Condensations of Tychonoff universal topological algebras

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Abstract. Let $(L, \mathcal{T})$ be a Tychonoff (regular) paratopological group or algebra over a field or ring K or a topological semigroup. If $\mathrm{nw}(L, \mathcal{T}) \leq \tau$ and $\mathrm{nw}(K) \leq \tau$, then there exists a Tychonoff (regular) topology $\mathcal{T}^* \subseteq \mathcal{T}$ such that $w(L, \mathcal{T}^*) \leq \tau$ and $(L, \mathcal{T}^*)$ is a paratopological group, algebra over K or a topological semigroup respectively.

Keywords: universal algebra, paratopological group, topological group

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1. Introduction

The concept of network was defined by Arhangel'skiĭ in 1959 for the study of topological spaces in general. This notion, however, is also very useful in topological algebra. In [1] Arhangel'skiĭ proved the following theorem: every Hausdorff topological group G of network weight $\leq \tau$ can be *condensed* isomorphically onto a Hausdorff topological group G^* of weight $\leq \tau$. That is, given a Hausdorff topological group G of network weight $\leq \tau$, one can find a Hausdorff group topology of weight $\leq \tau$ on G coarser than the original topology. Shakhmatov presented in [5] a general construction that makes it possible to prove similar assertions for topological algebras like linear topological spaces, topological rings, modules, fields, and so forth. These topological algebras "contain" the structure of topological group. So, the Hausdorff separation axiom implies complete regularity of these topological algebras. However, many Hausdorff topological algebras do not necessarily satisfy the axiom of regularity (see Example 7). In this paper we modify Shakhmatov's construction in such a way that it can be applied to regular or Tychonoff topological algebras preserving these separation properties under condensations.

Let us recall a few definitions. The weight $w(X)$ of a space X is $\aleph_0 \cdot \min\{|\mathcal{B}| : \mathcal{B}$ is a base of space $X\}$. A family $\mathcal{N}$ of subsets of a space X is called *network* for X if for every point $x \in X$ and every neighborhood V_x of x, there exists $M \in \mathcal{N}$ such that $x \in M \subseteq V_x$. The *network weight* $\mathrm{nw}(X)$ of X is $\aleph_0 \cdot \min\{|\mathcal{N}| : \mathcal{N}$ is a network of $X\}$. The Lindelöf number $l(X)$ of X is the smallest infinite cardinal τ such that every open cover of X has an open subcover of power $\leq \tau$.

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A condensation is a bijective continuous function. An isomorphic condensation of a topological ring (module, field, etc.) over other topological ring (module, field, etc.) is a condensation that is simultaneously an isomorphism of the corresponding algebraic structures.

2. Main result

In what follows all spaces are assumed to be Hausdorff. The symbol $\mathbb{N}$ denotes the set of nonnegative integers; τ is any infinite cardinal.

Suppose that X is a space, $\mathcal{K}$ is a class of spaces whose elements are external to X; and the spaces $K_1, \dots, K_m$ are in $\mathcal{K}$.

Definition 1 (Shakhmatov [5]). An *(n,m)-ary continuous operation on X* is a pair (j, D_j) consisting of the domain $D_j \subseteq X^n \times K_1 \times \cdots \times K_m$ of the operation j, and of a continuous mapping $j\colon D_j \to X$. The set D_j is considered with the topology induced from the product $X^n \times K_1 \times \cdots \times K_m$.

The necessity for considering operations that are defined not on the product $X^n \times K_1 \times \cdots \times K_m$, but on a subspace of it, arises naturally. For example, in the case of topological fields, the operation $j(x) = x^{-1}$ of taking inverse elements in the field K has domain $D_j = P \setminus \{0\}$, where 0 is the zero element of the field K. We start with the following auxiliary results.

Lemma 2. *If $\mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \cdots \subseteq \mathcal{T}_i \subseteq \mathcal{T}_{i+1} \subseteq \cdots$ is a sequence of topologies on a set X, then $\bigcup\{\mathcal{T}_i : i \in \mathbb{N}\}$ is the base for a topology $\mathcal{T}$ on X. Moreover*

(i) *if all topologies $\mathcal{T}_i$ are regular, then so is $\mathcal{T}$;*
(ii) *if all topologies $\mathcal{T}_i$ are Tychonoff, then so is $\mathcal{T}$;*
(iii) *if all topologies $\mathcal{T}_i$ have a base of size $\leq \tau$, then $\mathcal{T}$ also has a base of size $\leq \tau$.*

PROOF: It is evident that $\bigcup\{\mathcal{T}_i : i \in \mathbb{N}\}$ is the base for a topology on the set X because this set is closed under finite intersections. Since the cases for regular and Tychonoff topologies are similar, we will prove only that if all the topologies are Tychonoff, then $\mathcal{T}$ is Tychonoff. Let U be a set in $\mathcal{T}$ containing x. Then there exists an element V in $\bigcup\{\mathcal{T}_i : i \in \mathbb{N}\}$ such that $x \in V \subset U$. So, $V \in \mathcal{T}_i$ for some i. Now, there is a function $f\colon X \to [0,1]$ continuous with respect to the usual topology in $[0,1]$ and $\mathcal{T}_i$ in X such that such that $f(x) = 0$ and $f(y) = 1$ for all $y \in X \setminus V \supset X \setminus U$. Since $\mathcal{T}_i$ is coarser than $\mathcal{T}$, f is also continuous with respect to $\mathcal{T}$. Hence the topology $\mathcal{T}$ is Tychonoff.

To prove (iii), we only need to observe that if $\mathcal{B}_i$ is a base of $\mathcal{T}_i$ with $|\mathcal{T}_i| \leq \tau$, then the family $\mathcal{B}$ of finite intersections of $\bigcup_{i\in\omega} \mathcal{B}_i$ is a base of $\mathcal{T}$ and $|\mathcal{B}| \leq \tau$. □

In the proof of our main result, we need a lemma from [6].

Lemma (Shakhmatov [6]). *Let $\phi: X \to Z$ be a continuous mapping of a regular (Tychonoff) space X of netweight $\leq \tau$ to a space of weight $\leq \tau$. Then there are a regular (Tychonoff) space Q of weight $\leq \tau$, a condensation $i: X \to Q$ and a continuous map $q: Q \to Z$ such that $\phi = q \circ i$.*

In fact, if we take $(X, \mathcal{T})$ as X, $(X, \mathcal{T}')$ as Z and the identity map as ϕ, we get the following particular case of the last lemma.

Lemma 3. *Assume that τ is an infinite cardinal, X is a set, $\mathcal{T}$ is a topology on X that has a network of size $\leq \tau$ and $\mathcal{T}'$ is a topology on X that has a base of size $\leq \tau$ such that $\mathcal{T}' \subseteq \mathcal{T}$. Then one can find a topology $\mathcal{T}^*$ on X with the following properties:*

(i) $\mathcal{T}' \subseteq \mathcal{T}^* \subseteq \mathcal{T}$;
(ii) $\mathcal{T}^*$ *has a base of size* $\leq \tau$;
(iii) *if* $\mathcal{T}$ *is regular, then so is* $\mathcal{T}^*$;
(iv) *if* $\mathcal{T}$ *is Tychonoff, then so is* $\mathcal{T}^*$.

Finally, an analysis of the original proof of the main result in [5] shows that the initial topology $\mathcal{T}_0$ can be chosen arbitrarily except for the restriction $\mathcal{T}_0 \subseteq \mathcal{T}$. (We only use the fact that $\mathcal{T}_0$ is Hausdorff to assure that the topology $\mathcal{T}^* \subseteq \mathcal{T}_0$ is Hausdorff.) Therefore the proof of the main result from [5] also gives the following:

Lemma 4. *Assume that τ is an infinite cardinal, X is a set, $\mathcal{T}$ is a topology on X that has a network weight $\leq \tau$ and $\mathcal{T}'$ is a topology on X that has weight $\leq \tau$ such that $\mathcal{T}' \subseteq \mathcal{T}$. Let $\mathcal{K}$ be a class of topological spaces. Suppose that there are specified $\leq \tau$ continuous operations on the space $(X, \mathcal{T})$ in the sense of Definition* 1. *Assume also that each space $K \in \mathcal{K}$ has a network weight $\leq \tau$. Then one can find a topology $\mathcal{T}^*$ on X with the following properties:*

(i) $\mathcal{T}' \subseteq \mathcal{T}^* \subseteq \mathcal{T}$,
(ii) $\mathcal{T}^*$ *has a base of size* $\leq \tau$,
(iii) *all operations remain continuous on the space* $(X, \mathcal{T}^*)$.

Theorem 5. *Let $(X, \mathcal{T})$ be a (Tychonoff) regular space and $\mathcal{K}$ be a class of topological spaces. Suppose that there are specified $\leq \tau$ continuous operations on the space $(X, \mathcal{T})$ in the sense of Definition* 1. *If* $\mathrm{nw}(X, \mathcal{T}) \leq \tau$ *and all* $K \in \mathcal{K}$ *satisfy* $\mathrm{nw}(K) \leq \tau$, *then there exists a condensation $i: (X, \mathcal{T}) \to (X, \mathcal{T}^*)$, where $\mathcal{T}^*$ is a coarser (Tychonoff) regular topology on X such that all operations remain continuous on the space $(X, \mathcal{T}^*)$ and $w(X, \mathcal{T}^*) \leq \tau$.*

Proof: We will construct by induction two sequence of topologies $\{\mathcal{T}_i : i \in \mathbb{N}\}$ and $\{\mathcal{T}_i' : i \in \mathbb{N}\}$ satisfying the following conditions for each $n \in \mathbb{N}$:

(1) $\mathcal{T}_{i-1} \subseteq \mathcal{T}_i \subseteq \mathcal{T}_i' \subseteq \mathcal{T}$;
(2) both $\mathcal{T}_i$ and $\mathcal{T}_i'$ have weight $\leq \tau$;
(3) all operations remain continuous on the space $(X, \mathcal{T}_i)$;
(4) if $\mathcal{T}$ is (Tychonoff) regular, then so is $\mathcal{T}_i'$.

For $i = 0$, let $\mathcal{T}_0 = \{\emptyset, X\}$ be the indiscrete topology on X. Use Lemma 3 to get a topology $\mathcal{T}_0'$ on X such that $\mathcal{T}_0 \subseteq \mathcal{T}_0' \subseteq \mathcal{T}$ and $\mathcal{T}_0'$ satisfies the same separation axiom that topology $\mathcal{T}$ does.

Suppose now that $\mathcal{T}_i$ and $\mathcal{T}_i'$ satisfying (1)–(4) have been constructed for all $i < n$ ($n \geq 1$). Use Lemma 4 to get a topology $\mathcal{T}_n$ on X of weight $\leq \tau$ satisfying (3) and $\mathcal{T}_{n-1}' \subseteq \mathcal{T}_n \subseteq \mathcal{T}$. Next use Lemma 2 to get a topology $\mathcal{T}_n'$ on X of weight $\leq \tau$ satisfying (4) and $\mathcal{T}_n \subseteq \mathcal{T}_n' \subseteq \mathcal{T}$. Then al conditions (1)–(4) are satisfied and the inductive step is finished.

For a family $\mathcal{B}$ of subsets of X closed under finite intersections let $\langle \mathcal{B} \rangle$ denote the topology on X having $\mathcal{B}$ as its base. Finally, from (1) it follows that $\mathcal{T}^* = \langle \bigcup\{\mathcal{T}_i : i \in \mathbb{N}\}\rangle = \langle \bigcup\{\mathcal{T}_i' : i \in \mathbb{N}\}\rangle \subseteq \mathcal{T}$. From this and Lemma 2(iii) it follows that $\mathcal{T}^*$ is a topology on X of weight $\leq \tau$ with $\mathcal{T}^* \subseteq \mathcal{T}$. From $\mathcal{T}^* = \langle \bigcup\{\mathcal{T}_i : i \in \mathbb{N}\}\rangle$ and (3) we can conclude that all operations remain continuous on $(X, \mathcal{T}^*)$. Finally, $\mathcal{T}^* = \langle \bigcup\{\mathcal{T}_i' : i \in \mathbb{N}\}\rangle$, condition (4) and Lemma 2(i) and (ii) yield that $\mathcal{T}^*$ satisfies the same separation axioms that $\mathcal{T}$ does. $\square$

A set G with a binary operation $\cdot$ and a topology τ is called a *paratopological group* if $(G, \cdot)$ is a group and the function $\varphi: G \times G \to G$ defined by $\varphi(x, y) = x \cdot y$ is continuous. If in addition the function $\psi: G \to G$ defined by $\psi(x) = x^{-1}$ is continuous, then $(G, \cdot, \tau)$ is a topological group. If (G, τ) is a paratopological group, then so is (G, τ^{-1}), where $\tau^{-1} = \{A \subseteq G : A^{-1} \in \tau\}$. Of course, $(G, \cdot, \tau)$ is a topological group if and only if $\tau = \tau^{-1}$. The set of real numbers $\mathbb{R}$ with the usual sum $+$ and the Sorgenfrey topology (where the basic open sets are of the form $[x, r)$ with $x\langle r)$ is a paratopological group. Now, we shall construct a non regular paratopological group. The proof of the next lemma is similar to that of the analogous result for topological groups, and therefore is omitted.

Lemma 6. *Let G be a Hausdorff paratopological group with identity e_G. There exists a local base $\mathcal{V}$ for e_G in G satisfying the following conditions:*

(1) $\bigcap \mathcal{V} = \{e_G\}$;
(2) *if U, $V \in \mathcal{V}$, then there exists $W \in V$ such that $W \subseteq V \cap U$;*
(3) *for any $U \in \mathcal{V}$ there exists $V \in \mathcal{V}$ such that $V^2 \subseteq U$;*
(4) *for every $U \in \mathcal{V}$ and $x \in U$, there exists $V \in \mathcal{V}$ such that $xV \subseteq U$;*
(5) *for every $U \in \mathcal{V}$ and $a \in G$, there exists $W \in \mathcal{V}$ such that $aWa^{-1} \subseteq U$.*

Conversely, if we have a group G and a non-empty family $\mathcal{V}$ of subsets of G satisfying (1)–(5)*, then each of the families $\{xU : U \in \mathcal{V}, x \in G\}$ and $\{Ux : U \in \mathcal{V}, x \in G\}$ forms a base for a Hausdorff paratopological group topology on G.*

Example 7. Consider the Euclidean plane E. The family of neighborhoods of the identity $(0,0)$ is $\mathcal{V} = \{U_\varepsilon : \varepsilon > 0\}$, where $U_\varepsilon = \{(x, y) : x^2 + y^2 < \varepsilon^2, y > 0\} \cup \{(0,0)\}$. Observe that the family $\mathcal{V}$ satisfies (1)–(5) of Lemma 6. Hence, E with the topology τ generated by this family is a Hausdorff paratopological group. The τ-closed set $A = \{(x, y) : y \leq 0\} \setminus \{0, 0\}$ and the point $(0,0)$ cannot be separated by disjoint open sets. So, the paratopological group (E, τ) is not regular.

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