Smooth, very smooth and strongly smooth points in Musielak-Orlicz function spaces equipped with the Luxemburg norm

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Abstract. First, we extend the criteria for smooth points of $S(L_M)$ from [22] to the whole class of Musielak-Orlicz spaces. Next, we present criteria for very smooth and strongly smooth points of $S(L_M)$.

Keywords: smooth points, very smooth points, strongly smooth points, Musielak-Orlicz function spaces, Luxemburg norm

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1. Introduction

Let us start with some notations and definitions. In the whole paper X denotes a real Banach space and X^* denotes its dual space. $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ stand for the set of natural numbers, the set of reals and positive reals, respectively. By (T, Σ, μ) we denote a measure space with μ being monotonic and σ-finite. The letter M stands for a Musielak-Orlicz function, i.e. M is a mapping from $T \times \mathbb{R}$ into $[0, +\infty]$ satisfying the following conditions:

(i) there is a null set $A \in \Sigma$ such that for any $t \in T \setminus A$, $M(t, \cdot)$ is an Orlicz function, i.e. $M(t, 0) = 0$, $M(t, \cdot)$ is continuous at zero and left continuous on $(0, \infty)$, $M(t, \cdot)$ is convex and even on $\mathbb{R}$ and $M(t, u) \to \infty$ as $u \to \infty$,

(ii) for any $u \in \mathbb{R}$, $M(\cdot, u)$ is a Σ-measurable function on T.

Let us denote by $L^0 = L^0(T, \Sigma, \mu)$ the space of all (equivalence classes of) Σ-measurable functions $x : T \to \mathbb{R}$. Given any Musielak-Orlicz function M, we define on L^0 a convex modular ϱ_M by

$$\varrho_M(x) = \int_T M(t, x(t))\, d\mu$$

and a Musielak-Orlicz space L_M by

$$L_M = \{x \in L^0 : \varrho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

We denote by N the Musielak-Orlicz function complementary to M in the sense of Young, i.e.

$$N(t,v) = \sup_{u\geq 0}\{u|v| - M(t,u)\}$$

for all $u \in \mathbb{R}$ and $t \in T \setminus A$. We define in L_M two norms; the Luxemburg norm

$$\|x\|_M = \inf\{\lambda > 0 : \varrho_M(x/\lambda) \leq 1\}$$

and the Amemiya-Orlicz norm

$$\|x\|_M^o = \inf_{k>0} \frac{1}{k}(1 + \varrho_M(kx)).$$

For simplicity, we write L_M and L_M^0 in place of $(L_M, \|\cdot\|_M)$ and $(L_M, \| \ \|_M^0)$, respectively. Let us denote by $K(x)$ the set of all $k > 0$ such that the infimum in the last formula is attained at k. L_M is a Banach space under either of these two norms (see [2], [15] and in the case of Orlicz spaces also [12], [13], [14] and [17]).

Let $p_-(t,u)$ and $p(t,u)$ denote the left and right derivative of $M(t,\cdot)$ at u, respectively, and let us denote for $t \in T$:

$$e(t) = \sup\{u > 0 : M(t,u) = 0\}, \quad b(t) = \sup\{u > 0 : M(t,u) < \infty\},$$
$$\tilde{e}(t) = \sup\{v > 0 : N(t,v) = 0\}, \quad \tilde{b}(t) = \sup\{v > 0 : N(t,v) < \infty\},$$
$$S_x = \{t \in T : x(t) \neq 0\}, \quad O_x = \{t \in T : x(t) = 0\} \text{ for } x \in L^0, \qquad \text{and}$$
$$\xi_M(x) = \inf\{c > 0 : \varrho_M(x/c) < \infty\} \text{ for } x \in L_M.$$

We say that M satisfies the $\triangle_2$-condition ($M \in \triangle_2$ for short) if there are a null set $B \in \Sigma$, a constant $K \geq 2$ and a nonnegative function $h \in L^0$ such that $\varrho_M(h) < \infty$ and $M(t,2u) \leq KM(t,u)$ for all $u \geq h(t)$ (see [2] and [15]).

It is well known that between various smoothness properties of X and respective rotundity properties of X^* there is an one-side duality. Namely, if X^* is rotund (weakly locally uniformly rotund) [locally uniformly rotund] then X is smooth (very smooth) [strongly smooth].

Let us recall these six notions. X is said to be rotund if for any $x \in S(X)$ (= the unit sphere of X) if $y, z \in S(X)$ and $2x = y + z$, then $y = z = x$. X is said to be weakly locally uniformly rotund (locally uniformly rotund) if for any $x \in S(X)$ and (x_n) in $S(X)$ such that $\|x_n + x\| \to 2$ there holds $x_n \to x$ weakly ($x_n \xrightarrow{w} x$ for short), respectively $x_n \to x$ strongly, i.e. $\|x_n - x\| \to 0$.

X is said to be smooth if for any $x \in S(X)$ there is only one support functional x^* at x. Recall that $x^* \in X^*$ is said to be a support functional at x if $\|x^*\| = 1$ and $x^*(x) = \|x\|$. We denote by $\operatorname{Grad}(x)$ the set of all support functionals at x. X is said to be strongly (very) smooth if it is smooth and for any $x \in S(X)$ and (x_n) in $S(X)$ the condition $\|x_n - x\| \to 0$ implies that $x_n^* \to x^*$ strongly (weakly), where $\{x^*\} = \operatorname{Grad}(x)$ and $\{x_n^*\} = \operatorname{Grad}(x_n)$ for $n = 1, 2, \ldots$.

Smoothness properties of Orlicz spaces and Musielak-Orlicz spaces were considered in [1], [3]–[5], [7]–[11], [18]–[19] and [22]–[23].

2. Results

We start with a criterion for smooth points of $S(L_M)$. Analogous criterion has been obtained in [22] but only for Musielak-Orlicz functions which are smooth at zero. Note that smoothness of M at zero is equivalent to the fact that $\tilde{e}(t) = 0$ for μ-a.e. $t \in T$.

Theorem 1. *A point $x \in S(L_M)$ is a smooth point if and only if:*
(a) $\xi_M(x) < 1$,
(b) $\mu\{t \in O_x : \tilde{e}(t) > 0\} = 0$,
(c) $\mu\{t \in S_x : p_-(t, |x(t)|) < p(t, |x(t)|)\} = 0$.

Proof: Assume without loss of generality that $x(t) \geq 0$ for μ-a.e. $t \in T$.

Necessity. The necessity of (a) can be proved in the same way as in [22]. Since (a) must be true we have that $\mathrm{Grad}(x) = \mathrm{RGrad}(x)$, where $\mathrm{RGrad}(x)$ denotes the set of all regular, i.e. order continuous functionals. Recall that $x^* \in (L_M)^*$ is said to be order continuous if $x^*(x_n) \to 0$ whenever $0 \leq x_n \searrow 0$ and that every such functional x^* is represented by some $y \in L_N^0$ (see [17]). We will prove that if $y \in \mathrm{Grad}(x)$, then $k(y) \neq \emptyset$, i.e. $\|y\|_N^0 = \frac{1}{k}(1 + \varrho_N(ky))$ for some $k > 0$. Otherwise

$$1 = \|y\|_N^0 = \lim_{k\to\infty} \frac{1}{k}(1 + \varrho_N(ky)) = \int_{S_y} y(t)b(t)\,d\mu = \int_T x(t)y(t)\,d\mu$$
$$= \int_{S_y} x(t)y(t)\,d\mu.$$

Since $x(t) \leq b(t)$ μ-a.e. in T, we have $x(t) = b(t)$ μ-a.e. in S_y.

It follows from $\xi_M(x) < 1$ that there exists $\lambda > 1$ such that $\xi_M(\lambda x) < \infty$. Thus

$$\infty > \xi_M(\lambda x) \geq \int_{S_y} M(t, \lambda x(t))\,d\mu = \int_{S_y} M(t, \lambda b(t))\,d\mu = \infty.$$

This is a contradiction, which proves that $k(y) \neq \emptyset$.

Now, we are ready to prove the necessity of (b). Assume that x is a smooth point of $S(L_M)$ and (b) is not true. Then $T_0 = \{t \in O_x : \tilde{e}(t) > 0\}$ is a set in Σ with $\mu(T_0) > 0$. Assume that $y \in \mathrm{Grad}(x)$ and $\|y\|_N^0 = \frac{1}{k}(1 + \varrho_N(ky))$. Take $z \in L^0$ such that $z(t) = y(t)$ for $t \notin T_0$, $kz(t) \leq \tilde{e}(t)$ and $z(t) \neq y(t)$ for $t \in T_0$. Then

$$\|z\|_N^0 \leq \frac{1}{k}(1 + \varrho_N(kz)) = \frac{1}{k}(1 + \int_{T\setminus T_0} N(t, ky(t))\,d\mu) \leq \frac{1}{k}(1 + \varrho_N(ky))$$
$$= \|y\|_N^0 = 1$$

and

$$\langle x, z\rangle = \int_T x(t)z(t)\,d\mu = \int_{S_x} x(t)z(t)\,d\mu = \int_{S_x} x(t)y(t)\,d\mu = 1.$$

So, $\|z\|_N^0 = 1$ and $z \in \mathrm{Grad}(x)$. But $z \neq y$, whence x is not a smooth point, a contradiction.

Assume that $x \in S(L_M)$ is a smooth point and (c) is not true, then $T_1 = \{t \in S_x : p_-(t, x(t)) < p(t, x(t))\}$ has positive measure. We may assume that $0 < \mu(T_1) < \mu(T)$. Take $y \in \mathrm{RGrad}(x)$ with $\|y\|_N^0 = \frac{1}{k}(1 + \varrho_N(ky))$ for some $k > 0$. It can be proved in the same way as for Orlicz spaces in [2, Theorem 1.78] that

$$\int_T N(t, p_-(t, x(t)))\, d\mu \leq \int_T N(t, ky(t))\, d\mu = k - 1 < \infty.$$

Let

$$y_1(t) = \begin{cases} p_-(t, x(t)) & \text{for } \ t \in S_x \\ 0 & \text{for } \ t \in 0_x \end{cases}$$

and y_2 be a measurable function with $y_2(t) = p_-(t, x(t))$ for $t \in S_x \setminus T_0$ and $y_2(t) \in (p_-(t, x(t)), p(t, x(t)))$ for $t \in T_0$ and satisfying $\varrho_N(y_2) < \infty$. Then $y_1, y_2 \in L_N^0$. Let $z_1 = y_1/\|y_1\|_N^0$ and $z_2 = y_1/\|y_2\|_N^0$. Then $z_1 \neq z_2$ and $z_1, z_2 \in S(L_N^0)$. Furthermore

$$\begin{aligned} 1 \geq \langle x, z_1 \rangle &= \frac{1}{\|y_1\|_N^0} \langle x, y_1 \rangle = \frac{1}{\|y_1\|_N^0} \int_T x(t) p_-(t, x(t))\, d\mu \\ &= \frac{1}{\|y_1\|_N^0} \int_T (M(t, x(t)) + N(t, p_-(t, x(t))))\, d\mu \\ &= \frac{1}{\|y_1\|_N^0}(1 + \varrho_N(y_1)) = \frac{1}{\|y_1\|_N^0}(1 + \varrho_N(\|y_1\| z_1)) \geq \|z_1\| = 1, \end{aligned}$$

whence we conclude that $\|z_1\|_N^0 = 1 = \langle x_1, z_1 \rangle$. So, $z_1 \in \mathrm{Grad}(x)$. Similarly, $z_2 \in \mathrm{Grad}(x)$, which means that x is not a smooth point, a contradiction.

Sufficiency. Let $f = y + \phi \in \mathrm{Grad}(x)$, where y and ϕ denote the regular and the singular part of f, respectively. By condition (a), $\phi = 0$ and $1 = \|y\|_N^0 = \frac{1}{k}(1 + \varrho_N(ky))$ for some $k > 0$ (see the beginning of the proof of the necessity). It can be proved in the same way as in [4, Theorem 1.5] for Orlicz spaces that

$$p_-(t, x(t)) \leq ky(t) \leq p(t, x(t)) \quad \text{for } \ t \in S_x. \tag{1}$$

Moreover, by $\|x\|_M = 1$ and $\xi_M(x) < 1$, we have $\varrho_M(x) = 1$. Therefore, the equality

$$\int_{O_x} x(t) ky(t)\, d\mu = \int_{O_x} (M(t, x(t)) + N(t, ky(t)))\, d\mu$$

yields that $N(t, ky(t)) = 0$ for $t \in O_x$. By condition (b), $y(t) = 0$ for $t \in O_x$ and by condition (c), $ky(t) = p(t, x(t))$ for $t \in S_x$, i.e. ky is unique. By

$$k = \|ky\|_N^0 = \|ky\chi_{S_x}\|_N^0 = \|p(\cdot, x(\cdot))\chi_{S_x}\|_N^0,$$

we obtain $k = \frac{1}{\|p(\cdot,x(\cdot))\|_N^0}$. Therefore

$$y(t) = \begin{cases} \frac{p(t,x(t))}{\|p(\cdot,x(\cdot))\chi_{S_x}\|_N^0} & \text{for } \ t \in S_x \\ 0 & \text{for } \ t \in 0_x, \end{cases}$$

which means that y is unique and so x is a smooth point, which finishes the proof. □

Corollary 1. *The space L_M is smooth if and only if:*

(a) $M \in \triangle_2$,
(b) $\tilde{e}(t) = 0$ *for μ-a.e.* $t \in T$,
(c) $p(t, \cdot)$ *is continuous function on* $\mathbb{R}$ *for μ-a.e.* $t \in T$.

Proof: This result follows from Theorem 1. We need only to show the necessity of condition (b) because the rest can be proved in the same way as in [22].

Assume that condition (b) is not satisfied, that is, the set $A = \{t \in T : \tilde{e}(t) > 0\}$ has positive measure. Then we can easily build $x \in S(L_M)$ with $\mu(O_x \cap A) > 0$. By Theorem 1, x is not a smooth point, which finishes the proof of the necessity of condition (b). □

In the proof of the next theorem the following result will be useful.

Proposition 1. *Let M be a Musielak-Orlicz function and N be its complementary function in the sense of Young. Let $N \in \Delta_2$, $x \in S(L_M)$, $y_n \in L_N^0$, $k(y_n) \neq \emptyset$, $(n = 1, 2, \ldots)$, and $\langle x, y_n\rangle \to 1$ as $n \to \infty$. Then for every $\varepsilon > 0$ there is $T_\varepsilon \in \Sigma$ with $\mu T_\varepsilon < \infty$ such that $\sup_n \varrho_N(y_n\chi_{T\setminus T_\varepsilon}) < \varepsilon$.*

Proof: Take $T_1 \subset T_2 \subset \ldots \subset T_i \subset T_{i+1} \subset \ldots$ with $\mu T_i < \infty$ for each $i \in \mathbb{N}$ and $\bigcup_i T_i = T$. We will prove that for any $\varepsilon > 0$ there is $i_\varepsilon \in \mathbb{N}$ such that $\sup_n \varrho_N(y_n\chi_{T\setminus T_{i_\varepsilon}}) < \varepsilon$. Otherwise, there is $\varepsilon > 0$ such that for any $i \in \mathbb{N}$ there is $n_i \in \mathbb{N}$ such that $\varrho_N(y_{n_i}\chi_{T\setminus T_{n_i}}) > \varepsilon$. We may assume that $n_i \to \infty$ as $i \to \infty$ because (n_i) is unbounded by the fact that the assumption $N \in \Delta_2$ yields that

$$\sup_{n \in N_0} \varrho(y_n\chi_{T\setminus T_{n_i}}) \to 0 \ \text{ as } \ i \to \infty$$

for any finite subset N_0 of $\mathbb{N}$. Choose $k_i \in k(y_{n_i})$. From $\xi_M(x) < 1$ it follows that there is $\lambda > 1$ satisfying $\varrho_M(\lambda x) < \infty$. This yields that for $i \to \infty$ there holds

$$\begin{aligned}
1 \leftarrow \langle x, y_{n_i}\rangle &= \frac{1}{k_i}\langle x, k_i y_{n_i}\rangle \\
&= \frac{1}{k_i}\Big(\int_{T_i} x(t)k_i y_{n_i}(t)\,d\mu + \int_{T\setminus T_i} x(t)k_i y_{n_i}(t)\,d\mu\Big) \\
&\le \frac{1}{k_i}\Big(\varrho_M(x\chi_{T_i}) + \varrho_N(k_i y_{n_i}\chi_{T_i}) + \frac{1}{\lambda}\varrho_M(\lambda x\chi_{T\setminus T_i}) + \frac{1}{\lambda}\varrho_N(k_i y_{n_i}\chi_{T\setminus T_i})\Big) \\
&\le \frac{1}{k_i}\Big(\varrho_M(x) + \varrho_N(k_i y_{n_i}) - (1-\frac{1}{\lambda})\varrho_N(k_i y_{n_i}\chi_{T\setminus T_i}) + \frac{1}{\lambda}\varrho_M(\lambda x\chi_{T\setminus T_i})\Big) \\
&\le \frac{1}{k_i}\Big(1 + \varrho_N(k_i y_{n_i}) - (1-\frac{1}{\lambda})\varrho_N(k_i y_{n_i}\chi_{T\setminus T_i}) + \frac{1}{\lambda}\varrho_M(\lambda x\chi_{T\setminus T_i}\Big) \\
&\le \|y_{n_i}\| - (1-\frac{1}{\lambda})\varepsilon + \frac{1}{\lambda}\varrho(\lambda x\chi_{T\setminus T_i}) \to 1 - (1-\frac{1}{\lambda})\varepsilon,
\end{aligned}$$

a contradiction finishing the proof. □

Theorem 2. *Let $x \in S_{L_M}$. Then the following assertions are equivalent:*

(1) *x is a strongly smooth point,*
(2) *x is a very smooth point,*
(3) *x is a smooth point and $N \in \triangle_2$.*

Proof: We still assume without loss of generality that $x \ge 0$. The implication (1) $\Rightarrow$ (2) is obvious. Let us prove that (2) $\Rightarrow$ (3). We need only to prove that (2) $\Rightarrow N \in \triangle_2$. Assume that condition (2) holds and $N \notin \triangle_2$. There is $z \in L_N^0$ with $\varrho_N(z) < \infty$ and $\xi_N(y - \frac{z}{k}) =: A > 0$, where y defines the unique support functional for x and $k > 0$ satisfies $1 = \|y\|_N^0 = \frac{1}{k}(1 + \varrho_N(ky))$. Indeed, if $\xi_N(y) = 0$, we take $z \in L_N^0 \setminus E_N^0$; if $\xi_N(y) > 0$, we take $z = 0$. Divide T into T_1, T_1' with $\mu(T_1) = \mu(T_1') = \frac{\mu(T)}{2}$, $T_1 \cap T_1' = \emptyset$. Lemma 1.67 from [2] is also true for Musielak-Orlicz spaces (without any change of the proof). Namely, for any partition $\{T_i\}_{i=1}^n$ of T and any $x \in L_N^0$, $\xi_N(x) = \max_i \xi_N(x\chi_{T_i})$. So, we may assume that $\xi_N(y - \frac{z}{k}) = \xi_N((y - \frac{z}{k})\chi_{T_1})$.

Divide T_1 into T_2, T_2' with $\mu(T_2) = \mu(T_2') = \frac{\mu(T_1)}{2}$, $T_2 \cap T_2' = \emptyset$. We may assume that

$$\xi_N(y - \frac{z}{k}) = \xi_N((y - \frac{z}{k})\chi_{T_1}) = \xi_N((y - \frac{z}{k})\chi_{T_2}).$$

Continuing this process by induction one can find a sequence $(T_n)_{n=1}^\infty$ of measurable sets in T such that $T \supset T_1 \supset T_2 \supset \cdots \supset T_n \supset \dots$, $\mu(T_n) = \frac{1}{2^n}\mu(T)$, and $\xi_N(y - \frac{z}{k}) = \xi_N((y - \frac{z}{k})\chi_{T_n})$ for $n = 1, 2, \dots$. Let

$$y_n(t) = \begin{cases} \frac{z(t)}{k} & \text{for } t \in T_n \\ y(t) & \text{for } t \in T \setminus T_n \end{cases} \qquad (n = 1, 2, \dots).$$

Then

$$\|y_n\|_N^0 \leq \frac{1}{k}(1+\varrho_N(ky_n)) \leq \frac{1}{k}(1+\varrho_N(ky)+\int_{T_N} N(t,z(t))\,d\mu)) \to \|y\|_N^0 = 1.$$

On the other hand

$$\langle x, y_n\rangle = \int_{T\setminus T_n} x(t)y(t)\,dt + \int_{T_n} x(t)z(t)\,dt \to \langle x, y\rangle = 1.$$

But

$$\xi_N(\min_{1<i\leq n} |y-y_i|) = \xi_N((y-\frac{z}{k})\chi_{T_n}) = \xi_N(y-\frac{z}{k}) = A.$$

Since Theorem 1.68 from [2] holds also for Musielak-Orlicz spaces, that is if (x_n) is a sequence in L_N^0, then $\langle x_n, \varphi\rangle \to 0$ for any singular functional $\varphi \in (L_N^0)^*$ if and only if $\lim_{m\to\infty} \xi_N(\min_{i\leq m}|y_i|) = 0$ for each subsequence (y_i) of (x_n), we conclude from the last condition that $y_n \not\to y$ weakly. This contradicts the fact that x is a very smooth point.

(3) $\Rightarrow$ (1). Assume that (3) holds. Since x is a smooth point, by Theorem 1 we conclude that $\xi_M(x) < 1$ and for $y \in L_N^0$ determining the unique support functional at x there is $k > 0$ such that $1 = \|y\|_N^0 = \frac{1}{k}(1+\varrho_N(ky))$. Moreover, $ky(t) = p(t, x(t))$ for $t \in S_x$ and $y(t) = 0$ for $t \in O_x$.

Assume that $f_n = y_n + \phi_n \in S(L_M^*)$, $f_n(x) \to 1$. In order to prove that $\|f_n - y\| \to 0$, we consider six steps.

I. Assume that $\xi_M(x) < 1-\theta < 1$. Take $z \in E_M$ such that $\|x-z\|_M < 1-\theta$. Then

$$\begin{aligned}1 \leftarrow f_n(x) =&< x, y_n > +\phi_n(x) \leq \|x\|_M\|y_n\|_N^0 + \|\phi_n\|\|x-z\|_M \\ &\leq \|y_n\|_N^0 + \|\phi_n\|(1-\theta) = \|f_n\| - \theta\|\phi_n\|.\end{aligned}$$

Therefore $\|\phi_n\| \to 0$, $\|y_n\|_N^0 \to 1$ and $\langle x, y_n\rangle \to 1$. Without loss of generality we assume in the following that $\|y_n\|_N^0 = 1$ for $n = 1, 2, \ldots$ and $\langle x, y_n\rangle \to 1$.

II. Let us prove that $k(y_n) \neq \emptyset$ for an infinite number of $n \in \mathbb{N}$, i.e. there are $k_n > 0$ such that

$$\|y_n\|_N^0 = \frac{1}{k_n}(1+\varrho_N(k_n y_n)).$$

Otherwise $\|y_n\|_N^0 = \int_T y_n(t)b(t)\,d\mu$ for infinite number of n. Since $\xi_M(x) < 1$, there is $\lambda > 1$ such that $\varrho_M(\lambda x) < \infty$. Hence $1 = \|y_n\|_N^0 = \int_T y_n(t)b(t)\,d\mu \geq \int_T y_n(t)\lambda x(t)\,d\mu \to \lambda$ as $n \to \infty$, which is a contradiction. So, we may assume in the following, that $k(y_n) \neq \emptyset$ for all $n \in \mathbb{N}$.

III. We will prove that

$$\tilde{k} = \sup_n k_n < \infty.$$

Otherwise, we may assume that $k_n \to \infty$, whence for $\lambda > 1$ such that $\varrho_M(\lambda x) < \infty$, we get

$$\begin{aligned} 1 \leftarrow \int_T x(t)y_n(t)\,d\mu &= \frac{1}{\lambda}\int_{S_{y_n}} \lambda x(t)y_n(t)\,d\mu \\ &\le \frac{1}{\lambda}\int_{S_{y_n}} b(t)y_n(t)\,d\mu = \frac{1}{\lambda}\int_{S_{y_n}} \lim_{v\to\infty} q(t,v)y_n(t)\,d\mu \\ &= \frac{1}{\lambda}\int_{S_{y_n}} \lim_{v\to\infty}\frac{N(t,v)}{v}y_n(t)\,d\mu = \frac{1}{\lambda}\lim_{n\to\infty}\frac{1}{k_n}(1+\varrho_N(k_n y_n)) \\ &= \frac{1}{\lambda}, \end{aligned}$$

a contradiction. Therefore $\tilde{k} < \infty$.

IV. Let us prove that

$$\lim_{\mu(E)\to 0}\left[\sup_n \int_E N(t,k_n y_n(t))\,d\mu\right] = 0. \tag{2}$$

Otherwise, there is $\varepsilon > 0$ such that

$$\lim_{\mu E\to 0}\left[\sup_n \int_E N(t,k_n y_n(t))\,d\mu\right] > \varepsilon.$$

Given $\eta_1 > 0$ there is $E_1 \in \Sigma$ with $\mu E_1 < \eta_1$ and $n_1 \in \mathbb{N}$ such that $\int_{E_1} N(t,k_{n_1}y_{n_1}(t))\,d\mu > \varepsilon$. By the absolute continuity of integral there is Θ_1 such that

$$\int_A N(t,k_n y_n(t))\,d\mu < \varepsilon$$

for any $A \in \Sigma$ with $\mu A < \Theta_1$ and $n = 1,2,\ldots,n_1$. Take $\eta_2 = \min(\eta_1/2,\Theta_1)$. Then there is $E_2 \in \Sigma$ with $\mu E_2 < \eta_2$ and $n_2 \in \mathbb{N}$ such that $\int_{E_2} N(t,k_{n_2}y_{n_2}(t))\,d\mu > \varepsilon$. Obviously, $n_2 > n_1$. Proceeding like that by induction, we can construct a sequence (η_i) of positive numbers with $\eta_1 > 2\eta_2 > 2^2\eta_3 > \ldots > 2^{n-1}\eta_n > \ldots$, a sequence (n_i) of natural numbers with $n_1 < n_2 < n_3 < \ldots$ and a sequence (E_i) in Σ with $\mu E_i < \eta_i$ such that

$$\int_{E_i} N(t,k_{n_i}y_{n_i}(t))\,d\mu > \varepsilon \quad (i = 1,2,\ldots).$$

Hence

$$\begin{aligned}1 \leftarrow \langle x, y_{n_i}\rangle &= \frac{1}{k_{n_i}}\Big(\int_{T\setminus E_i} k_{n_i}x(t)y_{n_i}(t)\,d\mu + \frac{1}{\lambda}\int_{E_i}\lambda x(t)k_{n_i}y_{n_i}(t)\,d\mu\Big)\\ &\le \frac{1}{k_{n_i}}\big(\varrho_M(x\chi_{T\setminus E_i}) + \varrho_N(k_{n_i}y_{n_i}\chi_{T\setminus E_i}) + \frac{1}{\lambda}\varrho_M(\lambda x\chi_{E_i})\\ &\quad + \frac{1}{\lambda}\varrho_N(k_{n_i}y_{n_i}\chi_{E_i})\big)\\ &\le \frac{1}{k_{n_i}}\big(\varrho_M(x) + \varrho_N(k_{n_i}y_{n_i}) - (1-\frac{1}{\lambda})\varrho_N(k_{n_i}y_{n_i}\chi_{E_i}) + \frac{1}{\lambda}\varrho_M(\lambda x\chi_{E_i})\big)\\ &\le \|y_{n_i}\| - (1-\frac{1}{\lambda})\frac{\varepsilon}{\tilde{k}} + \varrho_M(\lambda x\chi_{E_i}) \to 1 - (1-\frac{1}{\lambda})\frac{\varepsilon}{\tilde{k}}.\end{aligned}$$

This is a contradiction, so equality (2) holds.

V. Now, we will prove that

$$\lim_{n\to\infty} k_n y_n(t) = ky(t) = \begin{cases} p(t,x(t)) = p_-(t,x(t)) & \text{for } \; t \in S_x\\ 0 & \text{for } \; t\in O_x.\end{cases}$$

From

$$\begin{aligned}0 \leftarrow \|y_n\|_N^0 - < x, y_n > &= \frac{1}{k_n}(1+\varrho_N(k_ny_n)) - \frac{1}{k_n}\langle x, k_ny_n\rangle\\ &= \frac{1}{k_n}(\varrho_M(x) + \varrho_N(k_ny_n) - \langle x, k_ny_n\rangle)\\ &\ge \frac{1}{\tilde{k}}\int_T (M(t,x(t)) + N(t,k_ny_n(t)) - x(t)k_ny_n(t))\,d\mu\end{aligned}$$

it follows that

$$M(t,x(t)) + N(t,k_ny_n(t)) - x(t)k_ny_n(t) \to 0 \quad \mu\text{-a.e. in } T. \tag{3}$$

Notice that $p_-(t,x(t)) = p(t,x(t))$ for $t \in S_x$. Therefore, by the Young inequality, we can easily deduce that $k_ny_n(t) \to p(t,x(t))$ μ-a.e. in S_x. Using condition (b) in Theorem 1, we conclude that $y_n \to 0$ μ-a.e. in T.

VI. Finally, we will show that $\|y_n - y\|_N^0 \to 0$. By Proposition 1, we can assume that $\mu T < \infty$. Take an arbitrary $\varepsilon > 0$. By $N \in \triangle_2$ there exist $k > 0$ and a nonnegative function $\delta_0 \in L^1$ such that

$$N(t,\frac{v}{2}) \le kN(t,v) + \delta_0(t)$$

for μ-a.e. $t\in T$. Take $\eta > 0$ such that if $E \subset T$ and $\mu(E) < \eta$, then $\int_E \delta_0(t)\,d\mu < \frac{1}{4}$, $\int_E N(t,ky(t))\,d\mu < \frac{1}{4k}$ and $\int_E N(t,k_ny_n(t))\,d\mu < \frac{1}{4k}$ for any $n\in\mathbb{N}$ (the last one is possible by (2)).

Since $k_n y_n \to ky$ μ-a.e. in T, there is $T_0 \subset T$ such that $\mu(T \setminus T_0) < \eta$ and $N(t, k_n y_n(t) - ky(t)) \to 0$ uniformly in T_0. Hence

$$\int_{T_0} N(t, \frac{k_n y_n(t) - ky(t)}{2\varepsilon})\, d\mu < \frac{1}{2}$$

for n large enough. Therefore,

$$\begin{aligned}
\|k_n y_n - ky\|_N^0 &\le 2\varepsilon(1 + \int_T N(t, \frac{k_n y_n(t) - ky(t)}{2\varepsilon})\, d\mu) \\
&\le 2\varepsilon(1 + \int_{T_0} N(t, \frac{k_n y_n(t) - ky(t)}{2\varepsilon})\, d\mu) \\
&\quad + \frac{1}{2}\int_{T\setminus T_0} N(t, \frac{k_n y_n(t)}{\varepsilon} + N(t, \frac{ky(t)}{\varepsilon})\, d\mu) \\
&\le 2\varepsilon(1 + \frac{1}{2} + \frac{1}{2}\int_{T\setminus T_0} (kN(t, k_n y_n(t))\, d\mu \\
&\quad + \delta_0(t) + kN(t, y(t)) + \delta_0(t))\, d\mu) \\
&\le 4\varepsilon
\end{aligned}$$

for n large enough, which means that $\|k_n y_n - ky\|_N^0 \to 0$ as $n \to 0$. On the other hand $k_n = \|k_n y_n\|_N^0 \to \|ky\|_N^0 = k$ as $n \to \infty$. Thus $\|y_n - y\|_N^0 \to 0$ as $n \to \infty$, which completes the proof. □

Corollary 2. *The following are equivalent:*

(1) L_M *is strongly smooth,*
(2) L_M *is very smooth,*
(3) L_M *is smooth and* $N \in \triangle_2$.

Proof: It is an immediate consequence of Theorem 2. □

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