Triangularization of some families of operators on locally convex spaces

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Abstract. Some results concerning triangularization of some operators on locally convex spaces are established.

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1. Introduction

In all that follows X will denote a complex Hausdorff locally convex space. The topology t on X is induced by some system of seminorms $P = \{p_\alpha : \alpha \in \Delta\}$, where Δ is some index set. Let us denote by $\mathcal{L}(X)$ the set of all linear continuous operators on X, by $\mathcal{K}(X)$ the set of compact operators on X ($T \in \mathcal{K}(X)$ if there exists a neighborhood U such that $T(U)$ is a relatively compact set) and by $\mathcal{F}(X)$ the set of all finite rank operators. The topology of bounded convergence on $\mathcal{L}(X)$ will be denoted by τ_b. We shall denote by $\mathcal{R}(T)$ the range of T and by $\mathcal{N}(T)$ the null space of T. For a given $T \in \mathcal{L}(X)$ the number $\lambda \in \mathbb{C}$ is in the resolvent set of T iff $(\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The spectrum $\sigma(T)$ is the complement of the resolvent set. An operator T is quasinilpotent if $\sigma(T) = \{0\}$. A closed subspace M in X is an *invariant subspace* of an operator T if $T(M) \subseteq M$. A family of subspaces totally ordered by inclusion is called a *chain*.

Definition 1. *A family of linear operators $\mathcal{F} \subset \mathcal{L}(X)$ is triangularizable if there is a maximal subspace chain $\mathcal{C}$ (a triangular representation of $\mathcal{F}$) consisting of subspaces which are invariant under all $A \in \mathcal{F}$.*

For a given chain of subspaces $\mathcal{C}$ and $M \in \mathcal{C}$ let us denote by M_- the closure of the union of subspaces $L \in \mathcal{C}$ such that $L \subset M$ and $L \neq M$. If there is no such L then let $M_- = \{0\}$. Let a chain $\mathcal{C}$ be a triangular representation of a compact operator T; then by maximality it follows that $\dim(M/M_-) \leq 1$ ([9]) for each $M \in \mathcal{C}$. Let us take a subspace $M \in \mathcal{C}$ such that $M \neq M_-$ and a $z_M \in M \setminus M_-$. Since $\dim(M/M_-) \leq 1$ it follows that $Tz_M = y_M + d_M z_M$ for some $y_M \in M$ and $d_M \in \mathbb{C}$. It is easy to prove that d_M is independent of the choice of $z_M \in M \setminus M_-$. The number d_M is called the *diagonal coefficient* of T and M relative to $\mathcal{C}$. Let

$\mathcal{C}$ be a triangular representation of a pair $S,T \in \mathcal{K}(X)$; then for the diagonal coefficients the following properties hold: (i) $d_M(S+T) = d_M(S) + d_M(T)$, (ii) $d_M(\alpha T) = \alpha d_M(T)$ and (iii) $d_M(ST) = d_M(S)d_M(T)$, for $\alpha \in \mathbb{C}$ and $M \in \mathcal{C}$ (see e.g. [9]). These ideas are due to Ringrose in [12] in the Banach space case.

Let $\mathcal{S}$ be a collection of continuous operators on X and M, N two invariant subspaces for $\mathcal{S}$ and $N \subset M$. Then $\mathcal{S}$ induces a family $\widehat{\mathcal{S}}$ of operators on the quotient space M/N in the following manner: $\widehat{T}\widehat{x} = Tx + N$, $\widehat{x} = x + N$, where $T \in \mathcal{S}$. A property of a collection $\mathcal{S}$ of operators is said to be *inherited by quotients* if the collection $\widehat{\mathcal{S}}$ satisfies the same property for every distinct pair M, N, $M \subset N$ invariant for $\mathcal{S}$.

Many results concerning triangular representations of operators on normed spaces are known. Some generalizations to locally convex spaces are given in [9]. It must be stressed that there are great difficulties because many of the usual theorems in normed spaces are not valid in locally convex spaces.

2. Main results

An important tool in the proofs of the triangularizability is the following "triangularization lemma".

Lemma 1. *Let X be a locally convex space. Let $\mathcal{P}$ be a property of families of operators on X which is inherited by quotients. If every family satisfying $\mathcal{P}$ has a non-trivial invariant subspace, then every such a family is triangularizable.*

The proof of this lemma is the same as for normed spaces. (see e.g. [11]). In the sequel we first give some auxiliary results.

Lemma 2. *Let X be a barreled locally convex space and (T_n) a τ_b-convergent sequence in $\mathcal{L}(X)$. Then the set (T_n) is equicontinuous.*

Proof: It is easy to see that a τ_b-convergent sequence (T_n) is τ_b-bounded and then also strongly bounded and hence equicountinuous since X is barreled ([2]). □

Proposition 1. *Let X be a barreled locally convex space and (T_n) resp. (S_n) two sequences of operators from $\mathcal{L}(X)$ which are τ_b-convergent to T resp. to S, then (S_nT_n) is τ_b-convergent to ST.*

Proof: Let us choose a bounded set M and a seminorm $p_\alpha \in P$, which generate one of the seminorms q_α^M defining the topology τ_b. By Lemma 2, (S_n) is an equicountinuous set; hence there is some $p_\gamma \in P$ such that $p_\alpha(S_nx) \le p_\gamma(x)$, $n \in \mathbb{N}$, $x \in X$. Then

$$\begin{aligned} q_\alpha^M(S_nT_n - ST) &\le \sup_M p_\alpha(S_n(T_n - T)x) + \sup_M p_\alpha((S_n - S)Tx) \\ &\le \sup_M p_\gamma((T_n - T)x) + \sup_N p_\alpha((S_n - S)x) \end{aligned}$$

where $N = T(M)$ and, since both terms on the right hand side converge to zero, the left side converges, too. $\square$

By repeatedly using Proposition 1 we obtain

Corollary 1. *Let X be a barreled locally convex space and let $(T_n) \subset \mathcal{L}(X)$ be such that $T_n \xrightarrow{\tau_b} T$. Then $T_n^k \xrightarrow{\tau_b} T^k$ for each $k \in \mathbb{N}$.*

For a given locally convex space with the topology t, another topology on $\mathcal{L}(X)$ was introduced by Mendoza ([10]) in the following manner. A net (x_δ) is *u-bounded* (ultimately bounded), if for each $p_\alpha \in P$ there is some $r > 0$ and an index δ_0 such that $p_\alpha(x_\delta) \leq r$ for all $\delta \geq \delta_0$. The family of all u-bounded nets in X will be denoted by Γ_t. A net (T_λ) from $\mathcal{L}(X)$ is said to be Γ_t*-convergent* to T in $\mathcal{L}(X)$ if for any given $p_\alpha \in P$ and $\varepsilon > 0$ and each $(x_\delta) \in \Gamma_t$ there are some λ_0 and δ_0 such that

$$p_\alpha((T_\lambda - T)x_\delta) < \varepsilon, \quad \lambda \geq \lambda_0, \ \delta \geq \delta_0.$$

In [10] it is also proven that the Γ_t topology is stronger than the τ_b topology on $\mathcal{L}(X)$.

Lemma 3. *Let X be a locally convex space and $T \in \mathcal{K}(X)$. Then $\sigma(T) = \{0\}$ if and only if*

$$\lim_{k\to\infty} s^k T^k x = 0$$

for all $s \in \mathbb{C}$ and $x \in X$.

Proof: Let $\sigma(T) = \{0\}$. Since $T \in \mathcal{K}(X)$ it follows that

$$\inf\{|\lambda|, \ \lambda^{-k}T^k \xrightarrow{\Gamma_t} 0\} = 0$$

(see [10]) and it is easily seen that in this case we have $s^kT^k \xrightarrow{\Gamma_t} 0$ for each $s \in \mathbb{C}$. In particular, the stationary sequence (x) is u-bounded. Hence for each $p_\alpha \in P$ and $\varepsilon > 0$ we have $p_\alpha(s^kT^kx) < \varepsilon$ where $k \geq k_0$ for some k_0. Thus, $s^kT^kx \to 0$. Conversely, let $\sigma(T) \neq \{0\}$. Since $T \in \mathcal{K}(X)$ we have $Tx = \mu x$ for some nonzero $x \in X$ and $\mu \neq 0$ ([1]) and hence $(1/\mu)^kT^kx = x$ for each $k \in \mathbb{N}$. Consequently $(1/\mu)^kT^k \not\to 0$. $\square$

Although continuity of the spectrum or spectral radius does not hold in general, the following is true.

Theorem 1. *Let X be a barreled locally convex space and (T_n) a sequence of compact quasinilpotent operators which is τ_b-convergent to some compact operator T on X. Then T is a quasinilpotent operator too.*

Proof: By Lemma 3 for each $s \in \mathbb{C}, x \in X$ and $n \in \mathbb{N}$ one has $\lim_{k\to\infty} s^k T_n^k x = 0$. Let us choose arbitrary $s \in \mathbb{C}$, $x \in X$, $k \in \mathbb{N}$ and any $p_\alpha \in P$. Since $T_n \xrightarrow{\tau_b} T$ by Corollary 1, it follows also that $T_n^k \xrightarrow{\tau_b} T^k$. Hence there is some $n_0 \in \mathbb{N}$ such that

$$|s^k| p_\alpha((T^k - T_n^k)x) < 1/k, \ n \geq n_0.$$

Consequently $p_\alpha(s^k T^k x) \leq |s^k| p_\alpha((T^k - T_n^k)x) + p_\alpha(s^k T_n^k x) \leq 1/k + p_\alpha(s^k T_n^k x)$, $n \geq n_0$. Since all T_n are quasinilpotent, letting $k \to \infty$, by Lemma 3 the operator T is quasinilpotent too. □

Also, in locally convex spaces, we have

Definition 2. *An algebra of operators $\mathcal{A} \subset \mathcal{L}(X)$ is transitive if there is no subspace invariant for all operators from $\mathcal{A}$ other than $\{0\}$ and X.*

It is well known that an algebra $\mathcal{A}$ is transitive if and only if $\mathcal{A}x$ is dense in X for each $x \neq 0$ (see [7]).

Proposition 2. *Let X be a barreled complete locally convex space and $\mathcal{A}$ a transitive algebra of operators in $\mathcal{K}(X)$. Then there is some finite rank operator F and a sequence $(A_n) \subset \mathcal{A}$ such that $A_n \xrightarrow{\tau_b} F$.*

Proof: By Lomonosov's theorem ([8]) there exists $K \in \mathcal{A}$ such that $\lambda = 1$ is an eigenvalue of K. The spectrum $\sigma(K)$ consists of at most a denumerable number of points with accumulation point 0 ([1]). Let O_1 be an open set containing $\lambda = 1$ that does not meet $\sigma(K)$ in other points and let $f(\lambda)$ be an analytic function equal to 1 for $\lambda \in O_1$ and 0 otherwise. Let us choose $\varepsilon > 0$ such that a closed circle S_0 with radius ε around $\lambda = 1$ is contained in O_1 and S_i, $i = 1, 2, \dots, n$, closed circles with radii ε such that the union $\bigcup_1^n S_i$ covers the other points of $\sigma(K)$ and does not intersect O_1. Denote $D = \bigcup_0^n S_i$ and let γ be the boundary of D. By the Riesz functional calculus ([13]) for locally convex spaces there is an operator in $F \in \mathcal{L}(X)$ obtained by

$$F = f(K) = \frac{1}{2\pi i} \int_\gamma f(\lambda) R(\lambda, K) d\lambda$$

where $R(\lambda, K) = (\lambda I - K)^{-1}$ is the resolvent operator of K. As in the Banach space case, one can prove that $F^2 = F$ and that F is a finite rank operator. The function $f(\lambda)$ can be uniformly approximated on compact sets by polynomials. Let $P_m(\lambda)$ be a polynomial such that

$$\max_{\lambda \in D} |f(\lambda) - P_m(\lambda)| < \varepsilon.$$

It is clear that $P_m(K) \in \mathcal{A}$. For an arbitrary bounded set M and any $p_\alpha \in P$ we can estimate

$$q_\alpha^M(f(K) - P_m(K)) \leq \frac{1}{2\pi} \int_\gamma |f(\lambda) - P_m(\lambda)|. \sup_{x \in M} p_\alpha(R(\lambda, K)x) d|\lambda|$$
$$\leq \frac{\varepsilon}{2\pi} L C_\alpha,$$

where $C_\alpha = \sup_{\lambda \in D} \sup_{x \in M} p_\alpha(R(\lambda, K)x)$ and L is the length of γ. □

Lemma 4. *Let X be a locally convex space, $\mathcal{A}$ a transitive algebra in $\mathcal{L}(X)$ and $\mathcal{J}$ a nontrivial ideal in $\mathcal{A}$. Then $\mathcal{J}$ is transitive too.*

Proof: Let us choose arbitrary nonzero elements $x \in X$ and $A \in \mathcal{J}$. Since $\mathcal{A}x$ is dense in X, if $\mathcal{A}x \subset \mathcal{N}(A)$ then $\mathcal{N}(A) = X$. Thus, there is some $B \in \mathcal{A}$ such that $ABx \neq 0$ and by transitivity it follows that $\mathcal{A}(ABx)$ is dense in X. Obviously $\mathcal{A}AB \subset \mathcal{J}$, hence $\mathcal{J}x$ is dense too. □

For an algebra $\mathcal{A}$ of compact operators, let us denote by $\widetilde{\mathcal{A}}$ the set of all operators $A \in \mathcal{K}(X)$ which are τ_b-limits of some sequence (A_n) of operators from $\mathcal{A}$. By Proposition 1 it follows that $\widetilde{\mathcal{A}}$ is a closed algebra when X is barreled.

Proposition 3. *Let X be a complete barreled locally convex space and $\mathcal{A}$ a transitive algebra in $\mathcal{K}(X)$. Then the algebra $\widetilde{\mathcal{A}} \cap \mathcal{F}(X)$ is also transitive.*

Proof: By Proposition 1, $\widetilde{\mathcal{A}} \cap \mathcal{F}(X)$ is an ideal in $\widetilde{\mathcal{A}}$ and by Proposition 2 it is different from $\{0\}$. Clearly, $\widetilde{\mathcal{A}}$ is transitive too and by Lemma 4 the ideal $\widetilde{\mathcal{A}} \cap \mathcal{F}(X)$ is also transitive. □

Lemma 5. *Let T be a compact and quasinilpotent operator on the locally convex space X and N an invariant subspace of T. Then the operator $\hat{T}$ on the quotient space $\hat{X} := X/N$ is also compact and quasinilpotent.*

Proof: Let us denote by π the natural homomorpism of X onto $\hat{X}$. The topology of the locally convex space $\hat{X}$ is generated by the system of seminorms $\hat{P} = \{\hat{p}_\alpha, \alpha \in \Delta\}$, where $\hat{p}_\alpha(\hat{x}) = \inf\{p_\alpha(x + z), z \in N\}$ for $\hat{x} = x + N$ and $p_\alpha \in P$. Clearly π is continuous and by the relation $\hat{T}\pi = \pi T$ the compactness of $\hat{T}$ follows. Since T is quasinilpotent by Lemma 3, for any $s \in \mathbb{C}$, $x \in X$ and $p_\alpha \in P$ we have $p_\alpha(s^k T^k x) \stackrel{k\to\infty}{\longrightarrow} 0$. Note that

$$\hat{p}_\alpha(s^k \hat{T}^k \hat{x}) \leq p_\alpha(s^k T^k x), \quad \hat{x} = x + N.$$

Thus, the left hand side also tends to 0; hence $\hat{T}$ is (by Lemma 3) quasinilpotent. □

The main result is the following generalization of a result from [5].

Theorem 2. *An algebra $\mathcal{A}$ of compact operators on a complete barreled locally convex space is triangularizable if and only if $AB-BA$ is a quasinilpotent operator for each pair $A, B \in \mathcal{A}$.*

Proof: Let $\mathcal{A}$ be triangularizable and let $\mathcal{C}$ be a maximal chain of invariant subspaces of $\mathcal{A}$. Then $d_N(AB-BA) = d_N(A)d_N(B) - d_N(B)d_N(A) = 0$ (see [9]) for each $N \in \mathcal{C}$. Since a nonzero number λ is a diagonal coefficient if and only if it is an eigenvalue ([9]) and since the spectrum of a nonzero compact operator is not empty ([13]), $AB-BA$ is quasinilpotent. Let this supposition be fulfilled. By Lemma 5 the quasinilpotency is inherited by quotients. Thus by Lemma 1 it suffices to show that each such algebra has a nontrivial invariant subspace. If this were not the case, $\mathcal{A}$ would be transitive and then by Proposition 3 the algebra $\widetilde{\mathcal{A}} \cap \mathcal{F}(X)$ would be transitive too. By Proposition 1 and by Theorem 1 this algebra also satisfies the above condition. Let us choose an $F \in \widetilde{\mathcal{A}} \cap \mathcal{F}(X)$ with rank greater than 2. The restriction of the algebra $\widetilde{\mathcal{A}} \cap \mathcal{F}(X)$ to $\mathcal{R}(F)$ is equal to $\mathcal{L}(\mathcal{R}(F))$ by Burnside's theorem. Consequently, operators of the form FAF, $A \in \widetilde{\mathcal{A}}$, include the following operators

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus 0 \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus 0.$$

Then it is easy to see that $CD-DC$ is not quasinilpotent which is a contradiction. □

The following result is a generalization of [11].

Proposition 4. *Let X be a barreled locally convex space and (S_n), (T_n) two sequences of compact operators on X such that $S_n \xrightarrow{\mathcal{T}_b} S$ and $T_n \xrightarrow{\mathcal{T}_b} T$, where S and T are compact operators. Let, for each n, the pair $\{S_n, T_n\}$ be triangularizable. Then the pair $\{S, T\}$ is triangularizable too.*

Proof: If we take into account the result from [9] it is sufficient to prove that for each noncommutative polynomial f in two variables the operator $C := (ST - TS)f(S,T)$ is quasinilpotent. By the same result and by our hypothesis, the operator $C_n := (S_nT_n - T_nS_n)f(S_n, T_n)$ is quasinilpotent for each $n \in \mathbb{N}$. All operators C_n, $n \in \mathbb{N}$, and C are compact and by Proposition 1 it follows that $C_n \xrightarrow{\mathcal{T}_b} C$. By Theorem 1, C is quasinilpotent. □

The next result is a generalization of a result proved for Hilbert spaces in [4].

Theorem 3. *Let X be a locally convex space and $\mathcal{A}$ an algebra in $\mathcal{L}(X)$ such that for all $A \in \mathcal{A}$, $A^k = 0$ for some $k \in \mathbb{N}$. Then $\mathcal{A}$ is triangularizable.*

Proof: Clearly, the nilpotency is inherited by quotients. Thus, by Lemma 1 it suffices to prove that $\mathcal{A}$ has an invariant subspace. Choose $A \in \mathcal{A}$ such that

$A^{k-1} \neq 0$. Since $(A+zB)^k = 0$ for each $z \in \mathbb{C}$ and $B \in \mathcal{A}$ it follows that the coefficient of each z^n is zero. In particular, for $n = 1$ we obtain

$$BA^{k-1} = -A(BA^{k-2} + ABA^{k-3} + \cdots + A^{k-2}B)$$

and, consequently, $\mathcal{R}(BA^{k-1}) \subset \mathcal{R}(A)$. Let us show that $\overline{\mathcal{R}(A)} \neq X$. If $\overline{\mathcal{R}(A)} = X$ for any $y \in X$ there would exist a sequence $(x_\delta) \subset X$ such that $Ax_\delta \to y$. By continuity of A^{k-1} it follows that $0 = A^k x_\delta \to A^{k-1}y$. Thus A^{k-1} would be the zero operator, which is a contradiction. Choose now a nonzero $x \in \mathcal{R}(A^{k-1})$ and define $M = \{Bx, B \in \mathcal{A}\}$. We have two possibilities. a) $M \neq \{0\}$. Clearly, M is an invariant subspace for $\mathcal{A}$ and then $\overline{M}$ is invariant too. By the above inclusion it follows that $M \subset \mathcal{R}(A)$, hence $\overline{M} \neq X$. b) $M = \{0\}$. Letting $N = \{\lambda x, \lambda \in \mathbb{C}\}$, N is a proper closed invariant subspace for $\mathcal{A}$. □

Let us denote by $B_P(X)$ the family of all operators T on X for which $p_\alpha(Tx) \leq Cp_\alpha(x)$, $x \in X$, $p_\alpha \in P$, where C is independent of $p_\alpha \in P$. This is a normed algebra with the norm $\|T\|_P = \sup\{p_\alpha(Tx) : p_\alpha(x) \leq 1, x \in X, p_\alpha \in P\}$ and it is complete if X is complete (see eg. [6]).

Corollary 2. *Let $\mathcal{A}$ be a $\|.\|_P$-closed algebra of nilpotent operators in $B_P(X)$. Then $\mathcal{A}$ is triangularizable.*

Proof: By Grabiner ([3]), there exists a number $k \in \mathbb{N}$ such that $A^k = 0$ for all $A \in \mathcal{A}$, and the conclusion follows by the above theorem. □

References

[1] Edwards R.E., *Functional Analysis, Theory and Applications*, Holt, Reinehart and Winston, New York, 1965.

[2] Floret K., Wloka J., *Einführung in die Theorie der lokalkonvexen Räumen*, Springer-Verlag, Berlin, Heidelberg, New York, 1968.

[3] Grabiner S., *The nilpotency of Banach nil algebras*, Proc. Amer. Math. Soc. **21** (1969), 510.

[4] Hadwin D., Nordgren E., Radjabalipour M., Radjavi H., Rosenthal P., *On simultaneous triangularization of collection of operators*, Houston J. Math. **17** (1991), 581–602.

[5] Katavolos A., Radjavi H., *Simultaneous triangulation of operators on a Banach space*, J. London Math. Soc. **41** (1990), 547–554.

[6] Kramar E., *Invariant subspaces for some operators on locally convex spaces*, Comment. Math. Univ. Carolinae **38** (1997), 635–644.

[7] Litvinov G.L., Lomonosov V.I., *Density theorems in locally convex spaces and theirs applications* (in Russian), Trudi sem. vekt. i tenz. analiza **20** (1981), 210–227.

[8] Lomonosov V.I., *Invariant subspace of operators commuting with compact operators*, Funct. Anal. Appl. **7** (1973), 213–214.

[9] Ma T.W., *On rank one commutators and triangular representations*, Canad. Math. Bull. **29** (1986), 268–273.

[10] Mendoza R.V., *The (Γ, t)-topology on $L(E,E)$ and the spectrum of a bounded linear operator on a locally convex topological vector space*, Bol. Soc. Mat. Mexicana **3** (1997), 151–164.

[11] Radjavi H., Rosenthal P., *From local to global triangularization*, J. Funct. Anal. **147** (1997), 443–456.

[12] Ringrose J.R., *Super-diagonal forms for compact linear operators*, Proc. London Math. Soc. (3) **12** (1962), 367–384.

[13] Uss P., *Sur les opérateurs bornés dans les espaces localement convexes*, Studia Math. **37** (1971), 139–158.

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