Almost closed sets and topologies they determine

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Abstract. We prove that every countably compact AP-space is Fréchet-Urysohn. It is also established that if X is a paracompact space and $C_p(X)$ is AP, then X is a Hurewicz space. We show that every scattered space is WAP and give an example of a hereditarily WAP-space which is not an AP-space.

Keywords: AP-space, WAP-space, scattered space, countably compact space, function space, discretely generated space

Classification: 54A25, 54D55

0. Introduction

If X is a topological space and $F \subset X$, let us say that F is *almost closed* if $\overline{F}\backslash F$ is a one-point set. If $\overline{F}\backslash F = \{x\}$, we sometimes denote it by $F \to x$. It is natural to say that the topology of a space X is *determined by almost closed subspaces* if for any non-closed $A \subset X$ and any $x \in \overline{A}\backslash A$ there is an almost closed $F \subset A$ such that $F \to x$. The concept sounds as a purely topological one. However it was first introduced in a paper [PT] which dealt with categorical topology. The relevant spaces were called AP-spaces and the explanation was "Approximation by Points". Unfortunately, there is no way to see what points have to do with the mentioned approximations, which in fact are approximations (of the closures) by almost closed sets. Of course, saying AC-space (from Almost Closed) instead of AP-space gives really nothing better provided that there are quite a few papers in which the term AP-space is accepted. We also use it here but the point is that it is worth to find a better name for so natural a concept. Maybe these spaces could be called apy spaces (from ape) which still has nothing to do with the intuitive perception, but sounds the same and is fancy.

Another method to determine a topology by almost closed sets is to say that a subset A of a space X is closed if and only if $\overline{F} \subset A$ for any almost closed $F \subset A$. Such topologies could be called *weakly determined by almost closed sets.* The relevant spaces were introduced by P. Simon in [Si] and studied intensively under the name WAP-spaces from "Weak Approximation by Points". This paper is devoted to study AP-spaces and WAP-spaces (apy? and wapy? spaces).

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In [BY] it was proved that any compact AP-space is Fréchet-Urysohn. We prove that the same is true for countably compact AP-spaces. The properties AP and WAP are respective generalizations of Fréchet-Urysohn property and sequentiality, so it is natural to see how theorems about Fréchet-Urysohn property and sequentiality can be strengthened to be proved for AP- or WAP-spaces. For example, any sequential $C_p(X)$ is Fréchet-Urysohn. The analogous assertion cannot be proved for AP and WAP: in [BY] an example of a space X was given for which $C_p(X)$ is WAP but not AP. We prove here that if $C_p(X)$ is AP and X is a paracompact space, then X is a Hurewicz space. This result is new even for separable metric spaces. As a consequence, if $\mathbb{P}$ is the space of the irrationals, then $C_p(\mathbb{P})$ is a space with a countable network which is not AP. We prove that any stratifiable AP-space is M_1; this result has to do with the latest achievement of T. Mizokami and N. Shimane who proved, in particular, that any sequential stratifiable space is M_1. Of course, we refer the heavy part of the work to their paper [MSh] where a class (P) is introduced in which the coincidence $(M_1 = M_3)$ takes place and observe that AP-spaces are inside (P).

Until today nobody seemed to have cared whether any WAP-space is hereditarily WAP. The question turned out not to be so easy. We give a counterexample under CH and it is not clear at all whether a WAP-space which is not hereditarily WAP, exists in ZFC. The last result which we would like to mention is Theorem 2.7 which says that any scattered space is WAP. This, together with a theorem of A. Bella [Be] implies a well known result of Mrówka, Rajagopalan and Soundararajan [MRS] that any compact scattered space is pseudoradial. Another consequence is that there is a plenty of hereditarily WAP-spaces which need not necessarily be AP.

1. Notation and terminology

All spaces under consideration are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. As usual, $C_p(X)$ is the set of real-valued continuous functions on X endowed with the topology of pointwise convergence. This means that $C_p(X)$ has the subspace topology induced from $\mathbb{R}^X$.

If X is a space and $A \subset X$ then the AP-closure of A in X is the set $A \cup \{x \in \overline{A}\backslash A :$ there is an almost closed $F \subset A$ such that $F \to x\}$. A set is AP-closed if it is equal to its AP-closure. A space X is AP, if the AP-closure of any $A \subset X$ coincides with $\overline{A}$; the space X is WAP, if any AP-closed set in X is closed in X. A space X is *Fréchet-Urysohn* if for any $A \subset X$ and any $x \in \overline{A}$ there is a sequence $S \subset A$ which converges to x. The space X is *sequential*, if for any non-closed $A \subset X$ there is a sequence $S \subset A$ which converges to some $x \in \overline{A}\backslash A$. The space X is *discretely generated* if for any $A \subset X$ and any $x \in \overline{A}$ there is a discrete $D \subset A$ such that $x \in \overline{D}$. A space X is *scattered* if any subspace of X has an isolated point. It is said that X is a Hurewicz space if for any sequence $\{\gamma_n : n \in \omega\}$ of open covers of X, for each $n \in \omega$ there exists a finite subfamily $\mu_n \subset \gamma_n$ such that $\bigcup\{\mu_n : n \in \omega\}$ is a cover of X. Now, a space X is called *stratifiable* if for each closed $F \subset X$ and $n \in \omega$ a set $G(F,n) \in \tau(X)$ can be chosen so that $F \subset G(F,n)$, if $F \subset F'$ then $G(F,n) \subset G(F',n)$ for each $n \in \omega$, and $F = \bigcap\{\overline{G(F,n)} : n \in \omega\}$. Stratifiable spaces are also called M_3-spaces. A space is called M_1, if it has a σ-closure preserving base. A dense-in-itself space X is *submaximal* if all dense subsets of X are open in X. If X has no isolated points, but any stronger topology on X is not dense-in-itself, it is called *maximal*.

The rest of the notation is standard and follows [En].

2. Some known and some new facts about AP- and WAP-spaces

Although quite a few nontrivial statements have been proved about these two classes, there are still some basic questions which seemed to never have been asked about AP and WAP-spaces. Our first proposition is a compilation of most trivial facts about these concepts. Surely, all they are known by the specialists while some have never been formulated explicitly.

2.1 Proposition. (1) *Any subspace of an AP-space is an AP-space;*

(2) *any closed subspace of a WAP-space is a WAP-space;*

(3) *any AP-space is a WAP-space but not vice versa;*

(4) *any Fréchet-Urysohn space is an AP-space;*

(5) *any sequential space is a WAP-space;*

(6) *any sequential AP-space is Fréchet-Urysohn;*

(7) *any closed continuous image of an AP-space is an AP-space;*

(8) *any closed continuous image of an WAP-space is a WAP-space;*

(9) *a quotient image of an AP-space is not necessarily an AP-space;*

(10) *any space with a unique non-isolated point is AP;*

(11) *a non-WAP-space can be a union of two subspaces each one of which is AP;*

(12) *any infinite compact WAP-space has a nontrivial convergent sequence; hence the space $\beta\omega\backslash\omega$ is not WAP;*

(13) *it is independent of ZFC whether every countably tight compact space is a WAP-space.*

Proof: The properties (1)–(5) are absolutely evident. Let us prove (6). If X is a sequential AP-space, take any $A \subset X$ and $x \in \overline{A}\backslash A$. By AP property, there is an almost closed $F \subset A$ with $F \to x$. Since F is non-closed, there is a convergent sequence $S \subset F$ whose limit is some point y outside of F. Of course, y has to coincide with x and hence S is a sequence in A which converges to x.

To prove (7) and (8) assume that X is an AP-space (WAP-space) and $f : X \to Y$ is a closed continuous onto map. Suppose that $A \subset Y$ and $x \in \overline{A}\backslash A$. If $B = f^{-1}(A)$ then $f^{-1}(x) \cap \overline{B} \neq \emptyset$ (and therefore the set B is not closed in X) and thus there is an almost closed $C \subset B$ with $y \in \overline{C}$ for some $y \in f^{-1}(x)$

(for some $y \in X \backslash B$ respectively). It is easy to check that $D = f(C)$ is an almost closed subset of A and $x \in \overline{D}$ (or $f(y) \in \overline{D} \backslash A$ respectively).

To prove (9), take any sequential non-Fréchet-Urysohn space X. Then X is a quotient image of a metric space M ([Ar1]). Since any sequential AP-space is Fréchet-Urysohn by (6), we can conclude that a quotient image of a metric space is not necessarily AP. The item (10) is evident because in a space with a unique nonisolated point any non-closed set is almost closed.

Now, let D be the countable maximal space constructed by van Douwen [vD]. It was proved in [BY] that $X = D \times (\omega + 1)$ is not WAP. However, $X = Y \cup Z$ where $Y = D \times \{\omega\}$ and $Z = X \backslash Y$. The space Y is homeomorphic to D which is submaximal and hence AP ([BY]). The space Y is homeomorphic to a free union of countably many copies of D and hence is also submaximal. Thus a non-WAP-space X is representable as a union of two AP-subspaces, which proves (11).

To show that (12) holds, take any infinite compact WAP-space X. If X is scattered, then denote by I the set of isolated points of X. The set $X \backslash I$ is not empty and hence it has an isolated point x. Take a closed neighborhood V of the point x such that $V \cap ((X \backslash I) \backslash \{x\}) = \emptyset$. Then V is a compact space with the only non-isolated point x. Hence X a nontrivial convergent sequence with limit x. Now if X is not scattered, then it has a countably infinite dense in itself subspace Y. If Y is closed then it is an infinite metrizable compact space and hence has a non-trivial convergent sequence. If not, then there is an almost closed $F \subset Y$ such that $F \to x \in X \backslash Y$. The space $F \cup \{x\}$ is an infinite countable compact space. Therefore it is metrizable and has a non-trivial convergent sequence. This settles (12).

Finally, to see that (13) is true, observe that there are models of ZFC in which every countably tight compact space is sequential and hence WAP ([Ba]). On the other hand, under the Jensen's axiom $\diamondsuit$, Fedorchuk constructed in [Fe] a compact countably tight space X without convergent sequences. This space is not WAP by (12). $\square$

2.2 Theorem. *Let X be a countably compact AP-space. Then X is a Fréchet-Urysohn space.*

Proof: Assume that $A \subset X$ and $x \in \overline{A} \backslash A$. Fix an almost closed $P \subset A$ such that $x \in \overline{P}$. Take a maximal disjoint family γ of open subsets of P whose closures do not contain x. Then $x \in \overline{\bigcup \gamma}$ and hence there is an almost closed $Q \subset \bigcup \gamma$ such that $x \in \overline{Q}$. Let $\gamma' = \{U \in \gamma : U \cap Q \neq \emptyset\}$. It is easy to see that γ' is infinite. For each $U \in \gamma'$ take an $x_U \in U \cap Q$. The set $B = \{x_U : U \in \gamma'\}$ is discrete and $(\overline{B} \backslash B) \cap (\bigcup \gamma) = \emptyset$. This means B is closed in Q and therefore x has to be the only accumulation point of B in the countably compact space $Q \cup \{x\}$. Therefore $B \cup \{x\}$ is a countably compact infinite space with the only non-isolated point x. Such a space must be compact and hence there is a sequence in B which converges to x. $\square$

2.3 Corollary. *If X is a k-space which is AP, then it is Fréchet-Urysohn. In particular, any Čech-complete AP-space is Fréchet-Urysohn.*

PROOF: Let $A \subset X$ and $x \in \overline{A}$. Find an almost closed $F \subset A$ such that $x \in \overline{F}$. Since F is not closed, there is a compact $K \subset X$ with $K \cap F$ non-closed and hence $x \in \overline{K \cap F}$. By Theorem 2.2 the space K is Fréchet-Urysohn, so there is a sequence $S \subset K$ which converges to x. Clearly, $S \subset A$ and we are done. $\square$

The following result shows that there is some strange situation with an evident question about WAP-spaces. The question is whether any subspace of a WAP-space is a WAP-space. All authors remark that it is so for closed subspaces and there is an absolute silence about arbitrary ones. It turned out that it is difficult to construct a WAP-space which is not hereditarily WAP. At least, we only could do it under CH and the example is not evident at all. We will see later that the class of hereditarily WAP-spaces is pretty large, it contains, in particular, all scattered spaces.

2.4 Example. There exists a countably compact WAP-space X such that under the Continuum Hypothesis there is a dense $Y \subset X$ such that Y is not WAP.

PROOF: Denote by $\mathbb{D}$ the discrete two-point space. Our space X will be a dense subspace of $\mathbb{D}^{\omega_1}$. Let $\Sigma(0) = \{x \in \mathbb{D}^{\omega_1} : |x^{-1}(1)| \leqslant \omega\}$ and $\Sigma(1) = \{x \in \mathbb{D}^{\omega_1} : |x^{-1}(0)| \leqslant \omega\}$. Now the set $X = \Sigma(0) \cup \Sigma(1)$ is as promised. It is evident that X is a countably compact space of uncountable tightness and hence not AP by Theorem 2.2. Let us show that X is WAP.

Suppose that A is an AP-closed non-closed subset of X. Consider the sets $A_0 = A \cap \Sigma(0)$ and $A_1 = A \cap \Sigma(1)$. If some A_i is not closed in the Fréchet-Urysohn space $\Sigma(i)$, then there is a convergent sequence $S \subset A_i$ such that $S \to x$, where $x \in \Sigma(i)\backslash A_i$. Of course, $x \notin A$ and S is an almost closed set with $S \to x$. Thus, A is not AP-closed in X, a contradiction. This proves that A_i is closed in $\Sigma(i)$ for each $i = 0, 1$. This implies that each A_i is countably compact and for any $x \in \overline{A}\backslash A$ we have $x \in \Sigma(i)$ and $x \in \overline{A}_{1-i}$ for some $i \in \{0, 1\}$. We can assume without loss of generality that there is an $x \in \Sigma(0)\backslash A$ such that $x \in \overline{A}_1$. Fix some local base $\{U_\alpha : \alpha < \omega_1\}$ at the point x. Since A_1 is countably compact, for each $\beta < \omega_1$ it is possible to choose a point $x_\beta \in \bigcap\{\overline{U}_\alpha : \alpha < \beta\} \cap A_1$.

The set $F = \overline{\{x_\beta : \beta < \omega_1\}}\backslash\{x\}$ contains x in its closure and therefore it is almost closed in X. Observe that the transfinite sequence $S = \{x_\beta : \beta < \omega_1\}$ converges to x which implies that any point of F is in the closure of some initial segment of S. Since all initial segments of S are countable and the closure of any countable subset of $\Sigma(1)$ lies in $\Sigma(1)$, we have $F \subset \Sigma(1)$. Using once more Fréchet-Urysohn property of $\Sigma(1)$ we can see that the closure of any countable subset of A_1 is contained in A_1. As a consequence, $F \subset A_1$ is an almost closed subset of A such that $F \to x \in X\backslash A$. Thus, X is a WAP-space. Note that we constructed the space X and proved that it is WAP without using any set-theoretic assumptions outside of ZFC.

Now assume that CH holds. It was proved in [ASh] that under CH the space $\Sigma(0)$ has a dense Luzin subspace L. "Luzin" means, that all nowhere dense subspaces of L are countable. Let $Y = \Sigma(1) \cup L$. We claim that Y is not a WAP-space. To establish this, it suffices to show that L is AP-closed in Y. Of course, L is not closed since it is a proper dense subspace of Y.

Assume that $F \subset L$ is an almost closed subspace of Y with $F \to x \in \Sigma(1)$. Clearly, F is closed in L. If it contains some non-empty set $U \in \tau(L)$ then there is a set $V \in \tau(\mathbb{D}^{\omega_1})$ such that $V \cap L = U$. Since L is dense in $\mathbb{D}^{\omega_1}$ the set F is dense in V and therefore all points of an infinite set $V \cap \Sigma(1)$ are in the closure of F which is supposed to have only x in its closure. This contradiction shows that F is nowhere dense in L and hence countable. But the closure of any countable subset of $\Sigma(0)$ is contained in $\Sigma(0)$ so $x \notin \overline{F}$, a contradiction. □

The following result shows that AP-spaces are useful in what concerns the $M_3 = M_1$ problem.

2.5 Theorem. *Suppose that X is a stratifiable AP-space. Then X is an M_1-space.*

Proof: We are going to use a recent result of Mizokami and Shimane [MSh] which states that if a stratifiable space X has a property, they call (P), then X is M_1. A space X has the property (P) if for every $x \in X$ and every open $U \subset X$ such that $x \in \overline{U} \setminus U$ there exists a closure preserving closed network $\mathcal{N}(x, U)$ at the point x such that $\overline{G \cap U} = G$ for each $G \in \mathcal{N}(x, U)$.

In [MSh] it was proved that every sequential space has the property (P).

To see that every AP-space has property (P) take any open $U \subset X$ and any $x \in \overline{U} \setminus U$. There exists an almost closed $P \subset U$ with $x \in \overline{P}$. Apply Theorem 1 of [Gr] to fix a closed closure preserving quasi-base $\mathcal{B}$ at the point x. We claim that the family $\mathcal{N}(x, U) = \{\{x\} \cup (B \cap P) : B \in \mathcal{B}\}$ is the desired network at x. It is clear that $\mathcal{N}(x, U)$ is a network at x. If $G \in \mathcal{N}(x, U)$ then $G = \{x\} \cup (B \cap P)$ for some $B \in \mathcal{B}$. Since B is an element of a quasi-base, we have $x \in \operatorname{Int}(B)$. Thus, $x \in \overline{P \cap B}$ and $G = \{x\} \cup (G \cap U) \subset \overline{G \cap U}$.

Observe that any closure preserving family restricted to a closed set is closure preserving, which shows that $\mathcal{N}(x, U)$ is closure preserving and we are done. □

2.6 Remark. The property AP does not imply sequentiality in stratifiable spaces. To see this, note that any countable space with a unique non-isolated point is M_1 as well as AP but not necessarily sequential. On the other hand not all M_1-spaces are AP: a relevant example is the space $X = V(\omega) \times (\omega + 1) \times (\omega + 1)$, where $V(\omega)$ is the Fréchet-Urysohn fan. In [BY] it was proved that this space is not AP. However it is stratifiable being is a product of three stratifiable spaces. Since any countable stratifiable space is M_1 the space X is M_1. Finally, there are countable

AP-spaces which are not stratifiable: any submaximal countable space will do ([BY]).

2.7 Theorem. *Any scattered space is WAP.*

PROOF: Let X be a scattered space. For any subset $Y \subset X$ denote by $i(Y)$ the set of isolated points of the subspace Y. Assume that a non-closed set $A \subset X$ is AP-closed in X. Let $A_0 = i(A)$ and $B_0 = \overline{A}_0 \backslash A_0$. Evidently, A_0 is dense in A and B_0 is closed in X. If we have the sets A_α and B_α, let $A_{\alpha+1} = i(B_\alpha)$ and $B_{\alpha+1} = B_\alpha \backslash A_{\alpha+1}$. If β is a limit ordinal and we have A_α, B_α for each $\alpha < \beta$, we let $A_\beta = \emptyset$ and $B_\beta = \bigcap\{B_\alpha : \alpha < \beta\}$.

Since the space X is scattered, there exists an ordinal γ such that $\overline{A}_0 = \overline{A} = \bigcup\{A_\alpha : \alpha < \gamma\}$. Let $\beta = \min\{\alpha < \gamma : A_\alpha \cap (\overline{A} \backslash A) \neq \emptyset\}$. Then β is a non-limit ordinal, $\beta = \beta_0 + 1$.

Pick a point $x \in A_\beta \backslash A = i(B_{\beta_0}) \backslash A$. Since x is isolated in B_{β_0} there is an open $U \subset X$ such that $U \cap B_{\beta_0} = \{x\}$. Take any $V \in \tau(X)$ with $x \in V \subset \overline{V} \subset U$ and consider the set $F = (\overline{V} \cap A) \backslash \{x\}$. Since $x \in \overline{A}$ we have $x \in \overline{F}$. We claim that F is an almost closed set. Indeed, $\overline{F} \subset \overline{A} = \bigcup\{A_\alpha : \alpha \leqslant \beta_0\} \cup B_{\beta_0}$ and $\overline{F} \cap B_{\beta_0} \subset \overline{V} \cap B_{\beta_0} = \{x\}$. Therefore $\overline{F} \backslash \{x\} \subset \bigcup\{A_\alpha : \alpha \leqslant \beta_0\} \subset A$. It turns out that x is in the AP-closure of A, a contradiction. □

2.8 Corollary ([MRS])**.** *Any scattered compact space is pseudoradial.*

PROOF: We only need to apply Theorem 2.7 and A. Bella's result ([Be]) which says that any compact WAP-space is pseudoradial. □

2.9 Corollary. *Any scattered space is hereditarily WAP.*

It is known ([Ar2]), that any hereditarily sequential space is Fréchet-Urysohn and any hereditarily pseudoradial space is radial. The following result shows that the properties AP and WAP behave differently.

2.10 Corollary. *There exist hereditarily WAP-spaces which are not AP-spaces.*

PROOF: The space $\omega_1 + 1$ is not AP ([PT]). Being scattered, it is hereditarily WAP by Corollary 2.9. □

3. The spaces $C_p(X)$ and AP-property

We are going to prove that for paracompact X the AP-property of $C_p(X)$ implies X is Hurewicz. Another interesting feature of spaces $C_p(X)$ which have countable tightness and AP-property is that they are discretely generated, i.e., the closures are determined by the closures of discrete subsets.

3.1 Proposition. *Let $C_p(X)$ be an AP-space. Then any discrete family $\gamma \subset \tau^*(X)$ is countable.*

PROOF: Suppose that it is not so and fix a discrete family $\gamma = \{U_\alpha : \alpha < \omega_1\}$ of non-empty open subsets of X. Pick a point $x_\alpha \in U_\alpha$ for each $\alpha < \omega_1$ and choose a function $f_\alpha \in C_p(X)$ such that $f_\alpha(x_\alpha) = 1$ and $f_\alpha|(X\backslash U_\alpha) \equiv 0$. Given any function $f : A \to \mathbb{R}$, where $A = \{x_\alpha : \alpha < \omega_1\}$, let $\varphi(f)(x) = \sum\{f(x_\alpha) \cdot f_\alpha(x) : \alpha < \omega_1\}$ for every $x \in X$. It is a routine and standard procedure to verify that $\varphi(f) \in C_p(X)$ and $\varphi : \mathbb{R}^A \to C_p(X)$ is an embedding. As a consequence, $\mathbb{R}^{\omega_1}$ embeds into $C_p(X)$ and therefore it is an AP-space which is a contradiction with the fact that a non-AP-space $\omega_1 + 1$ embeds in $\mathbb{R}^{\omega_1}$. □

3.2 Corollary. *If X is paracompact and $C_p(X)$ is an AP-space, then X is Lindelöf.*

3.3 Lemma. *Suppose that X is normal and $C_p(X)$ is an AP-space. Assume that we have a sequence $\{\gamma_n : n \in \omega\}$ of open covers of X with the following properties:*

(1) *$\gamma_n = \{U^n_m : m \in \omega\}$ and $U^n_m \subset U^n_{m+1}$ for each $m \in \omega$;*
(2) *for each $n \in \omega$ there exists a closed cover $\mu_n = \{F^n_m : m \in \omega\}$ of the space X such that $F^n_m \subset U^n_m$ and $F^n_m \subset F^n_{m+1}$ for all $m \in \omega$.*

Then it is possible to choose $W_n \in \gamma_n$ for each $n \in \omega$ in such a way that $\{W_n : n \in \omega\}$ is an ω-cover of X.

Proof: For each pair (m, n) of natural numbers, choose an $f^n_m \in C_p(X)$ such that $f^n_m|F^n_m \equiv \frac{1}{n}$ and $f^n_m|(X\backslash U^n_m) \equiv 1$. It is clear that the sequence $S_n = \{f^n_m : m \in \omega\}$ converges to the function $h_n \equiv \frac{1}{n}$. As a consequence, the function $h \equiv 0$ is in the closure of the set $S = \bigcup\{S_n : n \in \omega\}$. Apply the AP-property of the space $C_p(X)$ to find an almost closed $F \subset S$ such that $h \in \overline{F}$. Observe that for any natural n the set $F_n = F \cap S_n$ cannot be infinite because otherwise $h_n \in \overline{F}\backslash F$. Therefore, for each $n \in \omega$ we have a natural $m(n)$ such that $F_n \subset \{f^n_m : m \leqslant m(n)\}$. For each $n \in \omega$ let $W_n = U^n_{m(n)}$. We claim that the family $\{W_n : n \in \omega\}$ is an ω-cover of X. Indeed, let K be a finite subset of X. Since $h \in \overline{F}$, there exists an $f^n_m \in F$ such that $f^n_m(x) < 1$ for every $x \in K$ and therefore $K \cap (X\backslash U^n_m) = \emptyset$. Consequently, $K \subset U^n_m \subset U^n_{m(n)} = W_n$ and we are done. □

3.4 Theorem. *Suppose that $C_p(X)$ is an AP-space and X is paracompact. Then X is a Hurewicz space.*

Proof: Apply Corollary 3.2 to conclude that X is a Lindelöf space. Let $\{\lambda_n : n \in \omega\}$ be a sequence of open covers of the space X. Since X is Lindelöf, without loss of generality, we may assume that each λ_n is countable; let $\{W^n_m : m \in \omega\}$ be an enumeration of λ_n for each natural n. Define $U^n_m = \bigcup\{W^n_i : i \leqslant m\}$ for all $n, m \in \omega$. It is clear that the family $\gamma_n = \{U^n_m : m \in \omega\}$ is a cover of the space X and $U^n_m \subset U^n_{m+1}$ for all $m, n \in \omega$.

It is a standard fact for Lindelöf spaces (see [En, 3.8.11]) that for each $n \in \omega$ there exists a precise closed shrinking $\{G^n_m : m \in \omega\}$ of the cover λ_n, i.e., $\{G^n_m : m \in \omega\}$ is a closed cover of X and $G^n_m \subset W^n_m$ for all $n, m \in \omega$. Now if $F^n_m = \bigcup\{G^n_i : i \leqslant m\}$, then the covers $\gamma_n = \{U^n_m : m \in \omega\}$ and $\mu_n = \{F^n_m : m \in \omega\}$

satisfy the hypothesis of Lemma 3.3. Therefore we can choose a $W_n \in \gamma_n$ so that $\{W_n : n \in \omega\}$ is a cover of X. Since each W_n is covered by finitely many elements of λ_n, there exist finite families $\nu_n \subset \lambda_n$ such that $\bigcup\{\nu_n : n \in \omega\} = X$. □

3.5 Corollary. *If X is a metrizable space for which $C_p(X)$ is AP, then X is separable and has Hurewicz property.*

3.6 Corollary. *If $\mathbb{P}$ is the space of the irrational numbers with its natural topology, then $C_p(\mathbb{P})$ is not an AP-space.*

3.7 Theorem. *Suppose that $C_p(X)$ is AP and has countable tightness. Then $C_p(X)$ is discretely generated in the sense that for every $A \subset C_p(X)$ and every $f \in \overline{A}$ there exists a discrete $D \subset A$ such that $f \in \overline{D}$.*

Proof: There will be no loss of generality if we assume that f is identically zero and $f \in \overline{A}\backslash A$. Since $C_p(X)$ is AP and countably tight, we may restrict ourselves to the case when A is countable and almost closed. Let $\{f_n : n \in \omega\}$ be some enumeration of A. There exists a sequence $\{\varepsilon_n : n \in \omega\}$ of positive reals such that $\varepsilon_n \to 0$ and $f_i + \varepsilon_i \neq f_j$, $f_i + \varepsilon_i \neq f$ for any distinct $i, j \in \omega$. Consider the set $B = \{g_n : n \in \omega\}$ where $g_n = f_n + \varepsilon_n$ for each $n \in \omega$. We claim that $f \in \overline{B}$. Indeed, let $U = [f, x_1, \dots, x_l, \varepsilon] = \{g \in C_p(X) : |g(x_i) - f(x_i)| < \varepsilon$ for all $i \leqslant l\}$ be a basic neighborhood of f. Since $f \in \overline{A}\backslash A$, the set $M = \{n \in \omega : f_n \in [f, x_1, \dots, x_l, \frac{\varepsilon}{2}]\}$ is infinite. Then for any $n \in M$ with $\varepsilon_n < \frac{\varepsilon}{2}$ we have $|g_n(x_i) - f(x_i)| = |f_n(x_i) + \varepsilon_n - f(x_i)| \leqslant |f_n(x_i) - f(x_i)| + \varepsilon_n < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $i \leqslant l$ and therefore $g_n \in U$.

Since $C_p(X)$ is AP and $f \in \overline{B}\backslash B$, there is an almost closed $P = \{g_{n_k} : k \in \omega\} \subset B$ such that $f \in \overline{P}\backslash P$. Observe that the subspace $D = \{f_{n_k} : k \in \omega\}$ is discrete. Indeed, for any $k \in \omega$ we have $f_{n_k} \notin P$ and hence there is a standard open set $W = [f_{n_k}, x_1, \dots, x_l, \varepsilon]$ such that $W \cap P = \emptyset$. If there are infinitely many $i \in \omega$ such that $f_{n_i} \in V = [f_{n_k}, x_1, \dots, x_l, \frac{\varepsilon}{2}]$ then there is a natural $i \neq k$ for which $\varepsilon_{n_i} < \frac{\varepsilon}{2}$. As a consequence, $|g_{n_i}(x_p) - f_{n_k}(x_p)| = |f_{n_i}(x_p) - f_{n_k}(x_p) + \varepsilon_i| < \varepsilon$ for all $p \leqslant l$ which implies $g_{n_i} \in W \cap P$, a contradiction. Thus, V is a neighborhood of f_{n_k} which intersects only finitely many elements of D. Therefore D is discrete and we only have to prove that $f \in \overline{D}$.

Let $U = [f, x_1, \dots, x_l, \varepsilon]$ be a basic neighborhood of f. Since $f \in \overline{P}\backslash P$, the set $N = \{k \in \omega : g_{n_k} \in [f, x_1, \dots, x_l, \frac{\varepsilon}{2}]\}$ is infinite. Pick any $k \in N$ such that $\varepsilon_{n_k} < \frac{\varepsilon}{2}$. Then $|f_{n_k}(x_i) - f(x_i)| = |g_{n_k}(x_i) - f(x_i) - \varepsilon_{n_k}| < \varepsilon$ for all $i \leqslant l$ which shows that $f_{n_k} \in U$. □

3.8 Corollary. *Any countable submaximal space Y is an AP-space which cannot be embedded into a $C_p(X)$ which is an AP-space.*

Proof: In [BY] it was proved that any submaximal space is AP and hence so is Y. It is evident that discrete generability is hereditary. It is also a standard fact that if a countable space embeds into a $C_p(Z)$ then there is a second countable space

X such that Y embeds into $C_p(X)$ and the latter one embeds in $C_p(Z)$. Since the tightness of $C_p(X)$ is countable, we may apply Theorem 3.7 to conclude that Y has to be discretely generated, which is impossible because in a submaximal space all discrete subspaces are closed. □

3.9 Remark. Under Martin's axiom there exists a maximal (and hence submaximal) topological group ([Ma]). Thus, it cannot be asserted in ZFC that a submaximal space cannot be embedded into a topological group which is an AP-space.

3.10 Remark. There exist AP-spaces which are not discretely generated: the van Douwen's maximal countable space V [vD] is an example because every submaximal space is AP ([BY]) and all discrete subspaces of V are closed. On the other hand, any sequential non-Fréchet-Urysohn space is discretely generated ([DTTW]) and fails to be AP by Proposition 2.1(6).

4. Open problems

Still there are some very natural questions left open. Here is the list.

4.1 Problem. *Does there exist in ZFC a WAP-space X such that for some $Y \subset X$ the subspace Y is not WAP?*

4.2 Problem. *Is it true that every subspace of a sequential space is WAP?*

4.3 Problem. *Suppose that $C_pC_p(X)$ is AP. Is it true that X is finite?*

4.4 Problem. *Let X be a WAP stratifiable space. Is it true that X is M_1?*

4.5 Problem. *Is $C_p(\mathbb{R}^\omega)$ a WAP space?*

4.6 Problem. *Is an open image of an AP space an AP-space? How about open images of WAP-spaces?*

4.7 Problem. *Is any pseudocompact AP-space Fréchet-Urysohn?*

4.8 Problem. *Suppose that $C_p(X)$ is an AP-space. Is $C_p(X)$ necessarily discretely generated?*

4.9 Problem. *Suppose X is a second countable space and $C_p(X)$ is AP. Are all finite powers of X Hurewicz spaces?*

4.10 Problem. *Suppose that $C_p(X) = A \cup B$, where A and B are AP-spaces. Is it true that $C_p(X)$ is AP?*

4.11 Problem. *Suppose that $C_p(X)$ is an AP-space. Is it true that $(C_p(X))^\omega$ is also an AP-space?*

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