The chromatic number of the product of two graphs is at least half the minimum of the fractional chromatic numbers of the factors

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Abstract. One consequence of Hedetniemi's conjecture on the chromatic number of the product of graphs is that the bound $\chi(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\}$ should always hold. We prove that $\chi(G \times H) \geq \frac{1}{2} \min\{\chi_f(G), \chi_f(H)\}$.

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One outstanding problem in graph theory is a formula concerning the chromatic number of the product of two graphs:

Conjecture 1 (Hedetniemi [2]). *For any two graphs G and H,*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

This formula seems natural and attractive; however it is remarkably bold compared to our current state of knowledge: El-Zahar and Sauer [1] proved that the chromatic number of the product of two 4-chromatic graphs is 4, but it is not yet established that there exists a number n such that the chromatic number of the product of any two n-chromatic graphs is at least 5. Poljak and Rödl [4] introduced the function

$$f(n) = \min\{\chi(G \times H) : \chi(G) \geq n, \chi(H) \geq n\},$$

and in [3], [6], we find proofs of the strange result that f either goes to infinity with n or is bounded by 9. An attempt to settle at least a fractional version of this problem led us to the result presented in the title:

Theorem 2. *For any two graphs G and H,*

$$\chi(G \times H) \geq \tfrac{1}{2} \min\{\chi_f(G), \chi_f(H)\}.$$

In particular, this shows that the function

$$f'(n) = \min\{\chi(G \times H) : \chi_f(G) \geq n, \chi_f(H) \geq n\},$$

goes to infinity with n, though it has no direct bearing on the Poljak-Rödl function. However, the argument seems to suggest that it may be possible to prove that the Poljak-Rödl function is unbounded using probabilistic methods. At least, this is the hope that the author wishes to share in presenting this note.

1. Basic concepts

The *product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, whose edges are all pairs $[(u_1, u_2), (v_1, v_2)]$ with $[u_1, v_1] \in E(G)$ and $[u_2, v_2] \in E(H)$. Colorings of G or H naturally induce colorings of $G \times H$ hence the inequality $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$ trivially holds.

For two graphs G and K, the *exponential graph* K^G has for vertices all the functions from $V(G)$ to $V(K)$, and two of these functions f, g are joined by an edge if $[f(u), g(v)] \in E(K)$ for all $[u, v] \in E(G)$. There is a natural correspondence between the n-colorings of $G \times H$ and the edge-preserving maps from H to ${K_n}^G$. Applications of this correspondence in the context of Hedetniemi's conjecture are given in [1], [6].

Let $\mathcal{I}(G)$ denote the family of all independent sets of a graph G. A function $\mu : \mathcal{I}(G) \mapsto [0, 1]$ is called a *fractional coloring* of G if we have $\sum_{u \in I} \mu(I) \geq 1$ for all $u \in V(G)$. The value $\sum_{I \in \mathcal{I}(G)} \mu(I)$ is called the *weight* of μ. Also, a function $\nu : V(G) \mapsto [0, 1]$ is called a *fractional clique* of G if $\sum_{u \in I} \nu(u) \leq 1$ for all $I \in \mathcal{I}(G)$. Its *weight* is $\sum_{u \in V(G)} \nu(u)$. The *fractional chromatic number* $\chi_f(G)$ of G is the common value of the minimum weight of a fractional coloring of G and the maximum weight of a fractional clique of G (see [5]). We have $\chi_f(G) \leq \chi(G)$ for any graph G. Also, if there exists an edge-preserving map from G to H, then $\chi_f(G) \leq \chi_f(H)$.

2. Proof of Theorem 2

Let G, H be graphs such that $\chi(G \times H) = n$ and $\chi_f(G) \geq 2n$. Any n-coloring $\phi : G \times H \mapsto K_n$ induces an edge-preserving map $\psi : H \mapsto {K_n}^G$ defined by $\psi(v) = h_v$, where $h_v(u) = \phi(u, v)$ for all $u \in V(G)$, $v \in V(H)$. Therefore $\chi_f(H) \leq \chi_f({K_n}^G)$, and it will suffice to show that $\chi_f({K_n}^G) \leq 2n$.

For $u \in V(G)$ and $1 \leq k \leq n$, put

$$I(u, k) = \{h \in {K_n}^G : h(u) = k = h(v) \text{ for some } [u, v] \in E(G)\}.$$

If $h \in I(u, k)$ and h' is adjacent to h in ${K_n}^G$, then $h'(v) \neq k$ for all $[u, v] \in E(G)$, thus $h' \notin I(u, k)$. This shows that $I(u, k)$ is an independent set.

Let $\nu : V(G) \mapsto [0, 1]$ be a fractional clique of weight $\chi_f(G)$. For $u \in V(G)$ and $1 \leq k \leq n$, put

$$\mu(I(u, k)) = \frac{2}{\chi_f(G)} \nu(u).$$

Then $\sum_{I \in \mathcal{I}({K_n}^G)} \mu(I) = 2n$. We will show that μ is a fractional coloring of ${K_n}^G$. For a function $h \in V({K_n}^G)$, let G_h be the subgraph of G induced by

$$V(G_h) = \{u \in V(G) : h(u) = h(v) \text{ for some } [u, v] \in E(G)\}.$$

Then,

$$\sum_{h \in I} \mu(I) = \sum_{u \in V(G_h)} \mu(I(u, h(u))) = \frac{2}{\chi_f(G)} \sum_{u \in V(G_h)} \nu(u).$$

Now, the restriction of h to $V(G) \setminus V(G_h)$ is a proper coloring of $G - G_h$ whence $\sum_{u \in V(G) \setminus V(G_h)} \nu(u) \leq n \leq \frac{\chi_f(G)}{2}$. Therefore $\sum_{h \in I} \mu(I) \geq 1$ and μ is a fractional coloring of ${K_n}^G$. This shows that $\chi_f({K_n}^G) \leq 2n$, and concludes the proof of Theorem 2. $\square$

Slight improvements on Theorem 2 are readily possible. Ideally, it would be nice to prove that the inequality

$$\chi(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\} \tag{1}$$

holds for all graphs G and H. At least, this is a desirable result in view of Conjecture 1. Note that Theorem 2 remains true in the context of directed graphs, with essentially the same proof. However it is shown in [4] that for any $n \geq 3$, there exist tournaments T_1, T_2 on $n + 1$ vertices such that $\chi(T_1 \times T_2) \leq n$. This shows that (1) does not always hold in the case of directed graphs.

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