the circumstances under which an almost* realcompact space is realcompact. We take a cue from Dykes [3]. First recall that a space X is a *cb space* if for every decreasing sequence of closed sets $\{F_n : n \in \mathbf{N}\}$ with $\bigcap_{n\in\mathbf{N}} F_n = \emptyset$, there exists a decreasing sequence of zero-sets $\{Z_n : n \in \mathbf{N}\}$ such that $F_n \subseteq Z_n$ for every n, and $\bigcap_{n\in\mathbf{N}} Z_n = \emptyset$. A space X is *weak cb* if for every decreasing sequence $\{F_n\}$ of non-empty regular closed subsets of X with empty intersection, there is a sequence of zero-sets $\{Z_n\}$ of X with empty intersection such that for each n, $F_n \subseteq Z_n$.

Definition 6. A space X is *almost weak cb* if for every decreasing sequence of cozero-sets $\{P_n : n \in \mathbf{N}\}$ with $\bigcap_{n\in\mathbf{N}} \mathrm{cl}_X P_n = \emptyset$, there exists a decreasing sequence of zero-sets $\{Z_n : n \in \mathbf{N}\}$ such that $P_n \subseteq Z_n$ for every n, and $\bigcap_{n\in\mathbf{N}} Z_n = \emptyset$.

Note 7. Clearly, if X is weak cb, then X is almost weak cb. Furthermore if X is weak Oz, then X is almost weak cb. (Recall that a space is Oz if every regular closed set is a zero set. If the closure of every cozero-set is a zero-set, we say the space is weak Oz.)

Proposition 8. *If X is almost* realcompact and almost weak cb, then X is realcompact.*

Proof: We proceed as did Dykes in [3, 1.2]. Let $\mathcal{F}$ be a free z-ultrafilter on X. Let $\mathcal{B} = \{P : P \text{ is cozero and there exists a } Z \in \mathcal{F} \text{ with } Z \subseteq P\}$. Let $\mathcal{G}$ be a cozero ultrafilter containing $\mathcal{B}$. We show that $\bigcap \bar{\mathcal{G}} = \emptyset$. Indeed, let $p \in X$. Since $\mathcal{F}$ is free, $p \in X - Z$ for some $Z \in \mathcal{F}$. X is completely regular, so there exists a cozero set Q and a zero-set $\hat{Z}$ such that $p \in Q \subseteq \hat{Z} \subseteq X - Z$. Thus $Z \subseteq X - \hat{Z}$ and so $X - \hat{Z} \in \mathcal{G}$. But $p \notin \mathrm{cl}_X(X - \hat{Z})$. Therefore $p \notin \bigcap \bar{\mathcal{G}}$.

Since X is almost* realcompact, $\bar{\mathcal{G}}$ must not have the countable intersection property. Thus there exists a collection $\{P_n : n \in \mathbf{N}\} \subseteq \mathcal{G}$ with $\bigcap_{n\in\mathbf{N}} \mathrm{cl}_X P_n = \emptyset$.

Let $Q_n = \bigcap\{P_i : i \leq n\}$. Then $\{Q_n : n \in \mathbf{N}\}$ is a decreasing sequence of cozero-sets with $\bigcap_{n\in\mathbf{N}} \mathrm{cl}_X Q_n = \emptyset$. Since X is almost weak cb, there exists a collection of zero-sets $\{Z_n : n \in \mathbf{N}\}$ with $\mathrm{cl}_X Q_n \subseteq Z_n$ for each n and $\bigcap_{n\in\mathbf{N}} Z_n = \emptyset$. We need to show that each Z_n meets every member of $\mathcal{F}$. If not, then there exists a set $Z \in \mathcal{F}$ with $Z \cap Z_n = \emptyset$ for some n. Then $Z \subseteq X - Z_n$ and so $X - Z_n \in \mathcal{G}$. But $Q_n \in \mathcal{G}$ (since $\mathcal{G}$ is a filter) and $(X - Z_n) \cap Q_n = \emptyset$, contradicting the fact that $\mathcal{G}$ is a filter. Thus $Z_n \in \mathcal{F}$ for each n, and so $\mathcal{F}$ fails to have the countable intersection property. Therefore X is realcompact. □

Since every pseudocompact space is known to be weak cb [12, 8.5d], Proposition 8 implies that every pseudocompact almost* realcompact space is compact. As we shall soon see, almost* realcompact is a relatively strong generalization of realcompactness. But it is not so strong as to entail almost realcompactness.

Example 9. *An almost* realcompact space need not be almost realcompact.*

Steve Watson notes that a "fringed plank" is almost* but not almost. We take the Tychonoff Plank and add a convergent sequence $\{x_{j,n} : n \in \omega\}$ to each point (ω_1, j) on the right edge. So now those points have their usual neighborhoods plus enough tails of those sequences to make it a topology. All the added points are isolated.

First observe that X is not almost realcompact. Indeed, note that the Tychonoff Plank (which is not almost realcompact) is closed in X. Thus X cannot be almost realcompact, because closed subspaces inherit almost realcompactness.

X is however almost* realcompact. Let $\mathcal{P}$ be an ultrafilter of cozero sets on X with ccip. If we let P be all the added points, then P is a Lindelöf (in fact, σ-compact) cozero-set, and so $\mathrm{cl}_X P$ is realcompact and hence almost* realcompact. So if $P \in \mathcal{P}$, then by Lemma 4, $\mathcal{P}$ is fixed. On the other hand, if $P \notin \mathcal{P}$, then there is $Q \in \mathcal{P}$ with $Q \cap P = \emptyset$. But then Q cannot meet the right edge since if (ω_1, j) is in Q, then so is some tail of the convergent sequence. But then some tail of each point in the right edge misses Q. So Q has to be contained in some countable subset of the plank which again by the lemma makes the ultrafilter converge.

It is not known at this time whether or not almost* realcompact is productive. But the above example shows that almost* realcompactness is not preserved by perfect maps. Let X be the fringed plank and Y the Tychonoff Plank. Let $f : X \to Y$ be the identity map on $Y \subset X$ while all the added points in each sequence go to the point to which the sequence converges. This map is perfect, X is almost* realcompact, while Y is not.

Nor is almost* realcompactness preserved inversely by perfect maps. We now take the range space Y to be the fringed plank while the domain X consists of the disjoint union of the Tychonoff Plank together with ω many copies of the convergent sequence. The obvious map between them is perfect. But X is not almost* realcompact since the Tychonoff Plank is now a support of X. (See Proposition 10.)

The same example also clearly demonstrates that almost* realcompactness is not closed hereditary. There is, however, at least one class of subsets that does inherit the property, the supports of X, that is, the closures of cozero-sets.

Proposition 10. *Let X be almost* realcompact. Then the supports of X are also almost* realcompact.*

PROOF: Let $A = \mathrm{cl}_X P$ where P is cozero in X, and let $\mathcal{U}$ be a cozero-set ultrafilter on A with ccip.

First note that if $\mathcal{V}$ is any cozero ultrafilter on X with $\mathcal{U} \restriction P \subseteq \mathcal{V}$, then $\mathcal{V}$ has ccip. Indeed, pick cozero $Q \in \mathcal{V}$. Then since $U \cap P$ is cozero for every $U \in \mathcal{U}$, $Q \cap U \cap P \neq \emptyset$ for every $U \in \mathcal{U}$. Thus $Q \cap U \cap P \in \mathcal{U} \restriction P$ and since $Q \cap U \cap P \subseteq Q$, $\mathcal{U} \restriction P$ must be a base for $\mathcal{V}$. Now pick some collection $\{V_n : n \in \omega\} \subseteq \mathcal{V}$. Since $\mathcal{U} \restriction P$ is a base, there exist $H_n \subseteq V_n$ for each $n \in \omega$ with $H_n \in \mathcal{U} \restriction P$. But $\mathcal{U}$ has ccip, so there is an $x \in \bigcap_{n\in\omega} \overline{H_n} \subseteq \bigcap_{n\in\omega} \overline{V_n}$ and so $\mathcal{V}$ has ccip as well.

It remains to note then that since X is almost* realcompact, $\bigcap \bar{\mathcal{V}} \neq \emptyset$. Since $\bigcap_{V\in\mathcal{V}} \mathrm{cl}_X V \subseteq \bigcap_{U\in\mathcal{U}} \mathrm{cl}_X U$ it follows that $A = \mathrm{cl}_X P$ must be almost* realcompact. □

Let $C_K(X)$ denote the collection of continuous functions on X with compact support, and let $C_\psi(X)$ denote the collection of continuous functions on X with pseudocompact support. In [10], Mandelker discusses a generalization of realcompactness called ψ-compact: X is ψ-compact if and only if $C_\psi(X) = C_K(X)$. Since pseudocompact spaces are weak cb, and since supports inherit almost* realcompactness, the next proposition is an immediate corollary:

Proposition 11. *Every almost* realcompact space is ψ-compact.*

Recall that a space X if *c-realcompact* iff for every $p \in \beta X - X$ there exists a normal lower semi-continuous (nlsc) function f on βX such that $f(p) = 0$ and f is positive on X. It is known that almost realcompact spaces are c-realcompact [4, 3.3]. A nearly identical proof establishes that the same is true for almost* realcompact spaces.

Proposition 12. *Every almost* realcompact space is c-realcompact.*

PROOF: Pick $p \in \beta X - X$ and let $\mathcal{P}$ be an ultrafilter of cozero-sets on βX such that $\bigcap \bar{\mathcal{P}} = \{p\}$. Then $\mathcal{P} \restriction X$ is a cozero ultrafilter on X with $\bigcap \mathrm{cl}_X P = \emptyset$. By hypothesis there is a decreasing sequence $\{P_i : i \in \mathbf{N}\} \subset \mathcal{P}$ such that $\bigcap_{i\in\mathbf{N}} \mathrm{cl}_X(P_i \cap X) = \emptyset$. Define $f_i(x) = 0$ if $x \in \mathrm{cl}_X P_i$ and $f_i(x) = 1$ otherwise, with $0 \leq f \subset\leq 1$ for all $i \in \mathbf{N}$. Now let $f = \sum_{i\in\mathbf{N}} \frac{f_i}{2^i}$. Then f is nlsc, $f(p) = 0$ and f is positive on X. □

We were able to recover realcompactness from an almost* realcompact space by insisting that the space be almost weak cb. Is there a property we can add to almost* realcompact to achieve almost realcompactness? Yes.

Definition 13. X is said to be *almost weak Oz* if whenever $\mathcal{F}$ is an ultrafilter of open sets of X, then $\mathcal{G} = \{P : P \text{ is cozero and } P \in \mathcal{F}\}$ is an ultrafilter of cozero-sets.

Proposition 14. *If X is almost* realcompact and almost weak Oz, then X is almost realcompact.*

PROOF: Let $\mathcal{F}$ be an ultrafilter of open sets of X with $\bigcap \bar{\mathcal{F}} = \emptyset$, and let $\mathcal{G}$ be the ultrafilter of cozero sets guaranteed by the space being almost weak Oz. Note $\bigcap \bar{\mathcal{G}} = \emptyset$. To see this, pick $p \in X$. Then $p \in X - \mathrm{cl}_X F$ for some $F \in \mathcal{F}$. By complete regularity there is a cozero-set Q and a zero-set Z such that $p \in Q \subseteq Z \subseteq X - \mathrm{cl}_X F$. Thus $F \subseteq X - Z$ and so $X - Z \in \mathcal{G}$. But $p \notin \mathrm{cl}_X(X - Z)$, so $p \notin \bigcap \bar{\mathcal{G}}$.

Now since X is almost* realcompact, $\bar{\mathcal{G}}$ does not have the countable intersection property, so there is a collection $\{P_n : n \in \mathbf{N}\} \subseteq \mathcal{G}$ such that $\bigcap_{n \in \mathbf{N}} \mathrm{cl}_X P_n = \emptyset$. But each $P_n \in \mathcal{F}$, so $\bar{\mathcal{F}}$ does not have the countable intersection property either. Thus X is almost realcompact. □

Note 15. Just as it follows from Proposition 8 that the Mysior plane is an example of a space that is not almost weak cb, it follows immediately from Proposition 14 that the "fringed" plank is not almost weak Oz.

Proposition 16. *If X is weak Oz, then X is almost weak Oz.*

PROOF: Suppose not. Then there is an ultrafilter $\mathcal{F}$ of open sets such that $\mathcal{G} = \{P \in \mathcal{F} : P \text{ is a cozero-set }\}$ is not an ultrafilter of cozero-sets. Note $\mathcal{G}$ is a filter, so $\mathcal{G}$ must be properly contained in some cozero ultrafilter, say $\mathcal{H}$. Thus there exists a cozero-set $P \in \mathcal{H}$ with $P \cap Q \neq \emptyset$ for every $Q \in \mathcal{G}$, but for which $P \cap U = \emptyset$ for some $U \in \mathcal{F}$. But then $\mathrm{cl}_X P \cap U = \emptyset$ and since X is weak Oz, $\mathrm{cl}_X P$ is a zero-set. Thus $R = X - \mathrm{cl}_X P$ is a cozero-set containing U. This means that $R \in \mathcal{F}$ and, since R is cozero, $R \in \mathcal{G}$ and consequently $\mathcal{H}$. But then P and R are members of $\mathcal{H}$ which fail to intersect, a contradiction since $\mathcal{H}$ is a filter. □

It is natural at this point to wonder whether almost weak Oz is a genuine weakening of weak Oz. To see that it is, we will first need two lemmas.

Lemma 17. *Let X be perfectly normal and locally compact. Then βX is almost weak Oz.*

PROOF: We begin as in the proof of Proposition 16. Suppose not. Then there is an ultrafilter $\mathcal{U}$ of open sets of βX such that $\mathcal{G} = \{P \in \mathcal{U} : P \text{ is a cozero-set of } \beta X\}$ is not an ultrafilter of cozero-sets. Note $\mathcal{G}$ is a filter, so $\mathcal{G}$ must be properly contained in some cozero ultrafilter, say $\mathcal{H}$. Thus there exists a cozero-set $P \in \mathcal{H}$ with $P \cap Q \neq \emptyset$ for every $Q \in \mathcal{G}$, but for which $P \cap U = \emptyset$ for some $U \in \mathcal{U}$. Since $P \notin \mathcal{U}$, and $\mathcal{U}$ is an open ultrafilter, $\beta X - \mathrm{cl}_{\beta X} P \in \mathcal{U}$.

Now X is locally compact, so X is open in βX [12, 4.3e]. Therefore $\mathcal{U} \cap X$ is an open ultrafilter on X, and, since every open set is a cozero-set in a perfectly normal space [5, 1.5.19], a cozero ultrafilter as well. Note that $X - \mathrm{cl}_{\beta X} P$ is a cozero subset of X and that $X - \mathrm{cl}_{\beta X} P \in \mathcal{U} \cap X$. Since X is z-embedded in βX, there exists a cozero-set R of βX which extends $X - \mathrm{cl}_{\beta X} P$. But now R meets every element of $\mathcal{U}$, and so must belong to $\mathcal{U}$, and consequently $\mathcal{G}$ and $\mathcal{H}$ as well. But $R \subseteq \beta X - \mathrm{cl}_{\beta X} P$, and therefore R is disjoint from P, contradicting the fact that $\mathcal{H}$ is a filter. $\square$

Lemma 18. *Let X be σ-compact, locally compact, and perfectly normal. Then βX is Oz if and only if βX is weak Oz.*

Proof: Necessity is clear. For sufficiency, let P be an open subset of βX. Now X is σ-compact, and therefore an F_σ-set of βX. Moreover, since X is locally compact, it is open in βX. Its complement, then, is a closed G_δ-set of βX and consequently a zero-set of βX [8, 3.11(b)]. X is therefore a cozero-set of βX. Now X is perfectly normal, so $P \cap X$ is open in X and hence a cozero-set of X. But X is a cozero-set of βX, so $P \cap X$ must be a cozero-set of βX as well. By hypothesis then, $\mathrm{cl}_{\beta X}(P \cap X)$ is a zero-set of βX. Our conclusion now follows immediately from the fact that $\mathrm{cl}_{\beta X}(P \cap X) = \mathrm{cl}_{\beta X} P$. $\square$

Note 19. An almost weak Oz space need not be weak Oz. $\mathbb{R}$ is perfectly normal and locally compact, and so by Lemma 17, $\beta\mathbb{R}$ must be almost weak Oz. On the other hand, $\beta\mathbb{R}$ is not Oz [1,3], and so fails to be weak Oz by Lemma 18.

We have already ruled out the possibility that an almost* realcompact space need be almost realcompact. The converse too is false.

Example 20. *The Dieudonné Plank D* [12, 6V] *is almost realcompact but not almost* realcompact.*

Recall that $D = (\omega_1+1) \times (\omega+1) - \{(\omega_1+1, \omega+1)\}$, with all the points not on the top or right edges isolated. The points on the right edge have neighborhoods containing tails. The points on the top edge have basic neighborhoods that also contain tails (not "rectangles"). Let $P_{n,\alpha}$ be a northeast corner of the plank bounded on the south by n and on the west by α. Let $\mathcal{P}$ be the cozero ultrafilter containing all the $P_{n,\alpha}$. Then $\mathcal{P}$ has empty intersection, but also ccip: any countable collection of cozero-sets that meets the $P_{n,\alpha}$'s will have to have a point on the top edge in the intersection of the closures.

There are however at least two properties which we may add to almost realcompact to capture almost* realcompact. The first is a strengthening of countably paracompact. Recall that we may characterize a space X as being *countably paracompact* [5, 5.2.1] iff for every decreasing sequence of closed sets $\{F_n : n \in \mathbf{N}\}$ with $\bigcap_{n\in\mathbf{N}} F_n = \emptyset$, there exists a decreasing sequence of open sets $\{G_n : n \in \mathbf{N}\}$ with $F_n \subseteq G_n$ for each n and $\bigcap_{n\in\mathbf{N}} \mathrm{cl}_X G_n = \emptyset$. Furthermore if X is normal we can say that a space is countably paracompact iff for every decreasing sequence of closed sets $\{F_n : n \in \mathbf{N}\}$ with $\bigcap_{n\in\mathbf{N}} F_n = \emptyset$, there exists a decreasing sequence of open sets $\{G_n : n \in \mathbf{N}\}$ with $F_n \subseteq G_n$ for each n and $\bigcap_{n\in\mathbf{N}} G_n = \emptyset$.

Definition 21. X is said to be *super countably paracompact* if for every decreasing sequence of closed sets $\{F_n : n \in \mathbf{N}\}$ with $\bigcap_{n\in\mathbf{N}} F_n = \emptyset$, there exists a decreasing sequence of cozero-sets $\{P_n : n \in \mathbf{N}\}$ with $F_n \subseteq P_n$ for each n and $\bigcap_{n\in\mathbf{N}} \mathrm{cl}_X P_n = \emptyset$.

Note 22. If X is super countably paracompact, then X is countably paracompact. If X is normal and countably paracompact, then X is super countably paracompact. If X is almost weak cb and super countably paracompact, then X is a cb space.

Proposition 23. *If X is almost realcompact and super countably paracompact, then X is almost* realcompact.*

Proof: We proceed once more with the usual technique. Let $\mathcal{F}$ be an ultrafilter of cozero-sets with $\bigcap \bar{\mathcal{F}} = \emptyset$. Let

$$\mathcal{B} = \{U : U \text{ is open and there exists an } F \in \mathcal{F} \text{ with } F \subseteq U\}.$$

Let $\mathcal{G}$ be an open ultrafilter containing $\mathcal{B}$. We show that $\bigcap \bar{\mathcal{G}} = \emptyset$. Indeed, let $p \in X$. Then $p \in X - \mathrm{cl}_X F$ for some $F \in \mathcal{F}$. X is regular, so there exists an open set U such that $p \in U \subseteq \mathrm{cl}_X U \subseteq X - \mathrm{cl}_X F$. Thus $F \subseteq X - \mathrm{cl}_X U$ and so $X - \mathrm{cl}_X U \in \mathcal{G}$. But $p \notin \mathrm{cl}_X(X - \mathrm{cl}_X U)$, so $p \notin \bigcap \bar{\mathcal{G}}$.

Since X is almost realcompact, $\bar{\mathcal{G}}$ does not have the countable intersection property. That is, there exists a collection $\{U_n : n \in \mathbf{N}\} \subseteq \mathcal{G}$ with $\bigcap_{n\in\mathbf{N}} \mathrm{cl}_X U_n = \emptyset$. Let $G_n = \bigcap\{U_i : i \leq n\}$. Then $\{G_n : n \in \mathbf{N}\} \subseteq \mathcal{G}$ is a decreasing sequence of open sets with $\bigcap_{n\in\mathbf{N}} \mathrm{cl}_X G_n = \emptyset$. Since X has Property Q, there exists a collection of cozero-sets $\{P_n : n \in \mathbf{N}\}$ with $\mathrm{cl}_X G_n \subseteq P_n$ for each n and $\bigcap_{n\in\mathbf{N}} \mathrm{cl}_X P_n = \emptyset$. Now each P_n meets every member of $\mathcal{F}$. If not, suppose there exists a set $F \in \mathcal{F}$ with $F \cap P_n = \emptyset$ for some n. Then $F \subseteq X - \mathrm{cl}_X P_n$ and so $X - \mathrm{cl}_X P_n \in \mathcal{G}$. But then $G_n \cap (X - \mathrm{cl}_X P_n) = \emptyset$, contradicting the fact that $\mathcal{G}$ is a filter. Thus $P_n \in \mathcal{F}$ for each n, and so $\bar{\mathcal{F}}$ does not have the countable intersection property. Therefore X is almost* realcompact. □

Another way to retrieve almost* realcompact from almost realcompact establishes equivalence between the two.

Definition 24. We say that X is *rc=s* if every regular closed set of X is a support of X.

Note 25. If we note that regular open sets are cozero-sets in an Oz space, then it is easy to see that a space is Oz if and only if it is weak Oz and rc=s. Likewise, given that a clopen set is cozero and consequently a support, a space is extremally disconnected if and only if it is basically disconnected and rc=s. (Recall that a space is said to be *extremally disconnected* if every regular closed set is open and *basically disconnected* if every support is open.)

Clearly, if every regular closed set is a support, then an almost realcompact space will be almost* realcompact. That an almost* realcompact rc=s space is almost realcompact is a consequence of the following:

Proposition 26. *If X is rc=s, then X is almost weak Oz.*

Proof: Let $\mathcal{U}$ be an ultrafilter of open sets and P be a cozero-set that meets every cozero-set in $\mathcal{U}$. Let $V \in \mathcal{U}$. Now $\mathrm{cl}_X V = \mathrm{cl}_X Q$ where Q is a cozero-set. Since $V \in \mathcal{U}$ and $V \subseteq \mathrm{int}_X \mathrm{cl}_X V$, $\mathrm{int}_X \mathrm{cl}_X V \in \mathcal{U}$. Now Q is dense in $\mathrm{int}_X \mathrm{cl}_X V$ since they have the same closure, and so Q meets every member of $\mathcal{U}$. Thus P meets Q and consequently V, which implies that $P \in \mathcal{U}$. The subcollection of cozero-sets of $\mathcal{U}$ thus forms an ultrafilter. □

Note 27. The converse is not true. Since every basically disconnected space is weak Oz, it suffices to produce a basically disconnected space that is not extremally disconnected (see Note 25). [8, 4N] is such a space.

Note 28. The properties rc=s and super countably paracompact are independent. $\beta\mathbb{N} - \mathbb{N}$ is compact and hence super countably paracompact, but it follows from [9, 5.3(e) and 5.6] that $\beta\mathbb{N} - \mathbb{N}$ is not rc=s. On the other hand, the existence of an extremally disconnected Dowker space [2], demonstrates that rc=s does not even imply countably paracompact.

In conclusion, we would like to discuss the relationship of almost* realcompactness to the property which van der Slot has called $\aleph_1$-ultracompactness. A space X is said to be *m-ultracompact relative to a closed subbase* $\mathfrak{S}$ iff each ultrafilter $\mathcal{F}$ in X, for which $\mathcal{F} \cap \mathfrak{S}$ satisfies the m-intersection property, is convergent. A space X is then said to be *m-ultracompact* iff there is a closed subbase $\mathfrak{S}$ for X such that X is m-ultracompact relative to $\mathfrak{S}$. Frolík has shown that for regular spaces, $\aleph_1$-ultracompactness is equivalent to almost realcompactness [7, 2.7]. It follows immediately for the purposes of this paper that $\aleph_1$-ultracompactness is a property independent of almost* realcompactness.

This gives us the occasion to point to the existence of a remark in van der Slot's tract [14] which seems to be in error. On page 47, he prefaces a theorem by saying, "The following theorem generalizes the result stating that a completely regular space is realcompact iff for each maximal centered family $\mathcal{D}$ of cozerosets for which $\bar{\mathcal{D}}$ has cip, the intersection $\bigcap \bar{\mathcal{D}}$ is non empty." Van der Slot does not provide a citation for this claim, so its precise origins are unclear. In any case, our work indicates that the assertion is false.

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References

[1] Chigogidze A., *On a generalization of perfectly normal spaces*, Topology Appl. **13** (1982), 15–20.

[2] Dow A., van Mill J., *An extremally disconnected Dowker space*, Proc. Amer. Math. Soc. **86** (1982), 669–672.

[3] Dykes N., *Mappings and realcompact spaces*, Pacific J. Math. **31** (1969), 347–358.

[4] Dykes N., *Generalizations of realcompact spaces*, Pacific J. Math. **33** (1970), 571–581.

[5] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[6] Frolík Z., *A generalization of realcompact spaces*, Czech. Math. J. **13** (1963), 127–138.

[7] Frolík Z., *Prime filters with CIP*, Comment. Math. Univ. Carolinae **13.3** (1972), 553–575.

[8] Gillman L., Jerison M., *Rings of Continuous Functions*, University Series in Higher Math., Van Nostrand, Princeton New Jersey, 1960.

[9] Henriksen M., Jerison M., *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. **115** (3) (1965), 110–130.

[10] Mandelker M., *Supports of continuous functions*, Trans. Amer. Math. Soc. **156** (1971), 73–83.

[11] Mysior A., *A union of realcompact spaces*, Bull. Acad. Polon. Sci. Ser. Sci. Math. **XXIX** (3-4) (1981), 169–172.

[12] Porter J., Woods R.G., *Extensions and Absolutes of Hausdorff Spaces*, Springer-Verlag, New York, Heidelberg, Berlin, 1988.

[13] Schommer J., *A characterization of realcompactness*, Acta Mathematica Hungarica **72** (4) (1996), 319–322.

[14] van der Slot J., *Some Properties Related to Compactness*, Mathematical Centre Tracts 19, Mathematisch Centrum, Amsterdam, 1976.

[15] Weir M., *Hewitt-Nachbin Spaces*, North Holland Math. Studies, American Elsevier, New York, 1975.

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