Countable compactness and p-limits

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Abstract. For $\emptyset \neq M \subseteq \omega^*$, we say that X is quasi M-compact, if for every $f : \omega \to X$ there is $p \in M$ such that $\overline{f}(p) \in X$, where $\overline{f}$ is the Stone-Čech extension of f. In this context, a space X is countably compact iff X is quasi ω^*-compact. If X is quasi M-compact and M is either finite or countable discrete in ω^*, then all powers of X are countably compact. Assuming CH, we give an example of a countable subset $M \subseteq \omega^*$ and a quasi M-compact space X whose square is not countably compact, and show that in a model of A. Blass and S. Shelah every quasi M-compact space is p-compact (= quasi $\{p\}$-compact) for some $p \in \omega^*$, whenever $M \in [\omega^*]^{<\mathfrak{c}}$. We prove that if $\emptyset \notin \{T_\xi : \xi < 2^\mathfrak{c}\} \subseteq [\omega^*]^{<2^\mathfrak{c}}$, then there is a countably compact space X that is not quasi T_ξ-compact for every $\xi < 2^\mathfrak{c}$; hence, if $2^\mathfrak{c}$ is regular, then there is a countably compact space X such that X is not quasi M-compact for any $M \in [\omega^*]^{<2^\mathfrak{c}}$. We list some open problems.

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0. Introduction

All our spaces are Tychonoff. If $f : X \to Y$ is a continuous function, then $\overline{f} : \beta(X) \to \beta(Y)$ denotes the Stone-Čech extension of f. $\beta(\omega)$ is identified with the set of all ultrafilters on ω, and $\beta(\omega) \setminus \omega = \omega^*$ is the set of all free ultrafilters on ω. For $A \subseteq \omega$, $\hat{A} = \{p \in \beta(\omega) : A \in p\} = \mathrm{cl}_{\beta(\omega)} A$.

In the context of nonstandard analysis, the point $\overline{f}(p) \in X$, where $f : \omega \to X$ is a function and $p \in \omega^*$, has the following interpretation:

Definition 0.1 ([Be]). *Let $p \in \omega^*$ and let $(x_n)_{n<\omega}$ be a sequence in a space X. We say that x is the p-limit point of $(x_n)_{n<\omega}$, we write $x = p - \lim_{n\to\omega} x_n$, if for every neighborhood V of x, $\{n < \omega : x_n \in V\} \in p$.*

If $x = p - \lim_{n\to\omega} x_n$, then $x = \overline{f}(p)$, where $f : \omega \to X$ is defined by $f(n) = x_n$ for every $n < \omega$. It is known that, in the category of Tychonoff spaces, a space X is countably compact iff every sequence of points in X has a p-limit point for some $p \in \omega^*$: By using functions, X is countably compact iff for every $f : \omega \to X$ there is $p \in \omega^*$ such that $\overline{f}(p) \in X$. This last observation leads us to consider the following class of spaces.

Definition 0.2 ([Be]). *Let $p \in \omega^*$. A space X is said to be p-compact if for every sequence $(x_n)_{n<\omega}$ of points of X there is $x \in X$ such that $x = p - \lim_{n\to\omega} x_n$.*

Thus, a space X is p-compact, for $p \in \omega^*$, if $\overline{f}(p) \in X$ for every $f : \omega \to X$. It is shown in [GS] that all powers of a space X are countably compact iff there is $p \in \omega^*$ such that X is p-compact. A.R. Bernstein [Be] proved that p-compactness is preserved under arbitrary products, for every $p \in \omega^*$. Since countable compactness is not preserved under products, there are countably compact spaces which are not p-compact for any $p \in \omega^*$ (see [GJ]).

The following definition plays the main role in this paper:

Definition 0.3 ([G]). *Let $\emptyset \neq\subseteq \omega^*$. A space X is said to be quasi M-compact if for every $f : \omega \to X$ there is $p \in M$ such that $\overline{f}(p) \in X$.*

Thus, a space X is countably compact iff X is quasi ω^*-compact, and p-compactness agrees with quasi $\{p\}$-compactness. Given a countably compact space X, we may ask about that smallest cardinality of a nonempty subset $M \subseteq \omega^*$ such that for every $f : \omega \to X$ there is $p \in M$ such that $\overline{f}(p) \in X$. For instance, we mentioned above that if all the powers of a space X are countably compact, then set M may consist of just one single point. We show that if X is a countably compact space and one of its powers is not countably compact, then M cannot be neither finite and nor discrete. Under the assumption of CH, we give an Example of a countable subset M of ω^*, with one non-isolated point, and a countably compact space X such that X is quasi M-compact and fails to be p-compact for any $p \in \omega^*$. In a the models described in [BS1] and [BS2], we will prove that every quasi M-compact space is p-compact for some $p \in \omega^*$, provided that $M \subseteq \omega^*$ and $|M| < \mathfrak{c}$.

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1. Quasi M-compact spaces

Our first result is a particular case of Theorem 1.25 from [G].

Theorem 1.1. *Let $\emptyset \neq M \subseteq \omega^*$. If there is $f : \omega \to \omega$ and $p \in \omega^*$ such that $M \subseteq \overline{f}^{-1}(p)$, then every quasi M-compact space is p-compact.*

Proof: Let X be a quasi M-compact space and let $g : \omega \to X$ be a function. Consider the composition $g \circ f$. Since X is quasi M-compact, there is $r \in M$ such that $\overline{g}(\overline{f}(r)) \in X$ and then $\overline{g}(p) \in X$, because of $\overline{f}(r) = p$. Thus, X is p-compact. □

Theorem 1.2. *If X is quasi M-compact for some countable discrete subset $M \subseteq \omega^*$, then X is p-compact for some $p \in M$.*

Proof: Let $M \subseteq \omega^*$ be discrete and let X be a quasi M-space. Assume that X is not p-compact for any $p \in M$. Enumerate M as $\{p_n : n < \omega\}$ and let

$\{A_n : n < \omega\}$ be a partition of ω such that $A_n \in p_n$ for every $n < \omega$. By assumption, for every $n < \omega$, there is $f_n : \omega \to X$ such that $\overline{f_n}(p_n) \notin X$. Let us define $f : \omega \to X$ by $f|_{A_n} = f_n|_{A_n}$ for every $n < \omega$. Then, there is $m < \omega$ such that $\overline{f}(p_m) \in X$. But, by the definition of f, $\overline{f}(p_m) = \overline{f_m}(p_m)$ which is a contradiction since $\overline{f_m}(p_m) \notin X$. $\square$

In our first Example, we will need the following pre-orderings on ω^*: For $p, q \in \omega^*$, we say that $p \leq_{RK} q$ if there is a function $f : \omega \to \omega$ such that $\overline{f}(q) = p$, and $p \leq_{RF} q$ if there is an embedding $e : \omega \to \beta(\omega)$ such that $\overline{g}(p) = q$. If $p, q \in \omega^*$, then we say that $p \approx q$ if $p \leq_{RK} q$ and $q \leq_{RK} p$, and $p <_{RK} q$ (resp., $p <_{RF} q$) means that $p \leq_{RK} q$ (resp., $p \leq_{RF} q$) but p and q are not equivalent. The *type* of $p \in \omega^*$ is the set $T(p) = \{q \in \omega^* : p \approx q\}$. A RK-minimal ultrafilter on ω is usually called *selective*. We list the basic properties of these two pre-orderings that we shall use (proofs of these facts may be found in [Co], [CN], [Ku] and [vM]):

Lemma 1.3. *The following properties hold:*

1. $\leq_{RF} \subset \leq_{RK}$;
2. *for $p, q \in \omega^*$, $p \approx q$ iff there is a bijection $f : \omega \to \omega$ such that $\overline{f}(p) = q$;*
3. *let $f : \omega \to \omega$ and $p \in \omega^*$. Then, $p \approx \overline{f}(p)$ if and only if there is $A \in p$ such that $f|_A$ is one-to-one;*
4. *every weak P-point of ω^* is RF-minimal, and there are $2^{\mathfrak{c}}$-many weak P-points of ω^* which are not selective;*
5. *if $p \in \omega^*$ is selective and $f : \omega \to \omega^*$ is a function such that $\overline{f}(p) \notin f[\omega]$, then $p <_{RF} \overline{f}(p)$;*
6. *if $p \leq_{RF} r$ and $q \leq_{RF} r$, then p and q are RF-comparable;*
7. *if $f : \omega \to \omega^*$ is an embedding, then $p <_{RF} \overline{f}(p)$ for every $p \in \omega^*$;*
8. *if $X, Y \subseteq \omega^*$ are countable, then $\overline{X} \cap \overline{Y} = \emptyset$ iff $\overline{X} \cap Y = \emptyset$ and $X \cap \overline{Y} = \emptyset$. In particular, if X and Y are disjoint countable sets of weak P-points of ω^*, then $\overline{X} \cap \overline{Y} = \emptyset$.*

To state our preliminary results, we introduce the following notion: Let $F \in [\omega^*]^\omega$, let $e : \omega \to F$ be a function and let $p \in \omega^*$. Then, a function $f : \omega \to \omega^*$ is called a *(F, e, p)-function* if $q <_{RF} \overline{f}(q)$ for every $q \in F$, and $p <_{RF} \overline{f}(\overline{e}(p))$. Notice that if $F = \{p_n : n < \omega\}$ are RK-incomparable selective ultrafilters on ω, $p \in \omega^*$ and $e : \omega \to \omega^*$ is defined by $e(n) = p_n$ for all $n < \omega$, then every (F, e, p)-function satisfies that $\overline{f}(p_n) \neq \overline{f}(p_m)$ whenever $n < m < \omega$.

Lemma 1.4. *Let $\{p\} \cup \{p_n : n < \omega\}$ be pairwise RK-incomparable selective ultrafilters on ω, and let $e : \omega \to \omega^*$ be defined by $e(n) = p_n$ for every $n < \omega$. If $f : \omega \to \omega^*$ satisfies that $p_n <_{RF} \overline{f}(p_n)$ for every $n < \omega$, then f is a $(\{p_n : n < \omega\}, e, p)$-function.*

PROOF: We have to show that $p <_{RF} \overline{f}(\overline{e}(p))$. In fact, if $\overline{f}(\overline{e}(p)) \neq \overline{f}(p_n)$ for every $n < \omega$, then $\overline{f}(\overline{e}(p)) \notin \{\overline{f}(e(n)) : n < \omega\}$, and hence, by Lemma 1.3, $p <_{RF} \overline{f}(\overline{e}(p))$. Suppose that $\overline{f}(\overline{e}(p)) = \overline{f}(p_k)$ for some $k < \omega$. Our assumption implies that $\overline{f}(\overline{e}(p)) \neq \overline{f}(p_n)$ for every $n \in \omega - \{k\}$. Since $\overline{f}(\overline{e}(p))$ is an accumulation point of $\{\overline{f}(p_n) : n \in \omega \setminus \{k\}\}$. Then, we may find a pairwise disjoint family $\{B_m : m < \omega\}$ of subsets of ω such that $B_m \notin \overline{f}(p_k)$ for all $m < \omega$ and $\{\overline{f}(p_n) : n \in \omega \setminus \{k\}\} \subseteq \bigcup_{m<\omega} \widehat{B_m}$. Let $h : \omega \to \omega$ be the function defined by $h^{-1}(m) = B_m$ for each $m < \omega$. Then, $(\overline{h} \circ \overline{f} \circ e)[\omega \setminus \{k\}] \subseteq \omega$ and $\overline{h}(\overline{f}(\overline{e}(p))) \in \omega^*$. Hence, $\overline{h}(\overline{f}(\overline{e}(p))) \leq_{RK} p$. Since p is selective, $\overline{h}(\overline{f}(\overline{e}(p))) \approx p$. By applying Lemma 1.3, we may find $A \in p$ such that $\overline{h} \circ \overline{f} \circ e|_A$ is one-to-one. So, by the definition of h, the function $\overline{f} \circ e|_A$ is an embedding and hence $p <_{RF} \overline{f}(\overline{e}(p))$. But, by Lemma 1.3, this implies that p and p_k are RF-equivalent, which is a contradiction. Then, $\overline{f}(\overline{e}(p)) \neq \overline{f}(p_n)$ for all $n < \omega$. Therefore, f is a $(\{p_n : n < \omega\}, e, p)$-function. $\square$

Lemma 1.5. *Let $\{p\} \cup \{p_n : n < \omega\}$ be a set of pairwise RK-incomparable selective ultrafilters on ω, and let $e : \omega \to \omega^*$ be defined by $e(n) = p_n$ for every $n < \omega$. For a subspace X of ω^*, the following are equivalent:*

1. *X is quasi $(\{\overline{e}(p)\} \cup \{p_n : n < \omega\})$-compact;*
2. *for every $(\{p_n : n < \omega\}, e, p)$-function $f : \omega \to X$ there is $q \in \{\overline{e}(p)\} \cup \{p_n : n < \omega\}$ such that $\overline{f}(q) \in X$.*

Proof: The implication (1) $\Rightarrow$ (2) is evident.

(2) $\Rightarrow$ (1). Observe that e is an embedding and hence $\overline{e}(p) \neq p_n$ for all $n < \omega$. Put $M = \{\overline{e}(p)\} \cup \{p_n : n < \omega\}$. Let us assume that $f : \omega \to X$ is a function such that $\overline{f}(q) \notin X$ for every $q \in M$. Then, in particular, $\overline{f}(p_n) \notin f[\omega]$ for every $n < \omega$. Thus, by Lemma 1.3, $p_n <_{RF} \overline{f}(p_n)$ for each $n < \omega$. So, by Lemma 1.4, f is a $(\{p_n : n < \omega\}, e, p)$-function. By assumption, there is $q \in M$ such that $\overline{f}(q) \in X$, which is a contradiction. $\square$

Example 1.6. *Let $\{p\} \cup \{p_n : n < \omega\}$ be a set of pairwise RK-incomparable selective ultrafilters on ω, and let $e : \omega \to \omega^*$ be defined by $e(n) = p_n$ for every $n < \omega$. Then, there is a quasi $(\{\overline{e}(p)\} \cup \{p_n : n < \omega\})$-compact space that is not q-compact for any $q \in \omega^*$.*

Proof: Let $\{q_n : n < \omega\}$ be a set of selective ultrafilters on ω such that $\{p\} \cup \{p_n : n < \omega\} \cup \{q_n : n < \omega\}$ are pairwise RK-incomparable. Notice that $\overline{e}(p)$ is an accumulation point of $\{p_n : n < \omega\}$. Put $F = M_0 = \{p_n : n < \omega\}$ and $N_0 = \{q_n : n < \omega\}$. It follows from Lemma 1.3 that $\overline{M_0} \cap \overline{N_0} = \emptyset$. By transfinite induction, for each $0 < \nu < \omega_1$ we may define $M_\nu, N_\nu \subseteq \omega^*$ as follows:

1. $M_\nu = \{\overline{f}(\overline{e}(p)) : f : \omega \to \bigcup_{\mu<\nu}(M_\mu \cup N_\mu)$ is an (F, e, p)-function and $\{n < \omega : f(n) \in \bigcup_{\mu<\nu} M_\mu\} \in \overline{e}(p)\}$.
2. $N_\nu = \{\overline{f}(p_k) : f : \omega \to \bigcup_{\mu<\nu}(M_\mu \cup N_\mu)$ is an (F, e, p)-function and $\{n < \omega : f(n) \notin \bigcup_{\mu<\nu} M_\mu\} \in \overline{e}(p) \cap p_k, k < \omega\}$.

We have that $M_\nu \subseteq \overline{M_0}$ and $N_\nu \subseteq \overline{M_0}$ for every $\nu < \omega_1$. Our space is $X = \bigcup_{\nu<\omega_1}(M_\nu \cup N_\nu)$. By definition and Lemma 1.5, X is quasi $(\{\overline{e}(p)\} \cup \{p_n : n < \omega\})$-compact. To prove that X is not q-compact for any $q \in \omega^*$ is enough to show that $X \times X$ is not countably compact (see [GS]). Assume that $X \times X$ is countably compact and let us consider the function $h : \omega \to X$ given by $h(n) = q_n$, for every $n < \omega$. Let $\sigma : \omega \to X \times X$ be defined by $\sigma(n) = (e(n), h(n)) = (p_n, q_n)$, for each $n < \omega$. It is clear that σ is an embedding. By assumption, there is $r \in \omega^*$ such that $\overline{\sigma}(r) \in X \times X$. Then, $\overline{e}(r), \overline{h}(r) \in X$, $r <_{RF} \overline{e}(r)$ and $r <_{RF} \overline{h}(r)$. We also have that $\overline{e}(r), \overline{h}(r) \notin M_0 \cup N_0$, $\overline{e}(r) \in \overline{M_0}$ and $\overline{h}(r) \in \overline{N_0}$. Let $\theta = \min\{\mu < \omega_1 : \overline{e}(r) \in M_\mu \cup N_\mu\}$ and $\lambda = \min\{\mu < \omega_1 : \overline{h}(r) \in M_\mu \cup N_\mu\}$. Hence, we must have that $\overline{e}(r) = \overline{f}(\overline{e}(p))$ and $\overline{h}(r) = \overline{g}(p_i)$, for some $i < \omega$, where $f : \omega \to \bigcup_{\mu<\theta}(M_\mu \cup N_\mu)$ is an (F, e, p)-function, $\{n < \omega : f(n) \in \bigcup_{\mu<\theta} M_\mu\} \in \overline{e}(p)$, $g : \omega \to \bigcup_{\mu<\lambda}(M_\mu \cup N_\mu)$ is an (F, e, p)-function and $\{n < \omega : g(n) \notin \bigcup_{\mu<\theta} M_\mu\} \in \overline{e}(p) \cap p_i$. Then, we have that r and p are RF-comparable and r and p_i are RF-comparable as well. Since p and p_i are RF-minimal, $p \leq_{RF} r$ and $p_i \leq_{RF} r$, but this implies, by Lemma 1.3, that p and p_i are RK-comparable, which contradicts our hypothesis. Therefore, $X \times X$ is not countably compact. □

We remark that in Example 1.6 the set $\{p_n : n < \omega\}$ is discrete and has $\overline{e}(p)$ as an accumulation point. A. Blass [Bl] proved, in ZFC, that if $\emptyset \neq M \subseteq \omega^*$ has cardinality $< \mathfrak{d}$ and every element of M is generated by $< \mathfrak{d}$ sets, then there is a finite-to-one function $f : \omega \to \omega$ such that $\overline{f}[M]$ is a free ultrafilter on ω, and hence, by Theorem 1.1, every quasi M-compact space is p-compact for some $p \in \omega^*$. This shows that Example 1.6 cannot take place in some models of ZFC.

Theorem 1.7. *There is a model of* ZFC *in which every quasi M-compact space is p-compact for some $p \in \omega^*$, whenever $M \in [\omega^*]^{<\mathfrak{c}}$.*

Proof: The authors of [BL] showed that in the models described in [BS1] and [BS2] the following combinatorial principle holds:

$(*)$ If $\mathcal{F}$ is any free filter on ω, then there is a finite-to-one function $f : \omega \to \omega$ such that $f[\mathcal{F}]$ is either the filter of cofinite sets or an ultrafilter.

Fix $M \in [\omega^*]^{<\mathfrak{c}}$ and put $\mathcal{F} = \bigcap\{q : q \in M\}$. By $(*)$, there is a finite-to-one function $f : \omega \to \omega$ such that either $f[\mathcal{F}]$ is the filter of cofinite sets or $f[\mathcal{F}]$ is an ultrafilter. If $f[\mathcal{F}]$ is the filter of cofinite sets, then $\overline{f}[M]$ would be dense in ω^*, which is impossible. So, $f[\mathcal{F}]$ must be an ultrafilter, say p, and then $M \subseteq \overline{f}^{-1}(p)$. According to Theorem 1.1, every quasi M-compact space is p-compact. □

It is a consequence of Theorem 1.7 that, under $(*)$, if a quasi M-compact space X is not p-compact for any $p \in \omega^*$, then $|M| \geq \mathfrak{c}$.

We turn out to the second example of this section.

Example 1.8. *If $\emptyset \notin \{T_\xi : \xi < 2^{\mathfrak{c}}\} \subseteq [\omega^*]^{<2^{\mathfrak{c}}}$, then there is a countably compact space X such that it is not quasi T_ξ-compact for any $\xi < 2^{\mathfrak{c}}$.*

Proof: We will use the following fact:

If X is a countable infinite subset of $\beta(\omega)$, then $|\overline{X}| = 2^{\mathfrak{c}}$.

It is well-known that there are $2^{\mathfrak{c}}$-many weak P-points in ω^* (see [vM]). We partition the set of weak P-points of ω^* in countable infinite subsets and enumerate them as $\{S_\xi : \xi < 2^{\mathfrak{c}}\}$. Now, for each $\xi < 2^{\mathfrak{c}}$, we fix a bijection $h_\xi : \omega \to S_\xi$. We shall use the standard method of constructing countably compact subspaces of ω^*. We put $Y_0 = \overline{S_0} - \overline{h_0}[T_0]$. Suppose that for each $\xi < \lambda < 2^{\mathfrak{c}}$ we have defined $Y_\xi \subseteq \omega^*$ such that

1. $Y_\xi \subseteq \bigcup\{\overline{X} : X \in [\bigcup_{\zeta \leq \xi} S_\zeta]^\omega\}$ for each $1 \leq \xi < \lambda$;
2. $Y_\xi \subseteq Y_\zeta$ whenever $\xi < \zeta < \lambda$;
3. every countable discrete infinite subset of Y_ξ has an accumulation point in $Y_{\xi+1}$ for each $\xi < \xi + 1 < \lambda$; and
4. $S_\xi \subseteq Y_\xi \subseteq \omega^* \setminus [(\bigcup_{\zeta \leq \xi} \overline{h_\zeta}[T_\zeta]) \cup (\bigcup_{\xi < \zeta < 2^{\mathfrak{c}}} \overline{S_\zeta})]$ for each $\xi < \lambda$.

Define $Y = S_\lambda \cup (\bigcup_{\xi < \lambda} Y_\xi)$. First notice that $Y \cap \overline{h_\xi}[T_\xi] = \emptyset$ for every $\xi < 2^{\mathfrak{c}}$. We enumerate all countable discrete infinite subsets of Y as $\{D_\theta : \theta < |Y|^\omega = \kappa\}$. Without loss of generality, we may assume that either $D_\theta \subseteq \bigcup_{\xi<\lambda} Y_\xi$ or $D_\theta \subseteq S_\lambda$. By Lemma 1.3, $\overline{D_\theta} \cap (\bigcup_{\lambda < \zeta < 2^{\mathfrak{c}}} \overline{S_\zeta}) = \emptyset$, for each $\theta < \kappa$, and $\overline{S_\zeta} \cap Y_\xi = \emptyset$ whenever $\xi < \zeta \leq \lambda$. For each $\theta < \kappa$, we choose $p_\theta \in \overline{D_\theta}$ as follows:

Suppose that $D_\theta \subseteq \bigcup_{\xi<\lambda} Y_\xi$. By 1, there is a countable subset I of λ such that $D_\theta \subseteq \overline{\bigcup_{\xi \in I} S_\xi}$. Since $|\bigcup_{\xi \in I} \overline{h_\xi}[T_\xi]| < 2^{\mathfrak{c}}$, we may choose $p_\theta \in \overline{D_\theta} \setminus \bigcup_{\xi \in I} \overline{h_\xi}[T_\xi]$ (by the fact).

If $D_\theta \subseteq S_\lambda$, then we pick any $p_\theta \in \overline{D_\theta} \setminus h[T_\lambda]$, this is possible by the fact.

Then, we define $Y_\lambda = Y \cup \{p_\theta : \theta < \kappa\}$. It is clear that Y_λ satisfies all the conditions. Finally, we put $X = \bigcup_{\lambda < 2^{\mathfrak{c}}} Y_\lambda$. By clauses 2 and 3 and the fact that $cf(2^{\mathfrak{c}}) > \omega$, X is countably compact and, by clause 4, X is not quasi T_ξ-compact for every $\xi < 2^{\mathfrak{c}}$. $\square$

In particular, if $2^{\mathfrak{c}} = (2^{\mathfrak{c}})^{<2^{\mathfrak{c}}}$, then there is a countably compact space X such that X is not quasi M-compact for any $M \in [\omega^*]^{<2^{\mathfrak{c}}}$: The equality $2^{\mathfrak{c}} = (2^{\mathfrak{c}})^{<2^{\mathfrak{c}}}$ holds when $2^{\mathfrak{c}}$ is a regular cardinal. It should be remark that if X is a countably compact space of size $\mathfrak{c}$, then there is $M \in [\omega^*]^{\leq \mathfrak{c}}$ such that X is quasi M-compact.

Question 1.9. For each cardinal $\kappa < 2^{\mathfrak{c}}$, is there a countably compact space X such that X^κ is countably compact, and X is not quasi M-compact for any $M \in [\omega^*]^{<2^{\mathfrak{c}}}$?

By making some minor changes, for each $1 < n < \omega$, we may construct a space X like in Example 1.8 with the additional property that X^n is countably compact.

Question 1.10. Is there a countably compact space X and $M \in [\omega^*]^{\omega_1}$ such that X is quasi M-compact, and X is not N-compact for any $N \in [\omega^*]^{\leq\omega}$?

References

[Be] Bernstein A.R., *A new kind of compactness for topological spaces*, Fund. Math. **66** (1970), 185–193.

[Bl] Blass A., *Near coherence of filters I: Cofinal equivalence of models of arithmetic*, Notre Dame J. Formal Logic **27** (1986), 579–591.

[BL] Blass A., Laflamme C., *Consistency results about filters and the number of inequivalent growth types*, J. Symbolic Logic **54** (1989), 50–56.

[BS1] Blass A., Shelah S., *Near coherence of filters III: A simplified consistency proof*, to appear.

[BS2] Blass A., Shelah S., *There may be simple $P_{\aleph_1}$ and $P_{\aleph_2}$ points and the Rudin-Keisler ordering may be downward directed*, Ann. Pure Appl. Logic **33** (1987), 213–243.

[Co] Comfort W.W., *Ultrafilters: some old and some new results*, Bull. Amer. Math. Soc. **83** (1977), 417–455.

[CN] Comfort W., Negrepontis S., *The Theory of Ultrafilters*, Springer-Verlag, Berlin, 1974.

[G] Garcia-Ferreira S., *Quasi M-compact spaces*, Czechoslovak Math. J. **46** (1996), 161–177.

[GJ] Gillman L., Jerison M., *Rings of continuous functions*, Lectures Notes in Mathematics No. 27, Springer-Verlag, 1976.

[GS] Ginsburg J., Saks V., *Some applications of ultrafilters in topology*, Pacific J. Math. **57** (1975), 403–418.

[Ku] Kunen K., *Weak P-points in ω^**, Colloquia Math. Soc. János Bolyai **23** (1978), North-Holland, Amsterdam, pp. 741–749.

[vM] van Mill J., *An introduction to $\beta(\omega)$*, in K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 503–567.

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