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The fixed point property in Musielak-Orlicz sequence spaces

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Abstract. In this paper, we give necessary and sufficient conditions for a point in a Musielak-Orlicz sequence space equipped with the Orlicz norm to be an **H**-point. We give necessary and sufficient conditions for a Musielak-Orlicz sequence space equipped with the Orlicz norm to have the *Kadec-Klee* property, the uniform *Kadec-Klee* property and to be nearly uniformly convex. We show that a Musielak-Orlicz sequence space equipped with the Orlicz norm has the fixed point property if and only if it is reflexive.

Keywords: nearly uniformly convex, uniform Kadec-Klee property, Kadec-Klee property, Musielak-Orlicz sequence space, fixed point property

Classification: 46E30, 46B20, 47H10

§1. Introduction

The *Kadec-Klee* property and the notion of nearly uniform convexity play important roles in some branches of mathematics. Some well known results involve these properties. In this paper, we will examine these properties in the Musielak-Orlicz sequence space equipped with the Orlicz norm.

Let $(X, \|\cdot\|)$ be a real Banach space and X^* the dual space of X. By $B(X)$ and $S(X)$ we denote the closed unit ball and the unit sphere of X, respectively. Let l^0 be the set of all real sequences.

Definition 1. A point $x \in S(X)$ is said to be an **H**-point if whenever $\{x_n\} \subset S(X)$ satisfies $x_n \xrightarrow{w} x$, then $x_n \to x$ in norm (see [18]).

A Banach space X is said to have the Kadec-Klee property if every point on $S(X)$ is an **H**-point (see [18]).

Recall that a sequence $\{x_n\}$ is said to be an ε-separate sequence for some $\varepsilon > 0$ if

$$\operatorname{sep}(\{x_n\}) := \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon.$$

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Definition 2. A Banach space X is said to have the uniform Kadec-Klee property (**UKK**) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if x is a weak limit of a norm one ε-separate sequence, then $\|x\| < 1 - \delta$.

The notion of nearly uniform convexity of a *Banach space* was introduced by Huff in [12]. It is an infinite dimensional counterpart of the classical uniform convexity.

Definition 3. A Banach space X is said to be nearly uniformly convex (**NUC**), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence $\{x_n\} \subset B(X) = \{x \in X : \|x\| \leq 1\}$ with $\operatorname{sep}(\{x_n\}) > \varepsilon$ we have

$$\operatorname{conv}(\{x_n\}) \cap (1-\delta)B(X) \neq \emptyset.$$

It is easy to see that every (**NUC**) space has the (**UKK**) property. Huff (see [12]) proved that X is (**NUC**) if and only if X is reflexive and has the (**UKK**) property.

Definition 4. A Banach space X contains an asymptotically isometric copy of l_1 if, for every sequence $\{\varepsilon_n\}$ decreasing to 0, there exists a sequence $\{x_n\} \subset S(X)$ such that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)\,|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sum_{n=1}^{\infty}|a_n|\,.$$

Definition 5. A Banach space X is called a Köthe sequence space if for every $x \in l^0$ and $y \in X$ satisfying $|x(i)| \leq |y(i)|$ for all $i \in N$ we have $x \in X$ and $\|x\| \leq \|y\|$.

Definition 6. A Köthe sequence space X is said to have an absolutely continuous norm if

$$\lim_{n\to\infty}\|(0,0,\cdots,0,x(n+1),x(n+2),\cdots)\| = 0$$

for any $x \in X$.

Put

$$X_a = \left\{x \in X : \lim_{n\to\infty}\|(0,0,\cdots,0,x(n+1),x(n+2),\cdots)\| = 0\right\}.$$

Definition 7. A Köthe sequence space X is said to have property (A) if $\lim\limits_{n\to\infty}\left\|\sum\limits_{i=1}^{n} x(i)e_i\right\| = \|x\|$ for all $x \in X$, where $e_i = (0,\cdots 0,\overset{i-th}{1},0,\cdots)$.

A map $\Phi : \mathbb{R} \to [0,\infty)$ is said to be an Orlicz function if Φ is vanishing only at 0, is even, convex and continuous on $[0,\infty)$ and $\lim\limits_{u\to\infty}\Phi(u) = \infty$ (see [17] and [1]). Note that the continuity of Φ on $[0,c)$ follows from the convexity of Φ.

A sequence $\Phi = (\Phi_i)$ of Orlicz functions is called to be a *Musielak-Orlicz* function. By $\Psi = (\Psi_i)$ we denote the complementary function of Φ in the sense of Young; that is

$$\Psi_i(v) = \sup\{|v|\,u - \Phi_i(u) : u \geq 0\}\,,\ i = 1, 2, \cdots .$$

For a given *Musielak-Orlicz* function Φ, we define a convex modular by $I_\Phi(x) = \sum\limits_{i=1}^{\infty} \Phi_i(x(i))$ for any $x \in l^0$. A linear space l_Φ defined by

$$l_\Phi = \left\{x \in l^0 :\ I_\Phi(kx) < \infty \ \text{ for some } \ k > 0\right\}$$

is called the *Musielak-Orlicz sequence space* generated by Φ. We consider l_Φ equipped with the *Luxemburg norm*

$$\|x\| = \inf\left\{k > 0 :\ I_\Phi\left(\frac{x}{k}\right) \leq 1\right\}$$

or equipped with the equivalent so called *Orlicz norm*

$$\|x\|^0 = \inf\left\{\frac{1}{k}\left(1 + I_\Phi(kx)\right) : k > 0\right\}.$$

To simplify notations, we put $l_\Phi = (l_\Phi, \|\cdot\|)$ and $l_\Phi^0 = (l_\Phi, \|\cdot\|^0)$. Both l_Φ and l_Φ^0 are Banach spaces (see [17] and [1]).

The set of all k's at which the infimum in the Orlicz norm is attained for $x \in l_\Phi^0$ $(x \neq 0)$ will be denoted by $K(x)$.

We say that an *Orlicz function* Φ satisfies the δ_2-condition ($\Phi \in \delta_2$ for short) if there exist constants $k \geq 2$ and $u_0 > 0$ and a sequence (c_i) of positive numbers such that $\sum\limits_{i=1}^{\infty} c_i < \infty$ and the inequality

$$\Phi_i(2u) \leq k\Phi_i(u) + c_i$$

holds for every $i \in \mathbb{N}$ and $|u| \leq u_0$.

For more details on Musielak-Orlicz sequence spaces, we refer to [17], [1], and [15].

§2. Results

In order to get our main results, we need to recall some known facts.

Lemma 1. *If $\Phi \in \delta_2$, then $\|x_n\|^0 \to 0$ if and only if $I_\Phi(x_n) \to 0$ (see [14]).*

Lemma 2. *Let $x \in l_\Phi^0$ be given. If $K(x) = \emptyset$, then $\|x\|^0 = \sum_{i=1}^{\infty} a_i |x(i)|$, where $a_i = \lim_{u\to\infty} \frac{\Phi_i(u)}{u}$.*

Proof: Put $f(k) = \frac{1}{k}(1 + I_\Phi(kx))$. Then $\lim_{k\to 0+} f(k) = +\infty$. Since $f(k)$ is continuous and $K(x) = \emptyset$, we have $\|x\|^0 = \lim_{k\to\infty} f(k) = \lim_{k\to\infty} \frac{I_\Phi(kx)}{k}$. Set $N(x) = \{i \in \mathbb{N} : x(i) \neq 0\}$. Then $a_i = \lim_{u\to\infty} \frac{\Phi_i(u)}{u}$ is finite for all $i \in N(x)$. If $\lim_{u\to\infty} \frac{\Phi_{i_0}(u)}{u} = \infty$ for some $i_0 \in N(x)$, then $\|x\|^0 = \lim_{k\to\infty} \frac{I_\Phi(kx)}{k} \geq \lim_{k\to\infty} \frac{\Phi_{i_0}(kx(i_0))}{k} = \infty$. Hence $\|x\|^0 = \lim_{k\to\infty} \frac{I_\Phi(kx)}{k} = \sum_{i=1}^{\infty} a_i |x(i)|$. □

Lemma 3. *Let X be a Köthe sequence space with property (A). If $x \in S(X)$ is an H-point, then $x \in X_a$.*

Proof: Assume x does not have an absolutely continuous norm, thus there exists $\varepsilon_0 > 0$ such that

$$\Big\| \sum_{i=n+1}^{\infty} x_0(i) e_i \Big\| \geq \varepsilon_0$$

for any $n \in \mathbb{N}$.

Take $n = 0$. Since X has property (A), there is $n_1 \in \mathbb{N}$ such that

$$\Big\| \sum_{i=1}^{n_1} x_0(i) e_i \Big\| \geq \frac{\varepsilon_0}{2}.$$

Notice that

$$\lim_{m\to\infty} \Big\| \sum_{i=n_1+1}^{m} x_0(i) e_i \Big\| = \Big\| \sum_{i=n_1+1}^{\infty} x_0(i) e_i \Big\| \geq \varepsilon_0,$$

and thus there exists $n_2 > n_1$ such that

$$\Big\| \sum_{i=n_1+1}^{n_2} x_0(i) e_i \Big\| \geq \frac{\varepsilon_0}{2}.$$

In this way, we get a sequence $\{n_i\}$ of natural numbers such that

$$\Big\| \sum_{i=n_i+1}^{n_{i+1}} x_0(i) e_i \Big\| \geq \frac{\varepsilon_0}{2}, i = 1, 2, \cdots.$$

Put $x_i = \sum\limits_{i=n_i+1}^{n_{i+1}} x_0(i)e_i$. Then

(1) $\|x_i\| \geq \frac{\varepsilon_0}{2}$ for all $i \in \mathbb{N}$;

(2) $x_i \xrightarrow{w} 0$ as $i \to \infty$. It is well known that for any *Köthe* space X we have

$$X^* = X' \oplus S,$$

where S is the space of all singular functionals over X, i.e., functionals which vanish on the subspace $X_a = \{x \in X : x \text{ has absolutely continuous norm}\}$ and $X' = \{y \in l^0 : \sum\limits_{i=1}^{\infty} x(i)y(i) < \infty \text{ for any } x \in X\}$ (see [16]). This means that every $f \in X^*$ is uniquely represented in the form

$$f = T_y + \varphi,$$

where $\varphi \in S$ and T_y is the functional generated by an element $y \in X'$ by the following formula

$$T_y(x) = \sum_{i=1}^{\infty} x(i)y(i)$$

for any $x \in X$.

Let $y \in X'$. We have

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} x_i(j)y(j) = \lim_{i\to\infty} \sum_{j=n_i+1}^{n_{i+1}} x_i(j)y(j) = 0.$$

Put $z_i = x - x_i$ for any $i \in \mathbb{N}$. Then $z_i \xrightarrow{w} x$, $\lim\limits_{i\to\infty} \|z_i\| = \|x\| = 1$ and $\|z_i - x\| = \|x_i\| \geq \frac{\varepsilon_0}{2}$.

This contradiction shows that x has an absolutely continuous norm. □

Theorem 1. *$x \in S(l^0_\Phi)$ is an* **H***-point if and only if $\Phi \in \delta_2$ or $K(x) = \emptyset$.*

Proof: Proof of necessity. Suppose that $x \in S(l^0_\Phi)$ is an **H**-point, $\Phi \notin \delta_2$ and $K(x) \neq \emptyset$. Since $\Phi \notin \delta_2$, there exists $x_0 \in S(l_\Phi)$ such that $I_\Phi(x_0) \leq 1$ and $I_\Phi(kx_0) = \infty$ for any $k > 1$ (see [11]). Take a sequence of natural numbers $m_n \uparrow \infty$ such that $\|\sum\limits_{i=m_n+1}^{m_{n+1}} x_0(i)e_i\| > \frac{1}{2}$. Put

$$x_n = (x(1), \cdots, x(m_n), \frac{x_0(m_n+1)}{k}, \cdots, \frac{x_0(m_{n+1})}{k}, x(m_{n+1}+1), \cdots)$$

for all $n \in \mathbb{N}$, where $k \in K(x)$.

Arguing as in Lemma 3, we can show that $x_n \xrightarrow{w} x$. From

$$\begin{aligned}\|x_n\|^0 &\le \frac{1}{k}\left(1 + I_\Phi(kx) + \sum_{i=m_n+1}^{m_{n+1}} \Phi_i(x_0(i))\right)\\ &= 1 + \frac{1}{k}\sum_{i=m_n+1}^{m_{n+1}} \Phi_i(x_0(i)) \to 1\end{aligned}$$

and

$$1 \leftarrow \left\| \sum_{i=1}^{m_n} x(i)e_i \right\|^0 \le \|x_n\|^0,$$

we have $\lim\limits_{n\to\infty} \|x_n\|^0 = 1$.

Since x is an **H**-point, by Lemma 2 there exists $n_0 \in \mathbb{N}$ such that $\left\| \sum\limits_{i=m_n+1}^{m_{n+1}} x(i)e_i \right\|^0 \le \frac{1}{4k}$ when $n > n_0$. Hence

$$\begin{aligned}\|x_n - x\|^0 &= \left\| \sum_{i=m_n+1}^{m_{n+1}} \left(\frac{1}{k}x_0(i) - x(i)\right)e_i \right\|^0\\ &\ge \frac{1}{k}\left\| \sum_{i=m_n+1}^{m_{n+1}} x_0(i)e_i \right\|^0 - \left\| \sum_{i=m_n+1}^{m_{n+1}} x(i)e_i \right\|^0\\ &\ge \frac{1}{k}\left\| \sum_{i=m_n+1}^{m_{n+1}} x_0(i)e_i \right\| - \left\| \sum_{i=m_n+1}^{m_{n+1}} x(i)e_i \right\|^0 \ge \frac{1}{2k} - \frac{1}{4k} = \frac{1}{4k}\end{aligned}$$

when $n > n_0$, a contradiction, since x is an **H**-point. Thus either $\Phi \in \delta_2$ or $K(x) = \emptyset$.

Sufficiency. Suppose that $\{x_n\} \subset S(l_\Phi^0)$ is such that $x_n \xrightarrow{w} x$.

First, we consider the case $K(x) = \emptyset$. There are two subcases to consider.

Subcase I. $K(x_n) = \emptyset$, $n = 1, 2, \cdots$. In this case, we have

$$\|x_n\|^0 = \sum_{i \in N(x_n)} a_i\,|x_n(i)| \quad\text{and}\quad \|x\|^0 = \sum_{i \in N(x)} a_i\,|x(i)|.$$

Using $x_n \xrightarrow{w} x$, we get $x_n(i) \to x(i)$ for all $i \in \mathbb{N}$ as $n \to \infty$. Since $x_n(i) \to x(i)$ for all $i \in \mathbb{N}$ as $n \to \infty$, it follows easily that $\lim\limits_{n\to\infty} \|x_n - x\|^0 = 0$.

Using $x_n \xrightarrow{w} x$, we get $x_n(i) \to x(i)$ for all $i \in \mathbb{N}$ as $n \to \infty$. Now it is well known that the weighted space $l^1(\{a_i\})$ has the Schur property. Therefore,

$\|x_n - x\|_{l^1(\{a_i\})} \to 0$. Moreover, for any $n \in \mathbb{N}$,

$$\|x_n - x\|^0 \le \sum_{i=1}^{\infty} a_i |x_n(i) - x(i)| = \|x_n - x\|_{l^1(\{a_i\})}.$$

Therefore $\lim\limits_{n\to\infty} \|x_n - x\|^0 = 0$ as $n \to \infty$.

Subcase II. $K(x_n) \neq \emptyset$, $n = 1, 2, \cdots$. We will show that $K(x_n) \neq \emptyset$ is impossible. Take $k_n \ge 1$ such that $\|x_n\|^0 = \frac{1}{k_n}(1 + I_\Phi(k_n x_n))$. We will prove $\lim\limits_{n\to\infty} k_n = \infty$. Assume there exists a subsequence $\{k_{n_j}\} \subset \{k_n\}$ and $k < \infty$ such that $\lim\limits_{j\to\infty} k_{n_j} = k$. Since $K(x) = \emptyset$, we have $\|x\|^0 < \frac{1}{k}(1 + I_\Phi(kx))$. Hence there is $i_0 \in \mathbb{N}$ such that

$$\|x\|^0 < \frac{1}{k}\left(1 + \sum_{i=1}^{i_0} \Phi_i(kx(i))\right).$$

Using $x_n(i) \to x(i)$ for all $i \in \mathbb{N}$, we obtain

$$\liminf_{n\to\infty} \|x_n\|^0 \ge \frac{1}{k}\left(1 + \sum_{i=1}^{i_0} \Phi_i(kx(i))\right) > 1.$$

This contradiction shows that $\lim\limits_{n\to\infty} k_n = \infty$. Take $i_1 \in \mathbb{N}$ such that $|x(i_1)| > 0$. Thus there exists $n_1 \in \mathbb{N}$ such that $|x_n(i_1)| \ge \frac{|x(i_1)|}{2}$ when $n > n_1$. Hence

$$\begin{aligned} 1 = \|x_n\|^0 &= \frac{1}{k_n}\left(1 + \sum_{i=1}^{\infty} \Phi_i(k_n x_n(i))\right) \\ &\ge \frac{1}{k_n}\Phi_{i_0}(k_n x_n(i_1)) \ge \frac{\Phi_{i_0}(\frac{k_n}{2}x(i_1))}{k_n} \to \infty, \end{aligned}$$

a contradiction. Thus $K(x_n) \neq \emptyset$ is impossible.

Lastly we consider the case $K(x) \neq \emptyset$ and $\Phi \in \delta_2$. It suffices to prove that $\{x_n\}$ has equi-absolutely continuous norm. Let $\varepsilon > 0$ be given. Again we consider the following two subcases:

Subcase I. $K(x_n) = \emptyset$, $n = 1, 2, \cdots$. Take $i_2 \in \mathbb{N}$ such that $\left\|\sum\limits_{i=1}^{i_2} x(i)e_i\right\|^0 > 1 - \varepsilon$. Using $x_n(i) \to x(i)$ for all $i \in \mathbb{N}$, there exists $n_2 \in \mathbb{N}$ such that $\left\|\sum\limits_{i=1}^{i_2} x_n(i)e_i\right\|^0 > 1 - \varepsilon$ when $n > n_2$. Hence

$$1 = \|x_n\|^0 = \left\|\sum_{i=1}^{i_2} x_n(i)e_i\right\|^0 + \left\|\sum_{i=i_2+1}^{\infty} x_n(i)e_i\right\|^0 > 1 - \varepsilon + \left\|\sum_{i=i_2+1}^{\infty} x_n(i)e_i\right\|^0,$$

where the second equality holds since $K(x_n) = \emptyset$. Thus $\left\| \sum\limits_{i=i_2+1}^{\infty} x_n(i)e_i \right\|^0 < \varepsilon$ when $n > n_2$.

Subcase II. $K(x_n) \neq \emptyset$ for $n = 1, 2, \cdots$. Since $\Phi \in \delta_2$, there exists $\delta > 0$ such that $\|z\|^0 < \varepsilon$ whenever $I_\Phi(z) < \delta$.

Take $i_3 \in \mathbb{N}$ such that $\left\| \sum\limits_{i=1}^{i_3} x(i)e_i \right\|^0 > 1 - \delta$. Using $x_n(i) \to x(i)$ for all $i \in \mathbb{N}$, there exists $n_3 \in \mathbb{N}$ such that $\left\| \sum\limits_{i=1}^{i_3} x_n(i)e_i \right\|^0 > 1 - \delta$ when $n > n_3$. Hence

$$
\begin{aligned}
1 = \|x_n\|^0 &= \frac{1}{k_n}\left(1 + \sum_{i=1}^{i_3} \Phi(k_n x_n(i)) + \sum_{i=i_3+1}^{\infty} \Phi(k_n x_n(i))\right) \\
&= \frac{1}{k_n}\left(1 + \sum_{i=1}^{i_3} \Phi(k_n x_n(i))\right) + \frac{1}{k_n} \sum_{i=i_3+1}^{\infty} \Phi(k_n x_n(i)) \\
&\geq \left\| \sum_{i=1}^{i_3} x_n(i)e_i \right\|^0 + \sum_{i=i_3+1}^{\infty} \Phi(x_n(i)) > 1 - \delta + \sum_{i=i_3+1}^{\infty} \Phi(x_n(i))
\end{aligned}
$$

when $n > n_3$. Thus $\sum\limits_{i=i_2+1}^{\infty} \Phi(x_n(i)) < \delta$ and $\left\| \sum\limits_{i=i_2+1}^{\infty} x_n(i)e_i \right\|^0 < \varepsilon$, as required. $\square$

Corollary 1. *A Musielak-Orlicz sequence space equipped with the Orlicz norm has the Kadec-Klee property if and only if* $\Phi \in \delta_2$.

Theorem 2. *A Musielak-Orlicz sequence space equipped with the Orlicz norm has the uniform Kadec-Klee property if and only if* $\Phi \in \delta_2$.

Proof: We only need to prove the sufficiency of the theorem. Let $\varepsilon > 0$ and $\{x_n\} \subset B(l_\Phi^0)$ with $\mathrm{sep}(\{x_n\}) > \varepsilon$ and $x_n \xrightarrow{w} x \in l_\Phi^0$. Now for any $m \in \mathbb{N}$ there is n_m such that $\mathrm{sep}(\{\sum\limits_{i=m}^{\infty} x_n(i)e_i\}) > \varepsilon$ for any $n \geq n_m$. This follows since $x_n \xrightarrow{w} x$ implies $x_n \to x$ coordinatewise and so we can make the coordinates $x_n(1), \ldots, x_n(m-1)$, differ by as little as we please for n sufficiently large. Thus $\mathrm{sep}(\{\sum\limits_{i=m}^{\infty} x_{n_k}(i)e_i\}) > \varepsilon$ for a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Relabelling the subsequence we may assume that it holds for all n.

Hence for $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$ such that $\left\| \sum\limits_{i=m}^{\infty} x_{n_m}(i)e_i \right\|^0 \geq \frac{\varepsilon}{2}$. Without loss of generality, we may assume that $\left\| \sum\limits_{i=m}^{\infty} x_n(i)e_i \right\|^0 \geq \frac{\varepsilon}{2}$ for all $m, n \in \mathbb{N}$.

Secondly, we assume that $\Phi \in \delta_2$. Then there exists $\varepsilon_1 \in (0,\varepsilon)$ such that $I_\Phi\Big(\sum\limits_{i=m}^{\infty} x_n(i)e_i\Big) > \varepsilon_1$ for all $m \in \mathbb{N}$.

Take m large enough such that $\Big\|\sum\limits_{i=1}^{m} x(i)e_i\Big\|^0 > \|x\|^0 - \frac{\varepsilon_1}{4}$. Using $x_n(i) \to x(i)$ for $i = 1,2,\cdots$, there exists $m_0 \in \mathbb{N}$ such that $\Big\|\sum\limits_{i=1}^{m} x_n(i)e_i\Big\|^0 > \|x\|^0 - \frac{\varepsilon_1}{4}$ when $n > m_0$.

We divide our proof into the following parts:

Part I. $K(x_n) \neq \emptyset$ for $n = 1,2,\cdots$. Hence

$$
\begin{aligned}
1 \geq \|x_n\|^0 &= \frac{1}{k_n}\Big(1 + \sum_{i=1}^{m} \Phi(k_n x_n(i)) + \sum_{i=m+1}^{\infty} \Phi(k_n x_n(i))\Big) \\
&= \frac{1}{k_n}\Big(1 + \sum_{i=1}^{m} \Phi(k_n x_n(i))\Big) + \frac{1}{k_n} \sum_{i=m+1}^{\infty} \Phi(k_n x_n(i)) \\
&\geq \Big\|\sum_{i=1}^{m} x_n(i)e_i\Big\|^0 + \sum_{i=m+1}^{\infty} \Phi(x_n(i)) > \|x\|^0 - \frac{\varepsilon_1}{4} + \varepsilon_1
\end{aligned}
$$

when $n > m_0$. Thus $\|x\|^0 < 1 - \frac{3\varepsilon_1}{4}$.

Part II. $K(x_n) = \emptyset$ $(n = 1,2,\cdots)$. Hence

$$
\begin{aligned}
1 \geq \|x_n\|^0 &= \Big\|\sum_{i=1}^{m} x_n(i)e_i\Big\|^0 + \Big\|\sum_{i=m+1}^{\infty} x_n(i)e_i\Big\|^0 \\
&\geq \|x\|^0 - \frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{2}
\end{aligned}
$$

when $n > m$. Thus $\|x\|^0 < 1 - \frac{\varepsilon_1}{4}$. □

Corollary 2. *A Musielak-Orlicz sequence space equipped with the Orlicz norm is nearly uniformly convex if and only if it is reflexive.*

Theorem 3. *A Musielak-Orlicz sequence space equipped with the Orlicz norm has the fixed point property if and only if it is reflexive.*

Proof: Since a reflexive Banach space with the **UKK** property has the fixed point property (see [7]), we only need to prove the necessity.

Suppose Φ fails to satisfy the δ_2-condition. Then l_Φ contains an isometric copy of l_∞ (see [15]). Hence l_Φ^0 does not have the fixed point property (see [6]).

Suppose Φ fails to satisfy the $\overline{\delta}_2$-condition. Then Ψ fails to satisfy the δ_2-condition. Hence there exists $x \in S(l_\Psi)$ such that $I_\Psi(\lambda x) = \infty$ for any $l > 1$

(see [11]). For every sequence (ε_n) decreasing to 0, there exist $0 = I_1 < I_2 < I_3 < \cdots$ such that

$$\left\| \sum_{i=I_n+1}^{I_{n+1}} x(i)e_i \right\| > 1 - \varepsilon_n, \quad \text{for all} \quad n \in \mathbb{N}.$$

Put $x_n = \sum\limits_{i=I_n+1}^{I_{n+1}} x(i)e_i$ for all $n \in \mathbb{N}$. Then there exist $z_n \in S(l_\Phi^0)$ such that $\langle z_n, x_n \rangle = \|x_n\|$ for all $n \in \mathbb{N}$. Hence for any $\alpha = (\alpha(n)) \in l_1$ and $\operatorname{supp} z_n \subseteq \operatorname{supp} x$, we have

$$\begin{aligned}\left\| \sum_{n=1}^{\infty} \alpha(n) z_n \right\|^0 &= \left\| \sum_{n=1}^{\infty} \alpha(n) z_n \right\|^0 \geq \left\langle \sum_{n=1}^{\infty} \alpha(n) z_n, \sum_{n=1}^{\infty} \operatorname{sign}(\alpha(n)) x_n \right\rangle \\ &= \sum_{n=1}^{\infty} |\alpha(n)| \langle z_n, x_n \rangle \geq (1 - \varepsilon_n) \sum_{n=1}^{\infty} |\alpha(n)|,\end{aligned}$$

and

$$\left\| \sum_{n=1}^{\infty} \alpha(n) z_n \right\|^0 \leq \sum_{n=1}^{\infty} |\alpha(n)| \, \|z_n\|^0 = \sum_{n=1}^{\infty} |\alpha(n)|.$$

Thus l_Φ^0 contains an asymptotically isometric copy of l_1. By Theorem 2 (see [6]), l_Φ^0 does not have the fixed point property. □

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