Multipoint boundary value problems for discrete equations

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Abstract. In this work we establish existence results for solutions to multipoint boundary value problems for second order difference equations with fully nonlinear boundary conditions involving two, three and four points. Our results are also applied to systems.

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1. Introduction

In this work we establish existence results for solutions to multipoint boundary value problems for second order difference equations with fully nonlinear boundary conditions involving two, three and four points. Our results are also applied to systems.

The problem under consideration is

$$y(i+1) - 2y(i) + y(i-1) = f(i, y(i)), \quad i = 1, \dots, n-1, \tag{1}$$

$$(0,0) = G(y(0), y(n), y(c), y(d)), \quad c, d \in \{1, \dots, n-1\}, \tag{2}$$

where f is continuous and nonlinear. The equation defining the boundary conditions G is continuous and may be nonlinear. Note that we use the backward difference $y(i) - y(i-1)$ for $i = 1, \dots, n$.

By a solution of (1) we mean a vector $\bar{y} = (y(0), \dots, y(n)) \in \mathbb{R}^{n+1}$ satisfying (1) for $i = 1, \dots, n-1$.

As stated by Agarwal in [1] and [2], the discretization of a two-point boundary value problem for ordinary differential equations gives rise to discrete problems and this can lead to certain fundamental changes. For example, two-point boundary conditions involving derivatives for the continuous case lead to three or four-point boundary conditions for the discrete problem and thus (2) deserves particular attention when $c = 1$ and $d = n - 1$. In [1] Agarwal provides some excellent examples illustrating that even though the continuous problem has a solution, its discrete analogue may not. Thus the question of existence of solutions to (1), (2) is naturally raised.

Motivated by [1], [2], [4], [5] and [6] in this paper we formulate some existence theorems for solutions to (1), (2).

The methods used throughout the work include discrete lower and discrete upper solutions, and we incorporate a degree-based relationship between the boundary conditions and the discrete lower and upper solutions, known as discrete compatibility. Once this method of discrete compatibility is introduced, most of the standard existence results in the literature for two-point boundary value problems will have analogues for discrete equations subject to four-point boundary conditions, provided that the appropriate assumptions concerning discrete lower and discrete upper solutions are made. The compatibility conditions are usually quite easy to identify.

As opposed to the approach of various fixed point theorems in [7] and [17], we employ degree theoretic arguments using homotopy methods.

Special cases of our theory include the boundary conditions

$$G = (y(0), y(n)) = (0,0), \tag{3}$$

$$G = (y(c) - y(0), y(n)) = (0,0),\ c \in \{1, \dots, n-1\}, \tag{4}$$

$$G = (y(c) - y(0), y(n) - y(d)) = (0,0),\ c, d \in \{1, \dots, n-1\}. \tag{5}$$

Our results also apply to the inhomogeneous cases of the above and moreover, our theory may be applied to the case where G is nonlinear.

We also apply our theory to systems of equations. These generalizations include the use of discrete lower and discrete upper solutions for discrete systems of equations.

For additional information regarding discrete equations we refer the reader to the excellent texts by Agarwal [3], Kelley and Peterson [11] or Lakshmikantham and Trigiante [13].

2. Definitions and preliminary results

We now introduce some notation. We denote the boundary of a set A by ∂A and the closure of A by $\bar{A}$. We denote $C(A; B)$ to be the space of continuous functions mapping from A to B endowed with the maximum norm. If $B = \mathbb{R}$ then we omit the B. For any vector $\bar{s} = \{s(i)\}_{i=0}^{n} \in \mathbb{R}^{n+1}$ we write $\bar{s} \le \bar{z}$ if $s(i) \le z(i)$ for all $i = 0, \dots, n$.

If A is a bounded, open subset of $\mathbb{R}^d$, $q \in \mathbb{R}^d$, $f \in C(\bar{A}; \mathbb{R}^d)$ and $q \notin f(\partial A)$ we denote the Brouwer degree of f at q relative to A by $d(f, A, q)$.

Our existence theorems require the use of discrete lower and discrete upper solutions.

Definition 1. We call $\bar{\alpha}$ ($\bar{\beta}$) a discrete lower (discrete upper) solution for (1) if

$$\begin{aligned} &\alpha(i+1) - 2\alpha(i) + \alpha(i-1) \ge f(i, \alpha(i)),\\ &(\beta(i+1) - 2\beta(i) + \beta(i-1) \le f(i, \beta(i)),) \ \text{ for all } \ i = 1, \dots, n-1. \end{aligned}$$

We say $\bar{\alpha}$ ($\bar{\beta}$) is a strict discrete lower (strict discrete upper) solution for (1) if the above inequalities are strict. We set $\Delta_d = (\alpha(0), \beta(0)) \times (\alpha(n), \beta(n))$ and we will refer to the pair $\bar{\alpha}$, $\bar{\beta}$ as nondegenerate if $\alpha(0) < \beta(0)$ and $\alpha(n) < \beta(n)$.

Existence proofs for boundary value problems commonly employ modifications on f. We will make the necessary modifications by using the following functions.

Definition 2. If $a \le b$ are given, let $\pi : \mathbb{R} \to [a, b]$ be (the retraction) given by $\pi(y, a, b) = \max\{\min\{b, y\}, a\}$.

Let $K \in C(\mathbb{R}; [-1, 1])$ satisfy

(i) $\quad K(t) = 1,\ t > 1,$

(iii) $\quad K(t) = t,\ -1 \le t \le 1,$

(iii) $\quad K(t) = -1,\ t < -1.$

If $a \le b$ are given, let $T \in C(\mathbb{R})$ be given by $T(y, a, b) = K(y - \pi(y, a, b))$. Let $l(i, y(i)) = f(i, \pi(y(i), \alpha(i), \beta(i)))$ for $i = 0, \dots, n$ and let

$$\begin{aligned} k(i, y(i)) &= (1 - |T(y(i), \alpha(i), \beta(i))|)\, l(i, y(i)) \\ &\quad + T(y(i), \alpha(i), \beta(i))\, (|l(i, y(i))| + 1)\,. \end{aligned}$$

Let

$$Q(i,j) = \begin{cases} (1-i)j, & \text{for } \ 0 \le j \le i \le n, \\ (1-j)i, & \text{for } \ 0 \le i \le j \le n, \end{cases}$$

and $\bar{w}(C, D)(i) = [C(n - i) + Di]/n$ for $i = 0, \dots, n$ where $C, D \in \mathbb{R}$. Define $\mathcal{C} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$\mathcal{C}(\bar{y})(i) = \sum_{j=1}^{n-1} Q(i,j) y(j), \quad \text{for } \ i = 0, \dots, n \ \text{ and } \quad \bar{y} \in \mathbb{R}^{n+1}.$$

Clearly $\mathcal{C}$ is continuous. Given $\bar{g} \in \mathbb{R}^{n+1}$ then $\bar{y}$ is a solution of

$$\begin{gathered} y(i+1) - 2y(i) + y(i-1) = g(i), \quad \text{for } \ i = 1, \dots, n-1, \\ y(0) = C, \quad y(n) = D, \end{gathered}$$

if and only if $\bar{y} = \mathcal{C}(\bar{g}) + \bar{w}(C, D)$.

3. Nonlinear boundary conditions

We now introduce the concept of discrete compatible boundary conditions for two, three and four points. The idea is a discrete analogue to the definition of compatibility for boundary value problems for ordinary differential equations from [15].

Definition 3. Let $G \in C(\bar{\Delta}_d \times \mathbb{R}^2; \mathbb{R}^2)$. We say G is strongly discrete compatible with $\bar{\alpha}$ and $\bar{\beta}$ if for all continuous functions $\phi_c : [\alpha(0), \beta(0)] \to [\alpha(c), \beta(c)]$ and $\phi_d : [\alpha(n), \beta(n)] \to [\alpha(d), \beta(d)]$ we have

$$G(C, D, \phi_c(C), \phi_d(D)) = \mathcal{G}(C, D) \neq (0,0) \quad \text{for all} \quad (C, D) \in \partial\Delta_d,$$
$$d(\mathcal{G}, \Delta_d, (0,0)) \neq 0.$$

The degree-based relationship between the boundary conditions and the discrete lower and discrete upper solutions which defines discrete compatibility applies to well-known boundary conditions already in the literature. Special cases include (3), (4) and (5) and moreover G may be nonlinear.

4. Existence of solutions

The following lemma mirrors standard results in the literature concerning solutions to the modified difference equation (see [3] or [5]).

Lemma 1. *Let $\bar{\alpha} \leq \bar{\beta}$ be nondegenerate discrete lower and upper solutions for* (1). *Let $k(i, y(i))$ be the modification to $f(i, y(i))$ given in Definition* 2. *Let $\bar{y} \in \mathbb{R}^{n+1}$ be any solution to*

$$y(i+1) - 2y(i) + y(i-1) = k(i, y(i)), \quad i = 1, \dots, n-1, \tag{6}$$

which satisfies $\alpha(0) \leq y(0) \leq \beta(0)$ and $\alpha(n) \leq y(n) \leq \beta(n)$. Then $\bar{y}$ satisfies $\bar{\alpha} \leq \bar{y} \leq \bar{\beta}$.

Proof: Now suppose $\bar{y}$ is a solution of (6). Suppose $y(i) < \alpha(i)$ for some $i \in \{0, \dots, n\}$. From our assumptions we may assume that $i \in \{1, \dots, n-1\}$ and that $\alpha(i) - y(i)$ attains its positive maximum at $i = m$ for some $m \in \{1, \dots, n-1\}$. Thus

$$\alpha(m+1) - 2\alpha(m) + \alpha(m-1) - [y(m+1) - 2y(m) + y(m-1)] \leq 0 \tag{7}$$

and $\alpha(m) - y(m) > 0$. Therefore

$$\begin{aligned}
&y(m+1) - 2y(m) + y(m-1) \\
&= k(m, y(m)) \\
&= (1 - |T(y(m), \alpha(m), \beta(m))|) l(m, y(m)) \\
&\quad + T(y(m), \alpha(m), \beta(m))(|l(m, y(m))| + 1) \\
&= (1 - |K(y(m) - \alpha(m))|) f(m, \alpha(m)) \\
&\quad + K(y(m) - \alpha(m))(|f(m, \alpha(m))| + 1) \\
&< f(m, \alpha(m)) \leq \alpha(m+1) - 2\alpha(m) + y(m-1)
\end{aligned}$$

which contradicts (7). Thus $\alpha(i) \leq y(i)$ for all $i = 0, \dots, n$ and similarly $\bar{y} \leq \bar{\beta}$. $\square$

Remark 1. It follows from Lemma 1 that any solution to (6) is also a solution to (1).

Lemma 2. *Let the assumptions of Lemma* 1 *hold. If* G *is strongly discrete compatible with* $\bar{\alpha}$ *and* $\bar{\beta}$ *then any solution* $\bar{y}$ *to* (6) *and* (2) *satisfies* $y(0) \neq \alpha(0)$, $\beta(0)$ *and* $y(n) \neq \alpha(n)$, $\beta(n)$.

PROOF: Assume $y(0) = \alpha(0)$. Let ϕ_c and ϕ_d be continuous functions as in Definition 3. Take $C = y(0) = \alpha(0)$; thus we have $\mathcal{G}(C, D) = \mathcal{G}(\alpha(0), D) = (0, 0)$ which is a contradiction to the compatibility conditions since $\mathcal{G} \neq (0, 0)$ on $\partial\Delta_d$. Thus $y(0) \neq \alpha(0)$. The other cases follow in a similar fashion. □

We now present our first existence theorem.

Theorem 1. *Let* $f \in C(\{0, 1, \dots, n\} \times \mathbb{R}; \mathbb{R})$. *Assume that there exist non-degenerate discrete lower and discrete upper solutions* $\bar{\alpha} \leq \bar{\beta}$ *for* (1) *and that* $G \in C(\bar{\Delta}_d \times \mathbb{R}^2; \mathbb{R}^2)$ *is strongly discrete compatible with* $\bar{\alpha}$ *and* $\bar{\beta}$. *Then there exists a solution* $\bar{y}$ *to problem* (1), (2) *with* $\bar{\alpha} \leq \bar{y} \leq \bar{\beta}$.

PROOF: Our proof draws on discrete variants of the ideas from [14] and [15]. Consider

$$y(i+1) - 2y(i) + y(i-1) = k(i, y(i)), \quad i = 1, \dots, n-1, \tag{8}$$

where $k(i, y(i))$ is the modification to $f(i, y(i))$ given in Definition 2. Consider (8) together with the boundary conditions (2). From Lemma 1 a solution, if it exists, actually lies in the region where f is unmodified and hence is a solution to (1), (2) as well.

Take $\bar{\alpha}_\varepsilon \in \mathbb{R}^{n+1}$ with $\alpha_\varepsilon(i) = \min\{\alpha(i) : i = 0, \dots, n\} - 1$ and $\bar{\beta}_\varepsilon \in \mathbb{R}^{n+1}$ with $\beta_\varepsilon(i) = \max\{\beta(i) : i = 0, \dots, n\} + 1$. Now

$$\begin{aligned} T(\alpha_\varepsilon(i), \alpha(i), \beta(i)) &= K(\alpha_\varepsilon(i) - \pi(\alpha_\varepsilon(i), \alpha(i), \beta(i))) \\ &= K(\alpha_\varepsilon(i) - \alpha(i)) = K(-1) = -1. \end{aligned}$$

Hence

$$\begin{aligned} \alpha_\varepsilon(i+1) - 2\alpha_\varepsilon(i) + \alpha_\varepsilon(i-1) = 0 &> -(|l(i, \alpha_\varepsilon(i)| + 1) \\ &= k(i, \alpha_\varepsilon(i)), \text{ for } i = 1, \dots, n-1. \end{aligned}$$

Thus $\bar{\alpha}_\varepsilon$ is a strict discrete lower solution for (8). Similarly, $\bar{\beta}_\varepsilon$ is a strict discrete upper solution for (8). We show problem (8) and (2) has a solution with $(y(0), y(n)) \in \bar{\Delta}_d$. Since f and k agree in this region this is the required solution of problem (1) and (2). Set

$$\Omega_\varepsilon = \{\bar{y} \in \mathbb{R}^{n+1} : \bar{\alpha}_\varepsilon < \bar{y} < \bar{\beta}_\varepsilon\}$$

and $\Gamma_\varepsilon = \Omega_\varepsilon \times \Delta_d$. Define $\mathcal{K} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$\mathcal{K}(\bar{y})(i) = k(i, y(i)), \quad \text{for} \quad i = 0, \dots, n \quad \text{and} \quad \bar{y} \in \mathbb{R}^{n+1}.$$

Let ϕ_c and ϕ_d be continuous functions as in Definition 3. Define $\mathcal{H} : \bar{\Gamma}_\varepsilon \times [0,1] \to \mathbb{R}^{n+3}$ by

$$\mathcal{H}(\bar{y}, C, D, \lambda) = \big(\bar{y} - 3\lambda \bar{w}(C, D) - (1 - 3\lambda)(\bar{\alpha}_\varepsilon + \bar{\beta}_\varepsilon)/2, \mathcal{G}(C, D)\big)$$

for $0 \le \lambda \le 1/3$,

$$\mathcal{H}(\bar{y}, C, D, \lambda) = (\bar{y} - 3(\lambda - 1/3)\mathcal{C}K(\bar{y}) - \bar{w}(C, D), \mathcal{G}(C, D))$$

for $1/3 \le \lambda \le 2/3$, and

$$\mathcal{H}(\bar{y}, C, D, \lambda) = (\bar{y} - \mathcal{C}K(\bar{y}) - \bar{w}(C, D), \mathcal{S}(\bar{y}, C, D, \lambda))$$

for $2/3 \le \lambda \le 1$, where

$$\mathcal{S}(\bar{y}, C, D, \lambda) = G(C, D, 3(\lambda - 2/3)(y(c), y(d)) + 3(1 - \lambda)(\phi_c(C), \phi_d(D))).$$

Clearly $\mathcal{H}$ is continuous.
It is easy to see that $\bar{y}$ is a solution of problem (8), (2) with $(\bar{y}, y(0), y(n)) \in \Gamma_\varepsilon$ if and only if

$$\mathcal{H}(\bar{y}, y(0), y(n), 1) = \mathbf{0}.$$

Now if there is a solution with $(\bar{y}, y(0), y(n)) \in \partial\Gamma_\varepsilon$ then there is nothing to prove so we assume there is no solution in $\partial\Gamma_\varepsilon$. We show $\mathcal{H}$ is an admissible homotopy for the Brouwer degree on Γ_ε at 0. We argue by contradiction and assume solutions exist to $\mathcal{H}(\bar{y}, C, D, \lambda) = \mathbf{0}$ with $\lambda \in [0,1]$ and $(\bar{y}, C, D) \in \partial\Gamma_\varepsilon$. We investigate the cases $\lambda \in [2/3, 1]$ and $[1/3, 2/3)$; the case $\lambda \in [0, 1/3)$ is trivial because $\mathcal{G}(C, D) \ne (0,0)$ with $(C, D) \in \partial\Delta_d$ and $\bar{y} - 3\lambda\bar{w}(C, D) - (1 - 3\lambda)(\bar{\alpha}_\varepsilon + \bar{\beta}_\varepsilon)/2 \ne \mathbf{0}$ for $\bar{y} \in \partial\Omega_\varepsilon$.

Case (i) $\lambda \in [2/3, 1]$.

By assumption there is no solution with $\lambda = 1$, so we assume there is a solution $(\bar{y}, C, D)$ with $\lambda \in [2/3, 1)$, $\bar{\alpha} \le \bar{y} \le \bar{\beta}$, $y(0) = C$ and $y(n) = D$.

Firstly, let us assume that $(C, D) \in \partial\Delta_d$. If $\alpha(0) = y(0)$ then we see that $\mathcal{S} \ne (0,0)$ which is a contradiction. The other cases follow similarly. Thus $(C, D) \notin \partial\Delta_d$.

Secondly, let us assume that $\bar{y} \in \partial\Omega_\varepsilon$. Assume $y(i) = \alpha_\varepsilon(i)$ for some $i \in \{0, \dots, n\}$. Since $y(0) \ge \alpha(0) > \alpha_\varepsilon(0)$ and $y(n) \ge \alpha(n) > \alpha_\varepsilon(n)$ it follows that $i \in \{1, \dots, n-1\}$. Thus $y(i+1) - 2y(i) + y(i-1) \ge \alpha_\varepsilon(i+1) - 2\alpha_\varepsilon(i) + \alpha_\varepsilon(i-1) = 0$. However, from the definition of k we have

$$y(i+1) - 2y(i) + y(i-1) = k(i, y(i)) = -(|f(i, \alpha(i))| + 1) < 0,$$

which is a contradiction. Thus $y(i) \neq \alpha_\varepsilon(i)$ for any $i \in \{0, \dots, n\}$. Similarly, the assumption $y(i) = \beta_\varepsilon(i)$ for some $i \in \{0, \dots, n\}$ also leads to a contradiction. Thus $\bar{y} \notin \partial\Omega_\varepsilon$ and there are no solutions of $\mathcal{H}(\bar{y}, C, D, \lambda) = \mathbf{0}$ with $\lambda \in [2/3, 1]$ and $(\bar{y}, C, D) \in \partial\Gamma_\varepsilon$.

Case (ii) $\lambda \in [1/3, 2/3]$.

Since G is strongly discrete compatible, from the previous lemma there are no solutions $(\bar{y}, C, D)$ to $\mathcal{H} = \mathbf{0}$ with $(C, D) \in \partial\Delta_d$. The proof of the case $\bar{y} \in \partial\Omega_\varepsilon$ leads to a contradiction in a similar way as for $\lambda \in [2/3, 1)$.

Thus $\mathcal{H}$ is an admissible homotopy for the degree and since $\mathcal{H}(\cdot, 0) = (I - \bar{b}, \mathcal{G})$ where I is the identity on $\mathbb{R}^{n+1}$ and $\bar{b} \in \Omega_\varepsilon$ is a constant vector, it follows that

$$\begin{aligned} d(\mathcal{H}(\cdot, 1), \Gamma_\varepsilon, \mathbf{0}) &= d(\mathcal{H}(\cdot, 0), \Gamma_\varepsilon, \mathbf{0}) \\ &= d(\mathcal{G}, \Delta_d, (0, 0)) \neq 0. \end{aligned}$$

Thus there is a solution $(\bar{y}, C, D) \in \Gamma_\varepsilon$ of $\mathcal{H}(\bar{y}, C, D, 1) = \mathbf{0}$ and hence a solution $\bar{y} \in \mathbb{R}^{n+1}$ to problem (1) and (2). This concludes our proof. □

Remark 2. If $G = (y(0), y(n)) = (0, 0)$ then Theorem 1 coincides with certain results in [2] and [5].

5. Some special boundary conditions

We now give conditions for discrete compatibility for special cases of G. The following is a discrete analogue of [15, Lemma 6].

Lemma 3. *Let $\bar{\alpha} \leq \bar{\beta}$ be nondegenerate discrete lower and discrete upper solutions respectively for* (1) *and let the boundary conditions be given by*

$$G = (y(c) - y(0), y(n) - y(d)) = (0, 0), \; c, d \in \{1, \dots, n-1\}. \tag{9}$$

Then G is strongly discrete compatible if

$$\alpha(0) < \alpha(c), \; \alpha(n) < \alpha(d), \; \beta(0) > \beta(c), \; \beta(n) > \beta(d). \tag{10}$$

Proof: Suppose (10) holds. Let $G = (G_0, G_1)$, where

$$G_0(C, D) = y(c) - C, \quad \text{and} \quad G_1(C, D) = D - y(d).$$

Let ϕ_c and ϕ_d be given as in Definition 3. Then

$$\mathcal{G}_0(C, D) = \phi_c(C) - C, \; \mathcal{G}_1(C, D) = D - \phi_d(D).$$

This implies

$$\mathcal{G}_0(\alpha(0), D) > 0, \; \mathcal{G}_0(\beta(0), D) < 0, \; \mathcal{G}_1(C, \alpha(n)) < 0, \; \mathcal{G}_1(C, \beta(n)) > 0$$

and, hence, $d(\mathcal{G}, \Delta_d, (0, 0)) = 1 \neq 0$. Thus G is strongly discrete compatible. □

Lemma 4. *Let $\bar{\alpha} \leq \bar{\beta}$ be nondegenerate discrete lower and discrete upper solutions, respectively, for* (1) *and let the boundary conditions be given by*

$$G = (y(c) - y(0), y(n) - y(d)) = (0,0), \; c, d \in \{1, \dots, n-1\}. \tag{11}$$

If G is strongly discrete compatible then

$$\alpha(0) \leq \alpha(c), \; \alpha(n) \leq \alpha(d), \; \beta(0) \geq \beta(c), \; \beta(n) \geq \beta(d). \tag{12}$$

Proof: Assume $\alpha(0) > \alpha(c)$. Let $\phi_c(C) = \alpha(c)$ and $\phi_d(D) = \alpha(d)$ for all $(C, D) \in \bar{\Delta}_d$. Then

$$\begin{aligned} \mathcal{G}(C,D) &= (\phi_c(C) - C, D - \phi_d(D)) \\ &= (\alpha(c) - C, D - \alpha(d)). \end{aligned}$$

This implies

$$d(\mathcal{G}, \Delta_d, (0,0)) = d(\mathcal{G}_0, (\alpha(0), \beta(0)), 0) d(\mathcal{G}_1, (\alpha(n), \beta(n)), 0) = 0,$$

since $\mathcal{G}_0(C) < 0$ for all $C \in [\alpha(0), \beta(0)]$ and thus $\alpha(0) \leq \alpha(c)$. Similarly, we get $\beta(0) \geq \beta(c)$, $\alpha(n) \leq \alpha(d)$ and $\beta(n) \geq \beta(d)$. □

6. Application to systems

We now apply some theory from the previous section to systems of equations. Consider

$$\vec{y}(i+1) - 2\vec{y}(i) + \vec{y}(i-1) = \vec{f}(i, \vec{y}(i)), \quad i = 1, \dots, n-1, \tag{13}$$

where $\vec{f} \in C(\{0, 1, \dots, n\} \times \mathbb{R}^d; \mathbb{R}^d)$ and $\vec{y} = (\vec{y}(0), \dots, \vec{y}(n)) \in \mathbb{R}^{(n+1)d}$.

For vectors $\vec{y}(i) \in \mathbb{R}^d$ we denote the k-th component by $y_k(i)$ for each $k = 1, \dots, d$.

Our first step is to extend the definitions of discrete lower and discrete upper solutions, following the ideas of [12] for the continuous case.

Definition 4. We call $\vec{\bar{\alpha}}$ ($\vec{\bar{\beta}}$) a discrete lower (discrete upper) solution for (13) if for each $i = 1, \dots, n-1$ and $k = 1, \dots, d$

$$\begin{aligned} \alpha_k(i+1) - 2\alpha_k(i) + \alpha_k(i-1) &\geq f_k(i, \vec{\sigma}(i)), \\ &\qquad \text{for all } \; \vec{\alpha}(i) \leq \vec{\sigma}(i) \; \text{ with } \; \sigma_k(i) = \alpha_k(i), \\ \beta_k(i+1) - 2\beta_k(i) + \beta_k(i-1) &\leq f_k(i, \vec{\sigma}(i)), \\ &\qquad \text{for all } \; \vec{\beta}(i) \geq \vec{\sigma}(i) \; \text{ with } \; \sigma_k(i) = \beta_k(i). \end{aligned}$$

Theorem 2. *Let $\vec{f} \in C(\{0,1,\dots,n\} \times \mathbb{R}^d; \mathbb{R}^d)$. Assume that there exist non-degenerate discrete lower and discrete upper solutions $\vec{\alpha} \le \vec{\beta}$ for (13) and that $G \in C(\bar{\Delta}_d \times \mathbb{R}^{2d}; \mathbb{R}^{2d})$ is strongly discrete compatible with $\vec{\alpha}$ and $\vec{\beta}$. Then there exists a solution $\vec{y}$ to problem (13), (2) with $\vec{\alpha} \le \vec{y} \le \vec{\beta}$.*

PROOF: The proof follows similar lines as that of Theorem 1 and is hence omitted. $\square$

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