Comment.Math.Univ.Carolin. 42,2 (2001)331–343

The Laplace derivative

R.E. Svetic

Abstract. A function $f : \mathbb{R} \to \mathbb{R}$ is said to have the n-th Laplace derivative on the right at x if f is continuous in a right neighborhood of x and there exist real numbers $\alpha_0, \dots, \alpha_{n-1}$ such that $s^{n+1} \int_0^\delta e^{-st}[f(x+t) - \sum_{i=0}^{n-1} \alpha_i t^i/i!]\, dt$ converges as $s \to +\infty$ for some $\delta > 0$. There is a corresponding definition on the left. The function is said to have the n-th Laplace derivative at x when these two are equal, the common value is denoted by $f_{\langle n \rangle}(x)$.

In this work we establish the basic properties of this new derivative and show that, by an example, it is more general than the generalized Peano derivative; hence the Laplace derivative generalizes the Peano and ordinary derivatives.

Keywords: Peano derivative, generalized Peano derivative, Laplace derivative, Laplace transform, Tauberian theorem

Classification: Primary 26A24; Secondary 26A21, 26A48, 40E05, 44A10

1. Introduction

The Peano derivative and its generalizations have received considerable attention, see for example [1], and [3]–[16].

Lee's generalized Peano derivative [8] is of interest in this work. A continuous function f has the generalized Peano derivative at a point x, denoted by $f_{[1]}(x)$, if some k-th primitive of f has the $(1+k)$-th Peano derivative at x. In [14] it was shown that this implies that

$$\lim_{s \to +\infty} s^{k+2} \int_0^\delta e^{-st}[f^{(-k)}(x+t) - f(x)t^k/k!]\, dt = f_{[1]}(x)$$

for every $\delta > 0$, where $f^{(-k)}$ is the k-th primitive of f to be defined in Section 2. Integrating by parts k times shows that

$$\lim_{s \to +\infty} s^2 \int_0^\delta e^{-st}[f(x+t) - f(x)]\, dt = f_{[1]}(x). \tag{1}$$

The statement "$f_{[1]}(x)$ exists implies (1)" is an Abelian theorem in Laplace transform theory. In this work we show that (as is usually the case) the converse

The author gratefully acknowledges support by GAANN grant number PA200A70522.

does not hold. Namely, that $s^2 \int_0^\delta e^{-st}[f(x+t)-f(x)]\,dt$ may converge as $s \to +\infty$ for some $\delta > 0$ even though $f_{[1]}(x)$ does not exist.

This suggests that the limit behavior of the Laplace transform could be used to define a generalized derivative. We use this idea to define the Laplace derivative of order $n \in \mathbb{Z}^+$ and show that it generalizes the generalized Peano derivative.

2. Preliminaries

We denote the real numbers and integers by $\mathbb{R}$ and $\mathbb{Z}$ respectively. Then $\mathbb{R}^+$, $\mathbb{Z}^+$ and $\mathbb{R}_+$, $\mathbb{Z}_+$ denote the positive and non-negative elements respectively. When we say exists, we mean exists finite. Unless otherwise specified, function means real valued function of a real variable, and given two functions f and g, we write $f \sim g$ as $x \to x_0^+$ if $\lim_{x \to x_0^+} f(x)/g(x) = 1$.

We use $f^{(-k)}$ to denote the particular k-th primitive of a function f given by

$$f^{(-k)}(x) = \int_\xi^x f^{(-k+1)}(t)\,dt \quad \text{for} \quad k \in \mathbb{Z}^+ \quad \text{and} \quad x \in \mathbb{R},$$

where $f^{(0)} = f$ and $\xi \in \mathbb{R}$ is fixed. There is no loss in generality since, when a result mentioned in this work depends on a primitive, it will be true that the result is independent of which primitive is taken (see [8]).

It will happen that $\sin(x)$ or $\cos(x)$ appears in an expression according to whether a certain integer is even or odd. It will simplify the exposition if we adopt the convention that the notation $\mathcal{S}(x)$ will denote the appropriate trigonometric function.

3. The Laplace derivative

We say that a function f has the *n-th Peano derivative* (*PD*) *at* x, $n \in \mathbb{Z}^+$, if there exist real numbers $f_{(1)}(x), \ldots, f_{(n-1)}(x)$ such that

$$\frac{f(x+h) - f(x) - f_{(1)}(x)h - \cdots - f_{(n-1)}(x)h^{n-1}/(n-1)!}{h^n/n!}$$

converges as $h \to 0$. In this case the limit is denoted by $f_{(n)}(x)$; for convenience we define $f_{(0)}(x) := f(x)$ (see [13]).

We say that f has the n-th *generalized Peano derivative* (*GPD*) *at* x, $n \in \mathbb{Z}^+$ if there exists a nonnegative integer k such that a k-th primitive of f has the $(n+k)$-th Peano derivative at x. We denote the result by $f_{[n]}(x)$ and for convenience we define $f_{[0]}(x) := f(x)$ (see [8], [10]). From our definition of the primitive, it follows that this is equivalent to saying that f has the n-th GPD at x if there exists a nonnegative integer k and real numbers $f_{[1]}(x), \dots, f_{[n-1]}(x)$ such that

$$(2)\qquad \frac{\begin{array}{r} f^{(-k)}(x+h) - f(x)h^k/k! - f_{[1]}(x)h^{1+k}/(1+k)! - \cdots \\ \cdots - f_{[n-1]}(x)h^{n-1+k}/(n-1+k)! \end{array}}{h^{n+k}/(n+k)!}$$

converges as $h \to 0$.

The $\liminf$ ($\limsup$) as $h \to 0$ of the quotient in (2) will be denoted by $l_k f_{(n)}(x)$ ($u_k f_{(n)}(x)$) or even l_k (u_k) when there is no possibility of confusion. Notice that l_0 and u_0 are just the n-th Peano derivatives of f and the standard proof of l'Hospital's rule gives the monotonicity property ([10]):

$$(3)\qquad l_0 \le l_1 \le l_2 \le \cdots \le u_2 \le u_1 \le u_0.$$

It follows that we can write $\lim_{k\to\infty} l_k = l$ and $\lim_{k\to\infty} u_k = u$. Then Theorem 2 in [14] implies that f has the n-th GPD at x if and only if $l = u \in \mathbb{R}$; in this case $f_{[n]}(x) = l$. In the proof of Theorem 2 we see that $l = u \in \mathbb{R}$ implies that

$$(4)\qquad s^{n+1}\int_0^\delta e^{-st}[f(x+t) - \sum_{i=0}^{n-1} f_{[i]}(x)t^i/i!]\,dt \quad \begin{array}{l}\text{converges as } s \to +\infty \\ \text{for some } \delta \in \mathbb{R}^+\end{array}$$

and that the limit is l. The converse, namely that (4) implies that $l = u \in \mathbb{R}$, was not considered in [14]. The main result of this work is that the converse is not true and is most easily stated after we make the following definitions.

Definition 3.1. *A function $f : \mathbb{R} \to \mathbb{R}$ has the n-th Laplace derivative on the right [left] at x if f is continuous in a right [left] neighborhood of x and there exist numbers $\alpha_0, \dots, \alpha_{n-1}$ such that*
$s^{n+1}\int_0^\delta e^{-st}[f(x+t) - \sum_{i=0}^{n-1}\alpha_i t^i/i!]\,dt[(-s)^n s\int_{-\delta}^0 e^{st}[f(x+t) - \sum_{i=0}^{n-1}\alpha_i t^i/i!]\,dt]$
converges as $s \to +\infty$ for some $\delta > 0$. In this case the limit is denoted by $f_{\langle n,+\rangle}(x)$ [$f_{\langle n,-\rangle}(x)$].

Definition 3.2. *A continuous function $f : \mathbb{R} \to \mathbb{R}$ has the n-th Laplace derivative at x if $f_{\langle n,+\rangle}(x) = f_{\langle n,-\rangle}(x)$. In this case the common value is denoted by $f_{\langle n\rangle}(x)$.*

We may now state the main result.

Theorem 3.3. *There exists a continuous function which has the Laplace derivative at a point and does not have the generalized Peano derivative there. Furthermore, the Laplace derivative of this function is not the generalized Peano derivative of any continuous function.*

The function guaranteed by this theorem is such that (4) holds (with limit equal to 0) while $l = -\infty$ and $u = +\infty$. This is the only possibility; namely, if (4) holds and any l_k (or u_k) is finite, then the Tauberian theorem [2] implies that $l_{k+1} = u_{k+1}$ and that the common value equals the limit in (4).

Finally, we will show that, if the Laplace derivative exists at a point $x \in \mathbb{R}$, then it is well defined and that the associated numbers $\alpha_0, \dots, \alpha_{n-1}$ are uniquely determined with $\alpha_0 = f_{\langle 0 \rangle}(x) := f(x)$ and $\alpha_i = f_{\langle i \rangle}(x)$ for $i = 1, \dots, n-1$.

4. Proof of the main result

For each integer $m \geq 2$ define the function $\phi_m : \mathbb{R}_+ \to \mathbb{R}$ by $\phi_m(0) = 0$ and $\phi_m(x) = x^{[1/m+m+1]} \sin(x^{-1/m})$ otherwise. Since each ϕ_m is m-times continuously differentiable on $\mathbb{R}_+$, it makes sense to define constants $c_{m,j}$ for integer j, $0 \leq j \leq m$ by

$$\phi_m^{(j)}(x) \sim c_{m,j}\, x^{[1/m+(m-j)(1+1/m)]} \mathcal{S}(x^{-1/m}) \quad \text{as} \quad x \to 0^+.$$

Now for each integer $m \geq 2$ define $f_m : \mathbb{R} \to \mathbb{R}$ by

$$f_m(x) = \begin{cases} \phi_m^{(m)}(|x|)/\beta_m & \text{if } 0 \leq |x| \leq \alpha_m, \\ 0 & \text{if } \alpha_m < |x|, \end{cases}$$

where α_m is the largest zero of $\phi_m^{(m)}$ less than $1/m$ and β_m is a positive constant to be specified shortly. The functions defined in this way are continuous on $\mathbb{R}$ since $\phi_m^{(m)}(\alpha_m) = 0$ and have support in $[-1/m, 1/m]$. In order to simplify the exposition, we use this definition of f_m since it is adequate to prove the first part of the theorem. However, to prove the second part, we will need to modify the definition slightly; we do this later.

Notice that $f_m^{(-k)}(x) = (\phi_m^{(m)})^{(-k)}(x)/\beta_m = \phi_m^{(m-k)}(x)/\beta_m$, with $\xi = 0$, for $0 \leq k \leq m$ and $0 \leq x \leq \alpha_m$. Thus,

$$f_m^{(-k)}(x) \sim \frac{c_{m,m-k}}{\beta_m} x^{[1/m+k(1+1/m)]}\, \mathcal{S}(x^{-1/m}) \quad \text{as} \quad x \to 0^+. \tag{5}$$

Since $f_m(0) = 0$, the quotient in (2) takes the form $(1+k)! f_m^{(-k)}(x)/x^{1+k}$ for the $n = 1$ case. It is easy to check that the smallest integer k such that $l_k = u_k$ is m. Thus, we have that $(f_m)_{(1+m)}^{(-m)}(0) = (f_m)_{[1]}(0) = 0$. Using (1), we find that

$$\lim_{s \to +\infty} s^2 \int_0^\infty e^{-st} f_m(t)\, dt = 0. \tag{6}$$

In addition, since $s^2 \int_0^\infty e^{-st} f_m(t)\, dt$ is a continuous function of s that converges to zero as s tends to zero from the right, it is bounded on $\mathbb{R}_+$. Hence, we may choose each β_m such that $|f_m(x)| \le 1$ on $\mathbb{R}$ and $|s^2 \int_0^\infty e^{-st} f_m(t)\,dt| \le 1$ for $s \in \mathbb{R}_+$. We define our example function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{m=2}^{\infty} a_m f_m(x)$$

for all x where $\{a_m\}_{m=2}^{\infty}$ is a sequence of positive numbers which we initially require to satisfy the condition that

$$\text{(7)} \qquad a_2 \le 1 \ \text{ and } \ a_{m+1} < a_m/5 \ \text{ for each integer } \ m \ge 2.$$

Then, since $|f_m| \le 1$ for all m, the previous sum converges uniformly on $\mathbb{R}$ to a continuous function with support in $[-1,1]$ and $|f(x)| \le 5/4$ on $\mathbb{R}$.

Now we show that $f_{\langle 1 \rangle}(0) = 0$ and, since f is symmetric, it will suffice to show that $f_{\langle 1,+ \rangle}(0) = 0$. Let $s, \delta \in \mathbb{R}^+$ be arbitrary, then

$$s^2 \int_0^{\delta} e^{-st} f(t)\,dt = s^2 \int_0^{\delta} e^{-st} \sum_{m=2}^{\infty} a_m f_m(t)\,dt = \sum_{m=2}^{\infty} a_m\, s^2 \int_0^{\delta} e^{-st} f_m(t)\,dt$$

by the bounded convergence theorem. Furthermore, since $|s^2 \int_0^\delta e^{-st} f_m(t)\,dt| \le 1$ for all $s \in \mathbb{R}_+$, the convergence is uniform with respect to s. This and (6) imply that

$$f_{\langle 1,+ \rangle}(0) = \lim_{s \to +\infty} s^2 \int_0^{\delta} e^{-st} f(t)\,dt = \sum_{m=2}^{\infty} a_m \lim_{s \to +\infty} s^2 \int_0^{\delta} e^{-st} f_m(t)\,dt = 0.$$

Preparing to show that $f_{[1]}(0)$ does not exist, we introduce two additional sequences $\{x_m\}_{m=2}^{\infty}$ and $\{y_m\}_{m=2}^{\infty}$ satisfying

$$\text{(8)} \qquad 0 < \cdots < x_{m+1} < y_m < x_m < \cdots < y_2 < x_2 \le 1$$

and simultaneously complete the definition of the sequence $\{a_m\}_{m=2}^{\infty}$.

Set $x_1 = y_1 = 1 = a_2$. Then, for an integer $m \ge 2$, assume that $x_1, x_2, \ldots, x_{m-1}$ and $y_1, y_2, \ldots, y_{m-1}$ and $a_2, a_3, \ldots, a_m$ have been chosen. We will choose x_m, y_m and a_{m+1}.

First we choose $x_m \in (0, y_{m-1})$ such that for each $x \in (0, x_m]$ we have

$$\text{(9)} \qquad |a_{k+2} f_{k+2}^{(-k)}(x) + \cdots + a_{m-1} f_{m-1}^{(-k)}(x)| < \frac{1}{4} \frac{a_m\, |c_{m,m-k}|}{\beta_m} x^{[1/m + k(1+1/m)]}$$

for each integer k, $0 \le k \le m-3$ (an empty condition when $m = 2$, in which case we choose $x_2 = 1/2 < y_1 = 1$). This is possible for $m \ge 3$ as follows. Let m' and k be integers such that $2 \le m' < m$ and $0 \le k \le m'$. Then we have

$$f_{m'}^{(-k)}(x) \sim \frac{c_{m',m'-k}}{\beta_{m'}} x^{[1/m' + k(1+1/m')]}\, \mathcal{S}(x^{-1/m'}) \ \text{ as } \ x \to 0^+.$$

Since $m' < m$ implies that $1/m + k(1+1/m) < 1/m' + k(1+1/m')$, the left hand side of (9) can be made smaller than any fixed constant multiple of $x^{[1/m+k(1+1/m)]}$ for all sufficiently small x.

Now we choose $y_m \in (0, x_m)$ such that for each integer $k = 0, \ldots, m-2$ there exist points z^+ and z^- (depending on k) in (y_m, x_m) such that

$$a_m f_m^{(-k)}(z^+) > +\frac{3}{4}\frac{a_m\,|c_{m,m-k}|}{\beta_m}(z^+)^{[1/m+k(1+1/m)]} > +3m\frac{(z^+)^{1+k}}{(1+k)!}$$

and

(10)
$$a_m f_m^{(-k)}(z^-) < -\frac{3}{4}\frac{a_m\,|c_{m,m-k}|}{\beta_m}(z^-)^{[1/m+k(1+1/m)]} < -3m\frac{(z^-)^{1+k}}{(1+k)!}\,.$$

Property (5) implies the left pair of inequalities, while the right pair of inequalities follow from $x^{[1/m+k(1+1/m)]-(1+k)} \to +\infty$ as $x \to 0^+$ since $k+2 \leq m$ implies that $1/m + k(1+1/m) < 1+k$.

Now we choose a_{m+1} such that

$$0 < a_{m+1} < \frac{a_m}{5}\min\left\{1, \frac{|c_{m,0}|}{\beta_m}, \ldots, \frac{|c_{m,m}|}{\beta_m}\right\} y_m^{[1/m+m+1]}.$$

Then, for $x \in (y_m, x_m)$, we have

$$|a_{m+1}f_{m+1}^{(-k)}(x) + a_{m+2}f_{m+2}^{(-k)}(x) + \cdots| \leq \sum_{m'=m+1}^{\infty} a_{m'}|f_{m'}^{(-k)}(x)| \leq \sum_{m'=m+1}^{\infty} a_{m'}$$

since $|f_m| \leq 1$ implies that $|f_m^{(-k)}| \leq 1$ on $[0,1]$ for all $k \in \mathbb{Z}_+$. Hence

(11)
$$\begin{aligned} |a_{m+1}f_{m+1}^{(-k)}(x) + \cdots| &\leq a_{m+1} + (\frac{1}{5})a_{m+1} + (\frac{1}{5})^2 a_{m+1} + \cdots = \frac{5}{4}a_{m+1} \\ &< \frac{5}{4}\frac{a_m}{5}\min\{1, \frac{|c_{m,0}|}{\beta_m}, \ldots, \frac{|c_{m,m}|}{\beta_m}\} y_m^{[1/m+m+1]} \\ &< \frac{1}{4}\frac{a_m}{\beta_m}|c_{m,m-k}|x^{[1/m+k(1+1/m)]} \end{aligned}$$

since $x \in (y_m, x_m)$ and $k \leq m-2$. Thus, by the principle of recursive definition, we have sequences $\{a_m\}_{m=2}^{\infty}$, $\{x_m\}_{m=2}^{\infty}$ and $\{y_m\}_{m=2}^{\infty}$ which satisfy conditions (7), (8), (9), (10), and (11) for all integers $m \geq 2$.

This completes our preparations and, as before, it will suffice to show that $f_{[1,+]}(0)$ does not exist. Since $f(0) = 0$, the quotient in (2) takes the form

$(1+k)!f^{(-k)}(x)/x^{1+k}$, $k \in \mathbb{Z}_+$, and as previously mentioned we must show that $l_k = -\infty$ and $u_k = +\infty$ for each $k \in \mathbb{Z}_+$. Since the arguments are similar, we only show the latter; namely that

$$u_k = \limsup_{x\to 0^+} \frac{f^{(-k)}(x)}{x^{1+k}/(1+k)!} = +\infty \quad \text{for each} \quad k \in \mathbb{Z}_+.$$

Taking advantage of the monotonicity property, (3), it suffices to consider an arbitrary integer $k \geq 2$. Since $x \in [0,1]$, the bounded convergence theorem implies that we may integrate term by term to obtain

$$\begin{aligned} u_k &= \limsup_{x\to 0^+} \sum_{m=2}^{\infty} a_m \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!} \\ &\geq \sum_{m=2}^{k} a_m \liminf_{x\to 0^+} \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!} + \liminf_{x\to 0^+} a_{1+k} \frac{f_{1+k}^{(-k)}(x)}{x^{1+k}/(1+k)!} \\ &\qquad + \limsup_{x\to 0^+} \sum_{m=k+2}^{\infty} a_m \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!}\,. \end{aligned}$$

Using (5), we have $\lim_{x\to 0^+}(1+k)!f_m^{(-k)}(x)/x^{1+k} = 0$, for $m = 2,\dots,k$, and $\liminf_{x\to 0^+}(1+k)!f_{1+k}^{(-k)}(x)/x^{1+k} = -(1+k)!\,|c_{1+k,1}|/\beta_{1+k}$. Hence

$$\begin{aligned} u_k &\geq \limsup_{x\to 0^+} \sum_{m=k+2}^{\infty} a_m \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!} - a_{1+k}\frac{(1+k)!\,|c_{1+k,1}|}{\beta_{1+k}} \\ &\geq \limsup_{k+2<l\to\infty} \max_{x\in(y_l,x_l)} \left\{ \sum_{m=k+2}^{\infty} a_m \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!} \right\} - a_{1+k}\frac{(1+k)!\,|c_{1+k,1}|}{\beta_{1+k}}\,. \end{aligned}$$

To estimate this sum, let l be an integer such that $l > k+2$. Then

$$|a_{k+2}f_{k+2}^{(-k)}(x) + \cdots + a_{l-1}f_{l-1}^{(-k)}(x)| < \frac{1}{4}\frac{a_l|c_{l,l-k}|}{\beta_l}x^{[1/l+k(1+1/l)]}$$

for every $x \in (y_l, x_l)$ follows from (9). Furthermore,

$$|a_{l+1}f_{l+1}^{(-k)}(x) + \cdots| < \frac{1}{4}\frac{a_l|c_{l,l-k}|}{\beta_l}x^{[1/l+k(1+1/l)]}$$

for every $x \in (y_l, x_l)$ follows from (11).

According to (10) there exists a point $z^+ \in (y_l, x_l)$ such that

$$a_l f_l^{(-k)}(z^+) > +\frac{3}{4}\frac{a_l|c_{l,l-k}|}{\beta_l}(z^+)^{[1/l+k(1+1/l)]} > +3l\frac{(z^+)^{1+k}}{(1+k)!}\,.$$

Thus we have

$$\begin{aligned}\sum_{m=k+2}^{\infty} a_m f_m^{(-k)}(z^+) &\geq -\left|\sum_{m=k+2}^{l-1} a_m f_m^{(-k)}(z^+)\right| + a_l f_l^{(-k)}(z^+) \\ &\qquad - \left|\sum_{m=l+1}^{\infty} a_m f_m^{(-k)}(z^+)\right| \\ &\geq \left[-\frac{1}{4}+\frac{3}{4}-\frac{1}{4}\right]\frac{a_l|c_{l,l-k}|}{\beta_l}(z^+)^{[1/l+k(1+1/l)]} \\ &= \frac{1}{4}\frac{a_l|c_{l,l-k}|}{\beta_l}(z^+)^{[1/l+k(1+1/l)]} > l\frac{(z^+)^{1+k}}{(1+k)!}\,.\end{aligned}$$

Hence

$$\max_{x\in(y_l,x_l)}\left\{\sum_{m=k+2}^{\infty} a_m \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!}\right\} \geq \frac{l(z^+)^{1+k}/(1+k)!}{(z^+)^{1+k}/(1+k)!} = l.$$

Thus

$$u_k \geq \limsup_{k+2<l\to\infty} l - a_{1+k}\frac{(1+k)!\,|c_{1+k,1}|}{\beta_{1+k}} = +\infty.$$

In order to prove the second part of the theorem we need the functions f_m to be differentiable on $\mathbb{R}-\{0\}$. We modify the definition of f_m as follows.

First, define $\psi_{a,b}:\mathbb{R}_+\to\mathbb{R}$ for $a,b\in\mathbb{R}^+$, $a<b$, by

$$\psi_{a,b}(x) = \begin{cases} 1 & \text{if } 0\leq x<a, \\ F_{a,b}(x) & \text{if } a\leq x<b, \\ 0 & \text{if } b\leq x<\infty, \end{cases}$$

where $F_{a,b} = [2x^3-3ax^2-3bx^2+6abx+b^3-3ab^2]/(b-a)^3$. Observe that $\psi_{a,b}$ is continuously differentiable on $\mathbb{R}_+$.

Next, let α'_m be the largest zero of $\phi_m^{(m)}$ less than α_m. Finally, define

$$f_m(x) = \frac{\phi_m^{(m)}(|x|)\psi_{\alpha'_m,\alpha_m}(|x|)}{\beta_m}\,.$$

Now, each f_m is differentiable on $\mathbb{R} - \{0\}$ and has support contained in $[-1/m, 1/m]$. Thus in a neighborhood of any $x \neq 0$, f is the sum of at most finitely many non-zero differentiable functions and hence $f_{\langle 1 \rangle}(x) = f^{(1)}(x)$.

With this new definition we repeat the construction of f and observe that equation (5) (for $0 \le x \le \alpha'_m$) remains valid, as does the remainder of the proof of the first part, although the constants a_m, β_m, x_m, and y_m may change.

Proceeding toward a contradiction, suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous and such that $g_{[1]} = f_{\langle 1 \rangle}$ on $\mathbb{R}$. Since $f_{\langle 1 \rangle} = f^{(1)}$ on $\mathbb{R} - \{0\}$, g is differentiable on $\mathbb{R} - \{0\}$, and hence, without loss of generality, we may write $g = f$ on $\mathbb{R} - \{0\}$. Since f and g are continuous on $\mathbb{R}$, it must be $f \equiv g$ which is impossible since $f_{[1]}(0)$ does not exist. Hence no such g exists which shows that $f_{\langle 1 \rangle}$ is not the generalized Peano derivative of any continuous function. $\square$

5. Properties of the Laplace derivative

The Laplace derivative is well-defined as a result of the following lemma.

Lemma 5.1. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be integrable and $n \in \mathbb{Z}_+$. If there exists $0 < \delta_0 < b - a$ and $\alpha \in \mathbb{R}$ such that*

$$\lim_{s \to +\infty} s^{1+n} \int_0^{\delta_0} e^{-st} f(a+t)\, dt = \alpha,$$

then the same is true for each $0 < \delta < b - a$ replacing δ_0.

Proof: The result is a consequence of the fact that $\lim_{s \to +\infty} s^{1+n} \int_c^d e^{-st} f(a+t)\, dt = 0$ for every $0 < c < d < b - a$. $\square$

We need the following two lemmas to prove that the Laplace derivative is uniquely defined.

Lemma 5.2. *Let $\delta \in \mathbb{R}^+$ and $p, q \in \mathbb{Z}_+$. Then*

$$s^q \int_0^{\delta} e^{-st} t^p\, dt = p!\, s^{q-p-1} + \epsilon(s),$$

where $\epsilon(s) \to 0$ as $s \to +\infty$.

Proof: For $p, q \in \mathbb{Z}_+$, $s^q \int_0^{\delta} e^{-st} t^p\, dt = s^{q-p-1} \int_0^{s\delta} e^{-\tau} \tau^p\, d\tau$ for any $\delta \in \mathbb{R}^+$, where $\tau = st$. Since $\int_0^{\infty} e^{-\tau} \tau^p\, d\tau = p!$ and $\lim_{s \to +\infty} s^{q-p-1} \int_{s\delta}^{\infty} e^{-\tau} \tau^p\, d\tau = 0$, we have the desired result. $\square$

Lemma 5.3. *Let $n \in \mathbb{Z}_+$ and $\epsilon : \mathbb{R}_+ \to \mathbb{R}$ be continuous with $\epsilon(0) = 0$. Then for all $\delta_0 \in \mathbb{R}^+$*

$$\lim_{s\to+\infty} s^{1+n} \int_0^{\delta_0} e^{-st} t^n \epsilon(t)\, dt = 0.$$

Proof: Let $\delta_0 \in \mathbb{R}^+$ be given. By the continuity of $\epsilon(t)$ there exists $M \in \mathbb{R}^+$ such that $|\epsilon(t)|$ is bounded by M on $[0, \delta_0]$. Let $\epsilon \in \mathbb{R}^+$ be arbitrary and choose $\delta \in (0, \delta_0)$ such that $|\epsilon(t)| < \epsilon$ on $[0, \delta]$. Then we have

$$\begin{aligned} |s^{1+n} \int_0^{\delta_0} e^{-st} t^n \epsilon(t)\, dt| &\le s^{1+n} \int_0^{\delta_0} e^{-st} t^n |\epsilon(t)|\, dt \\ &\le s^{1+n} \int_0^{\delta} e^{-st} t^n \epsilon\, dt + s^{1+n} \int_\delta^{\delta_0} e^{-s\delta} \delta_0^n M\, dt. \end{aligned} \tag{12}$$

By Lemma 5.2 the right hand side of (12) converges to $n!\epsilon + 0$ as $s \to +\infty$. The result follows since ϵ was arbitrary. □

Theorem 5.4 (Uniqueness Theorem)**.** *Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be continuous, $x \in [a, b]$, and $n \in \mathbb{Z}^+$. If $f_{\langle n\rangle}(x)$ exists with associated numbers $\alpha_0, \dots, \alpha_{n-1}$, then the numbers are uniquely determined with $\alpha_0 = f(x) = f_{\langle 0\rangle}(x)$ and $\alpha_i = f_{\langle i\rangle}(x)$ for $i = 1, \dots, n-1$.*

Proof: There is no loss in generality in proving the theorem only for the right hand one-sided derivative at $x \in [a, b)$. We know that there exists $\delta \in \mathbb{R}^+$ and $\epsilon_x : \mathbb{R}^+ \to \mathbb{R}$ such that $\epsilon_x(s) \to 0$ as $s \to +\infty$ and

$$f_{\langle n,+\rangle}(x) + \epsilon_x(s) = s^{1+n} \int_0^{\delta} e^{-st} [f(x+t) - \sum_{i=0}^{n-1} \alpha_i t^i / i!]\, dt$$

for all $s \in \mathbb{R}^+$. Let m be an integer such that $1 \le m < n$ and $s \in \mathbb{R}^+$, then

$$\begin{aligned} \frac{f_{\langle n,+\rangle}(x)}{s^{n-m}} + \frac{\epsilon_x(s)}{s^{n-m}} &= s^{1+m} \int_0^{\delta} e^{-st} [f(x+t) - \sum_{i=0}^{m-1} \alpha_i t^i / i!]\, dt \\ &\quad - s^{1+m} \int_0^{\delta} e^{-st} \alpha_m t^m / m!\, dt - s^{1+m} \int_0^{\delta} e^{-st} \sum_{i=m+1}^{n-1} \alpha_i t^i / i!\, dt. \end{aligned}$$

Letting $s \to +\infty$, and using Lemma 5.2 we obtain that $0 = f_{\langle m,+\rangle}(x) - \alpha_m - 0$.

The $m = 0$ case, namely $\alpha_0 = f(0)$, follows from Lemma 5.3, with $n = 0$ and $\epsilon(t) = f(x+t) - f(x)$. □

Theorem 5.5 (Regularity Theorem)**.** *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be continuous, $x \in [a,b]$, and $n \in \mathbb{Z}^+$. Then,*

(i) *if $f_{[n]}(x)$ exists, then $f_{\langle n\rangle}(x)$ exists and $f_{[n]}(x) = f_{\langle n\rangle}(x)$;*

(ii) *if $f_{(n)}(x)$ exists, then $f_{\langle n\rangle}(x)$ exists and $f_{[n]}(x) = f_{(n)}(x) = f_{\langle n\rangle}(x)$;*

(iii) *if $f^{(n)}(x)$ exists, then $f_{\langle n\rangle}(x)$ exists and $f_{[n]}(x) = f_{(n)}(x) = f^{(n)}(x) = f_{\langle n\rangle}(x)$; and*

(iv) *for each $k, n \in \mathbb{Z}_+$, $f^{(-k)}_{\langle n+k\rangle}(x)$ exists if and only $f_{\langle n\rangle}(x)$ does, in which case they are equal.*

Proof: There is no loss in generality in proving the theorem only for the right hand one-sided derivatives at $x \in [a,b)$.

Beginning with (ii), suppose that $f_{(n,+)}(x)$ exists. Since $f(x) = f_{(0,+)}(x) = f_{\langle 0,+\rangle}(x)$ there is a positive integer $m \leq n$ such that $f_{(j,+)}(x) = f_{\langle j,+\rangle}(x)$ for $j = 0 \dots m-1$. Since $f_{(m,+)}(x)$ exists, there is a continuous $\epsilon_x : \mathbb{R}_+ \to \mathbb{R}$ such that

$$f(x+t) - \sum_{i=0}^{m-1} f_{\langle i,+\rangle}(x) t^i/i! = (t^m/m!)[f_{(m,+)}(x) + \epsilon_x(t)]$$

for all $t \in \mathbb{R}_+$, where $\epsilon_x(t) \to 0$ as $t \to 0^+$ and $\epsilon_x(0) = 0$. Then, using Lemmas 5.1, 5.2 and 5.3, there exists $0 < \delta_0 < b - x$ such that

$$\begin{aligned} f_{(m,+)}(x) &= \lim_{s\to+\infty} s^{1+m} \int_0^{\delta_0} e^{-st} t^m [f_{(m,+)}(x) + \epsilon_x(t)]/m!\, dt \\ &= \lim_{s\to+\infty} s^{1+m} \int_0^{\delta_0} e^{-st} [f(x+t) - \sum_{i=0}^{m-1} f_{\langle i,+\rangle}(x) t^i/i!]\, dt \\ &= f_{\langle m,+\rangle}(x). \end{aligned}$$

The proof of (ii) is completed by repeating this calculation $n - m + 1$ times and recalling the well known fact that the existence of $f_{(n,+)}(x)$ implies that $f_{[n,+]}(x) = f_{(n,+)}(x)$.

Statement (iii) is now clear since it is well known that if $f^{(n,+)}(x)$ exists then $f^{(n,+)}(x) = f_{(n,+)}(x)$ and, hence, (ii) completes the proof.

To see (i), assume that $f_{[n,+]}(x)$ exists. Then the definition of $f_{[n,+]}(x)$ implies that there exists $k \in \mathbb{Z}_+$ such that $f_{[n,+]}(x) = f^{(-k)}_{(n+k,+)}(x)$. Now, from (ii), we have that $f_{[n,+]}(x) = f^{(-k)}_{\langle n+k,+\rangle}(x)$ and the result will follow from (iv).

To see (iv), we proceed by induction on $n \in \mathbb{Z}_+$. Since f is continuous, (iii) implies the $n = 0$ case by observing that $f(x) = (f^{(-k)})^{(k,+)}(x) = f^{(-k)}_{\langle k,+\rangle}(x)$ for all $k \in \mathbb{Z}_+$.

Now, suppose that for some $n \in \mathbb{Z}^+$, (iv) holds for $0, \ldots, n-1$. We want to show that for each $k \in \mathbb{Z}_+$, $f^{(-k)}_{\langle n+k,+\rangle}(x)$ exists if and only $f_{\langle n,+\rangle}(x)$ does, in which case they are equal.

Suppose that $f^{(-k)}_{\langle n+k,+\rangle}(x)$ exists for some $k \in \mathbb{Z}^+$ since there is nothing to prove if $k = 0$. We have

$$f^{(-k)}_{\langle n+k,+\rangle}(x) = \lim_{s\to+\infty} s^{1+n+k} \int_0^\delta e^{-st}[f^{(-k)}(x+t) - \sum_{i=k}^{k+n-1} f^{(-k)}_{\langle i,+\rangle}(x)t^i/i!]\, dt$$

for some $0 < \delta < b - x$, where $f^{(-k)}_{\langle i,+\rangle}(x) = 0$, $i = 0 \ldots k-1$, by (iii) and our definition of $f^{(-k)}(x)$. If we integrate by parts and let s tend to infinity we obtain

$$\begin{aligned} f^{(-k)}_{\langle n+k,+\rangle}(x) &= \lim_{s\to+\infty} s^{n+k} \int_0^\delta e^{-st}[f^{(-k+1)}(x+t) - \sum_{i=k-1}^{k+n-2} f^{(-k+1)}_{\langle i,+\rangle}(x)t^i/i!]\, dt \\ &= f^{(-k+1)}_{\langle n+k-1,+\rangle}(x), \end{aligned}$$

where we have used our induction hypothesis to write $f^{(-k)}_{\langle i,+\rangle}(x) = f^{(-k+1)}_{\langle i-1,+\rangle}(x)$ for $i = k, \ldots, k+n-1$. Hence, integrating by parts k times, we find that $f^{(-k)}_{\langle n+k,+\rangle}(x) = f_{\langle n,+\rangle}(x)$ as required.

Conversely, suppose that $f_{\langle n,+\rangle}(x)$ exists. We have

$$f_{\langle n,+\rangle}(x) = \lim_{(s\to+\infty)} s^{1+n} \int_0^\delta e^{-st}[f(x+t) - \sum_{i=0}^{n-1} f_{\langle i,+\rangle}(x)t^i/i!]\, dt$$

for some $\delta \in \mathbb{R}^+$. Integrate by parts and let s tend to infinity to obtain that

$$\begin{aligned} f_{\langle n,+\rangle}(x) &= \lim_{(s\to+\infty)} s^{1+n+1} \int_0^\delta e^{-st}[f^{(-1)}(x+t) - \sum_{i=1}^{n} f^{(-1)}_{\langle i,+\rangle}(x)t^i/i!]\, dt \\ &= f^{(-1)}_{\langle n+1,+\rangle}(x). \end{aligned}$$

Integrating by parts k times we obtain that $f_{\langle n,+\rangle}(x) = f^{(-k)}_{\langle n+k,+\rangle}(x)$ as required. Thus, by induction, (iv) is true completing the proof of the theorem. □

References

[1] Bullen P.S., Mukhopadhyay S.N., *Properties of Baire*-1 Darboux functions and some mean value theorems for Peano derivatives*, Math. Japon. **36** (1991), 309–316.
[2] Doetsch G., *Handbuch der Laplace-Transformation*, Vol. 1, Birkhäuser Verlag, Basel, 1950.
[3] Fejzić H., *The Peano derivatives*, Doct. Dissertation, Michigan State University, 1992.
[4] Fejzić H., *On generalized Peano and Peano derivatives*, Fund. Math. **143** (1993), 55–74.
[5] Fejzić H., Rinne D., *Continuity properties of Peano derivatives in several variables*, Real Anal. Exchange **21** (1995–6), 292–298.
[6] Laczkovich M., *On the absolute Peano derivatives*, Ann. Univ. Sci. Budapest, Eötvös Sect. Math. **21** (1978), 83–97.
[7] Lee C.-M., *On the approximate Peano derivatives*, J. London Math. Soc. **12** (1976), 475–478.
[8] Lee C.-M., *Generalizations of Cesàro continuous functions and integrals of Perron type*, Trans. Amer. Math. Soc. **266** (1981), 461–481.
[9] Lee C.-M., *On absolute Peano derivatives*, Real Anal. Exchange **8** (1982–3), 228–243.
[10] Lee C.-M., *On generalized Peano derivatives*, Trans. Amer. Math. Soc. **275** (1983), 381–396.
[11] Lee C.-M., *On generalizations of exact Peano derivatives*, Contemp. Math. **42** (1985), 97–103.
[12] Mukhopadhyay S.N., Mitra S., *Measurability of Peano derivatives and approximate Peano derivatives*, Real Anal. Exchange **20** (1994–5), 768–775.
[13] Oliver H.W., *The exact Peano derivative*, Trans. Amer. Math. Soc. **76** (1954), 444–456.
[14] Svetic R.E., Volkmer H., *On the ultimate Peano derivative*, J. Math. Anal. Appl. **218** (1998), 439–452.
[15] Verblunsky S., *On the Peano derivatives*, Proc. London Math. Soc. **22** (1971), 313–324.
[16] Weil C.E., *The Peano notion of higher order differentiation*, Math. Japon. **42** (1995), 587–600.
[17] Widder D.V., *An Introduction to Transform Theory*, Academic Press, New York, 1971.

Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA

E-mail: rsvetic@math.msu.edu

(*Received* May 4, 2000)