Integrability for vector-valued minimizers of some variational integrals

Francesco Leonetti, Francesco Siepe

Abstract. We prove that the higher integrability of the data f, f_0 improves on the integrability of minimizers u of functionals $\mathcal{F}$, whose model is

$$\int_\Omega \left[|Du|^p + (\det(Du))^2 - \langle f, Du\rangle + \langle f_0, u\rangle\right] dx,$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ and $p \geq 2$.

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1. Introduction

Let us consider the following elliptic boundary value problem

$$\begin{cases} \operatorname{div}(aDu) = \operatorname{div}(f) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and $a = \{a_{ij}(x)\}$ is an elliptic matrix with measurable and bounded entries. In Stampacchia's book [14, Chapter 4] we can find the following regularity result for weak solutions $u \in W_0^{1,2}(\Omega)$ of (1.1) with $f \in L^q(\Omega)$:

$$\begin{cases} q > n & \Longrightarrow & u \in L^\infty(\Omega), \\ 2 < q < n & \Longrightarrow & u \in L^{q^*}(\Omega). \end{cases} \tag{1.2}$$

In (1.1) we have the boundary condition $u = 0$ and one single elliptic equation $\operatorname{div}(aDu) = \operatorname{div}(f)$. Let us consider the case of a system of N elliptic equations:

$$\begin{cases} \operatorname{div}(ADu) = \operatorname{div}(f) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and $A = \{A_{ij}^{\alpha\beta}(x)\}$ is elliptic with measurable and bounded entries. De Giorgi's counterexample shows that regularity (1.2) does not

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hold true any longer [4], [8, Chapter 2, Section 3]. However, if the matrix $A = \{A_{ij}^{\alpha\beta}(x,u)\}$ is "diagonal" for large values of u, that is $A_{ij}^{\alpha\beta}(x,u) = a_{ij}^{\alpha}(x,u)\delta^{\alpha\beta}$ for $|u| \geq R$, then (1.2) can be recovered ([13]). Solutions $u \in W_0^{1,2}(\Omega,\mathbb{R}^N)$ of (1.3) are minimizers of the functional

$$I(u) = \int_\Omega \langle ADu, Du\rangle\,dx - \int_\Omega \langle f, Du\rangle\,dx \tag{1.4}$$

provided the matrix $A = \{A_{ij}^{\alpha\beta}(x)\}$ is symmetric. Viceversa, minimizers $u \in W_0^{1,2}(\Omega,\mathbb{R}^N)$ of (1.4) are solutions to the boundary value problem (1.3). In this paper we consider more general functionals

$$I(u) = \int_\Omega G(x,u(x),Du(x))\,dx - \int_\Omega \langle f, Du\rangle\,dx \tag{1.5}$$

and we prove that the degree of integrability of f improves on the integrability of u as in (1.2). Because of De Giorgi's counterexample, we have to assume some restrictions on $G(x,u,Du)$. A simple model for our results is

$$G(x,u,Du) = |Du|^p + |\det(Du)|^2, \tag{1.6}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ and $p \geq 2$. The higher integrability of minimizers is achieved by using test functions built by means of truncation of only one component u^γ of our minimizer $u = (u^1,\dots,u^n)$. The truncation argument has been successfully employed in the scalar case $u : \Omega \to \mathbb{R}$ ([14], [1], [2], [7]) and in some special vector valued cases $u : \Omega \to \mathbb{R}^N$ ([13]). The leading part $|Du|^p$ in (1.6) is one of those special cases ([13]); the main feature of our model (1.6) is the presence of $|\det(Du)|^2$ and its good behaviour with respect to the truncation argument (see [11], [12], [5], [6]).

2. Statements and preliminary results

In this section we introduce some notations and we state the result which will be proved in the next section.

In the following Ω will always denote a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ and c a constant that may vary from line to line.

First of all, let us recall the definition of *weak* L^p-spaces, or Marcinkiewicz spaces (see [3, Chapter 1, Section 2], [9, Chapter 2, Section 5] or [10, Chapter 2, Section 18]):
for $p > 0$ we will say that $f \in L_w^p(\Omega)$ if and only if there exists a positive constant $k = k(f)$ such that

$$|\{x \in \Omega :\ |f(x)| > t\}| \leq \frac{k}{t^p} \tag{2.1}$$

for every $t > 0$, where $|E|$ is the n-dimensional Lebesgue measure of $E \subset \mathbb{R}^n$. We recall that if $f \in L_w^p$ for some $p > 1$, then $f \in L^q$ for every $1 \leq q < p$.

Later we will use the following result (see [3, Chapter 1, Lemma 2.1]).

Lemma 2.1. *Let $p > 1$. Then $f \in L^p_w(\Omega)$ if and only if for every measurable set $E \subset \Omega$, the following inequality holds*

$$\int_E |f|\, dx \le c|E|^{\frac{p-1}{p}}$$

for some constant $c > 0$.

We will also need the following technical result (see [14, Lemma 4.1]).

Lemma 2.2. *Let $s_0 > 0$ and let $\psi : (s_0, +\infty) \to [0, +\infty)$ be a decreasing function, such that for every h, k with $h > k > s_0$*

$$\psi(h) \le \frac{c}{(h-k)^\alpha} (\psi(k))^\beta,$$

where c, α, β are positive constants. Then

(i) *if $\beta > 1$ we have that $\psi(s_0 + d) = 0$, where*

$$d^\alpha = c\, 2^{\frac{\alpha\beta}{\beta-1}} (\psi(s_0))^{\beta-1};$$

(ii) *if $\beta < 1$ we have that*

$$\psi(h) \le 2^{\frac{\mu}{1-\beta}} \left[c^{\frac{1}{1-\beta}} + (2s_0)^\mu \psi(s_0) \right] h^{-\mu},$$

where $\mu = \frac{\alpha}{1-\beta}$.

For $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ we write Du for the Jacobian matrix $D_i^\alpha u$, $\alpha = 1, \dots, N$, $i = 1, \dots, n$, with N rows and n columns. We set $n \wedge N = \min\{n, N\}$ and consider the functional

$$\begin{aligned} \mathcal{F}(u) = \int_\Omega L(x, u(x), Du(x))\, dx + \sum_{s=1}^{n\wedge N} \int_\Omega g_s(|M_s Du(x)|)\, dx \\ - \int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N f_i^\alpha(x) D_i u^\alpha(x)\, dx + \int_\Omega \sum_{\alpha=1}^N f_0^\alpha(x) u^\alpha(x)\, dx, \end{aligned} \tag{2.2}$$

where $M_s Du(x)$ is the vector containing all the $s \times s$-minors taken from the $N \times n$ matrix $Du(x)$.
We assume that for every $s = 1 \dots, n \wedge N$, $g_s : [0, +\infty) \to \mathbb{R}$ is increasing and $g_s \ge 0$.

For the leading part L of the functional (2.2) we assume that $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is measurable with respect to $x \in \Omega$ and continuous with respect to

$(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$, with $L \geq 0$. Moreover, there exists $s_0 \geq 0$ and $p \in (1, n)$ such that

$$L(x, u, \xi) = \sum_{\alpha=1}^{N} \left(\sum_{i,j=1}^{n} a_{ij}^{\alpha}(x) \xi_i^{\alpha} \xi_j^{\alpha} \right)^{\frac{p}{2}} \qquad \text{if } |u| \geq s_0, \tag{2.3}$$

where the functions a_{ij}^{α} belong to $L^{\infty}(\Omega)$ and satisfy the following ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}^{\alpha}(x) \eta_i \eta_j \geq \nu |\eta|^2 \tag{2.4}$$

for every $\eta \in \mathbb{R}^n$, for any $\alpha = 1, \ldots, N$ and for some $\nu > 0$.

Finally, for the linear part of (2.2) we will assume that

$$f \in L^{p'}(\Omega, \mathbb{R}^{N\times n}), \tag{2.5}$$

$$f_0 \in L^{(p^*)'}(\Omega, \mathbb{R}^N) \tag{2.6}$$

where $r' = \frac{r}{r-1}$ and $p^* = \frac{np}{n-p}$.

Let us remark that assumptions (2.5)–(2.6) guarantee that $\langle f, Dv\rangle \in L^1(\Omega)$ and $\langle f_0, v\rangle \in L^1(\Omega)$, for every $v \in W_0^{1,p}(\Omega, \mathbb{R}^N)$.

A minimizer of functional (2.2) is a function $u : \Omega \to \mathbb{R}^N$ such that $u \in W_0^{1,p}(\Omega, \mathbb{R}^N)$, with $x \to L(x, u(x), Du(x)) \in L^1(\Omega)$ and $g_s(|M_s Du|) \in L^1(\Omega)$ $\forall\, s = 1, \ldots, n \wedge N$ and

$$\mathcal{F}(u) \leq \mathcal{F}(v) \qquad \forall\, v \in W_0^{1,p}(\Omega, \mathbb{R}^N). \tag{2.7}$$

Let us write the components of f and f_0 in the way

$$f(x) = (f^1(x), \ldots, f^N(x)) \quad \text{with } f^{\alpha}(x) \in \mathbb{R}^n$$

and

$$f_0(x) = (f_0^1(x), \ldots, f_0^N(x)) \quad \text{with } f_0^{\alpha}(x) \in \mathbb{R}.$$

Let us assume that there exists an index $\gamma \in \{1, \ldots, N\}$ and an exponent $q > p' = \frac{p}{p-1}$ such that

$$f^{\gamma} \in L_w^q(\Omega, \mathbb{R}^n), \qquad f_0^{\gamma} \in L_w^{q_*}(\Omega), \tag{2.8}$$

where $q_* = \frac{nq}{n+q}$. The main result of the paper is the following

Theorem 2.3. *Let $u = (u^1, \dots, u^N)$ be a minimizer of functional* (2.2), *under the previous assumptions. Then the component u^γ of our minimizer enjoys the following regularity:*

(i)
$$q > \frac{n}{p-1} \quad \Longrightarrow \quad u^\gamma \in L^\infty(\Omega);$$

(ii)
$$q < \frac{n}{p-1} \quad \Longrightarrow \quad u^\gamma \in L^m_w(\Omega),$$

where $m = [q(p-1)]^*$.

Remark 2.1. The previous theorem still holds true when

$$L(x,u,\xi) = \left(\sum_{\alpha=1}^{N} \sum_{i,j=1}^{n} a_{ij}^\alpha(x) \xi_i^\alpha \xi_j^\alpha \right)^{\frac{p}{2}} \qquad \text{for } |u| \geq s_0, \tag{2.9}$$

where $a_{ij}^\alpha \in L^\infty(\Omega)$ and satisfy (2.4), provided $p \geq 2$.

Example 2.1. For $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, that is $n = N$, let us write $u = (u^1, \dots, u^n)$. A functional model for (2.2) is

$$\begin{aligned} \mathcal{F}(u) = &\int_\Omega \sum_{\alpha=1}^{n} |Du^\alpha|^p \, dx + \int_\Omega |\det(Du)|^2 \, dx \\ &- \int_\Omega \langle f, Du \rangle \, dx + \int_\Omega \langle f_0, u \rangle \, dx, \end{aligned} \tag{2.10}$$

where $1 < p < n$. The structure (2.3) is easily checked.

Example 2.2. For $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, a functional model for (2.9) is

$$\begin{aligned} \mathcal{F}(u) = &\int_\Omega |Du|^p \, dx + \int_\Omega |\det(Du)|^2 \, dx \\ &- \int_\Omega \langle f, Du \rangle \, dx + \int_\Omega \langle f_0, u \rangle \, dx, \end{aligned} \tag{2.11}$$

where $2 \leq p < n$.

3. Proof of Theorem 2.3

Let $k > s_0$ and define $T_k : \mathbb{R} \to \mathbb{R}$ such that

$$T_k(s) = \begin{cases} -k & \text{if } s \leq -k, \\ s & \text{if } -k < s < k, \\ k & \text{if } k \leq s. \end{cases}$$

For γ as in our assumptions we consider $v : \Omega \to \mathbb{R}^N$ defined as follows

$$v^\alpha = \begin{cases} T_k(u^\gamma) & \text{if } \alpha = \gamma, \\ u^\alpha & \text{if } \alpha \neq \gamma. \end{cases} \tag{3.1}$$

Since $u^\gamma \in W_0^{1,p}(\Omega)$, it follows that $T_k(u^\gamma) \in W_0^{1,p}(\Omega)$ and

$$D(T_k(u^\gamma)) = Du^\gamma \ \chi_{\{|u^\gamma|<k\}},$$

where χ_E is the characteristic function of the set E, that is $\chi_E(x) = 1$ if x belongs to E, $\chi_E(x) = 0$ if x does not belong to E. Thus

$$Dv^\alpha = \begin{cases} Du^\gamma \ \chi_{\{|u^\gamma|<k\}} & \text{if } \alpha = \gamma, \\ Du^\alpha & \text{if } \alpha \neq \gamma. \end{cases} \tag{3.2}$$

Let $\varphi = v - u$; then $\varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ and

$$D\varphi^\gamma = Dv^\gamma - Du^\gamma = -Du^\gamma \ \chi_{\{|u^\gamma|\geq k\}}.$$

Thus, for almost every $x \in \{|u^\gamma| < k\}$ we have:

$$\begin{cases} v(x) = u(x), \\ Dv(x) = Du(x), \\ M_s Dv(x) = M_s Du(x) \quad \forall\ s = 1, \dots, n \wedge N, \\ L(x, v(x), Dv(x)) = L(x, u(x), Du(x)), \\ g_s(|M_s Dv(x)|) = g_s(|M_s Du(x)|) \quad \forall\ s = 1, \dots, n \wedge N, \end{cases} \tag{3.3}$$

while for a.e. $x \in \{|u^\gamma| \geq k\}$ it is easy to see that:

$$\begin{cases} |M_s Dv(x)| \leq |M_s Du(x)| \quad \forall\ s = 1, \dots, n \wedge N, \\ 0 \leq L(x, v(x), Dv(x)) \leq L(x, u(x), Du(x)), \\ 0 \leq g_s(|M_s Dv(x)|) \leq g_s(|M_s Du(x)|) \quad \forall\ s = 1, \dots, n \wedge N. \end{cases} \tag{3.4}$$

Hence $x \to L(x, v(x), Dv(x)) \in L^1(\Omega)$ and $g_s(|M_s Dv|) \in L^1(\Omega)$ for every $s = 1, \dots, n \wedge N$.

We use (2.7) with v as before.

We split Ω into the two subsets $\{|u^\gamma| \geq k\}$ and $\{|u^\gamma| < k\}$; recalling (3.3) we easily obtain

$$
\begin{aligned}
&\int_{\{|u^\gamma|\geq k\}} L(x,u,Du)\,dx + \sum_{s=1}^{n\wedge N} \int_{\{|u^\gamma|\geq k\}} g_s(|M_s Du|)\,dx \\
&\qquad - \int_{\{|u^\gamma|\geq k\}} \sum_{\alpha=1}^{N}\sum_{i=1}^{n} f_i^\alpha D_i u^\alpha\,dx + \int_{\{|u^\gamma|\geq k\}} \sum_{\alpha=1}^{N} f_0^\alpha u^\alpha\,dx \\
&\leq \int_{\{|u^\gamma|\geq k\}} L(x,v,Dv)\,dx + \sum_{s=1}^{n\wedge N} \int_{\{|u^\gamma|\geq k\}} g_s(|M_s Dv|)\,dx \\
&\qquad - \int_{\{|u^\gamma|\geq k\}} \sum_{\alpha=1}^{N}\sum_{i=1}^{n} f_i^\alpha D_i v^\alpha\,dx + \int_{\{|u^\gamma|\geq k\}} \sum_{\alpha=1}^{N} f_0^\alpha v^\alpha\,dx.
\end{aligned} \tag{3.5}
$$

Because of (3.4), for every $s = 1, \ldots, n\wedge N$ we have

$$
\int_{\{|u^\gamma|\geq k\}} g_s(|M_s Dv|)\,dx \leq \int_{\{|u^\gamma|\geq k\}} g_s(|M_s Du|)\,dx;
$$

thus, the integrals containing $M_s Du$ and $M_s Dv$ can be dropped in (3.5).
Using (3.1) and (3.2) we get

$$
\begin{aligned}
&\int_{\{|u^\gamma|\geq k\}} L(x,u,Du)\,dx - \int_{\{|u^\gamma|\geq k\}} \sum_{i=1}^{n} f_i^\gamma D_i u^\gamma\,dx + \int_{\{|u^\gamma|\geq k\}} f_0^\gamma u^\gamma\,dx \\
&\qquad \leq \int_{\{|u^\gamma|\geq k\}} L(x,v,Dv)\,dx + \int_{\{|u^\gamma|\geq k\}} f_0^\gamma T_k(u^\gamma)\,dx.
\end{aligned} \tag{3.6}
$$

Furthermore, since $k \geq s_0$, in the set $\{|u^\gamma| \geq k\}$ we get

$$
\begin{aligned}
&L(x,u(x),Du(x)) \\
&\geq L(x,v(x),Dv(x)) + \Big(\sum_{i,j=1}^{n} a_{ij}^\gamma(x) D_i u^\gamma(x) D_j u^\gamma(x) \Big)^{\frac{p}{2}}.
\end{aligned} \tag{3.7}
$$

Indeed, if $L(x,u,\xi)$ has the structure described in (2.3), then (3.7) holds with equality sign, while if $L(x,u,\xi)$ is the one of (2.9), then to obtain (3.7) we use the inequality $(x_1^2+x_2^2)^{\frac{p}{2}} \geq x_1^p + x_2^p$, which holds true for every $x_1, x_2 \geq 0$, provided $p \geq 2$.
Hence by (3.6) and (3.7) we have

$$
\begin{aligned}
&\int_{\{|u^\gamma|\geq k\}} \Big(\sum_{i,j=1}^{n} a_{ij}^\gamma D_i u^\gamma D_j u^\gamma \Big)^{\frac{p}{2}} dx \\
&\leq \int_{\{|u^\gamma|\geq k\}} \sum_{i=1}^{n} f_i^\gamma D_i u^\gamma\,dx + \int_{\{|u^\gamma|\geq k\}} f_0^\gamma \left[T_k(u^\gamma) - u^\gamma\right] dx.
\end{aligned} \tag{3.8}
$$

Now we use ellipticity condition (2.4) in (3.8) so that

$$\nu^{\frac{p}{2}} \int_{\{|u^\gamma|\geq k\}} |Du^\gamma|^p\,dx \leq \int_{\{|u^\gamma|\geq k\}} \sum_{i=1}^{n} f_i^\gamma D_i u^\gamma\,dx + \int_{\{|u^\gamma|\geq k\}} f_0^\gamma \varphi^\gamma\,dx, \tag{3.9}$$

where we recall that $\varphi = v - u$.
We observe that for almost every $x \in \{|u^\gamma| = k\}$ we have $Du^\gamma(x) = 0$ and $\varphi^\gamma = 0$. Then by applying Hölder inequality to the right hand side of (3.9) we have

$$\begin{aligned} \nu^{\frac{p}{2}} \int_{\{|u^\gamma|>k\}} |Du^\gamma|^p\,dx \leq & \Big(\int_{\{|u^\gamma|>k\}} |Du^\gamma|^p\,dx\Big)^{\frac{1}{p}} \Big(\int_{\{|u^\gamma|>k\}} |f^\gamma|^{p'}\,dx\Big)^{\frac{1}{p'}} \\ & + \Big(\int_{\{|u^\gamma|>k\}} |\varphi^\gamma|^{p^*}\,dx\Big)^{\frac{1}{p^*}} \Big(\int_{\{|u^\gamma|>k\}} |f_0^\gamma|^{(p^*)'}\,dx\Big)^{\frac{1}{(p^*)'}}. \end{aligned} \tag{3.10}$$

Now we use Sobolev inequality for the function φ^γ in Ω and we note that $D\varphi^\gamma = -Du^\gamma$ in $\{|u^\gamma| > k\}$, while $D\varphi^\gamma = 0$ in $\{|u^\gamma| \leq k\}$, so that by (3.10) we easily get

$$\begin{aligned} \nu^{\frac{p}{2}} \Big(\int_{\{|u^\gamma|>k\}} |Du^\gamma|^p\,dx\Big)^{\frac{1}{p'}} \leq & \Big(\int_{\{|u^\gamma|>k\}} |f^\gamma|^{p'}\,dx\Big)^{\frac{1}{p'}} \\ & + c\Big(\int_{\{|u^\gamma|>k\}} |f_0^\gamma|^{(p^*)'}\,dx\Big)^{\frac{1}{(p^*)'}}, \end{aligned} \tag{3.11}$$

where $c = c(n,p)$. We observe also that, again by Sobolev inequality

$$\begin{aligned} \int_{\{|u^\gamma|>k\}} |Du^\gamma|^p\,dx &= \int_{\{|u^\gamma|>k\}} |D\varphi^\gamma|^p\,dx = \int_{\Omega} |D\varphi^\gamma|^p\,dx \\ &\geq c\Big(\int_{\Omega} |\varphi^\gamma|^{p^*}\,dx\Big)^{\frac{p}{p^*}} = c\Big(\int_{\{|u^\gamma|>k\}} (|u^\gamma| - k)^{p^*}\,dx\Big)^{\frac{p}{p^*}}, \end{aligned}$$

with $c = c(n,p)$. Then (3.11) leads to

$$\begin{aligned} \int_{\{|u^\gamma|>k\}} (|u^\gamma| - k)^{p^*}\,dx \leq c\Big[& \Big(\int_{\{|u^\gamma|>k\}} |f^\gamma|^{p'}\,dx\Big)^{\frac{p^*}{p}} \\ & + \Big(\int_{\{|u^\gamma|>k\}} |f_0^\gamma|^{(p^*)'}\,dx\Big)^{\frac{p^*-1}{p-1}}\Big], \end{aligned} \tag{3.12}$$

where $c = c(n,p,\nu)$.

By the *weak integrability* assumptions (2.8) and by (2.1), we deduce that

$$|\{x \in \Omega : \; |f^\gamma|^{p'} > \sigma\}| = |\{x \in \Omega : \; |f^\gamma| > \sigma^{\frac{1}{p'}}\}| \le \frac{c_0(f^\gamma)}{\sigma^{\frac{q}{p'}}}$$

and

$$|\{x \in \Omega : \; |f_0^\gamma|^{(p^*)'} > \sigma\}| = |\{x \in \Omega : \; |f^\gamma| > \sigma^{\frac{1}{(p^*)'}}\}| \le \frac{c_0(f_0^\gamma)}{\sigma^{\frac{q_*}{(p^*)'}}} .$$

Then, by applying Lemma 2.1 to f^γ and f_0^γ we obtain that

$$\left(\int_{\{|u^\gamma|>k\}} |f^\gamma|^{p'} \, dx \right)^{\frac{p^*}{p}} \le c_1 |\{|u^\gamma| > k\}|^{\frac{p^*(q-p')}{pq}}, \tag{3.13}$$

where $c_1 = c_1(f^\gamma, n, p, q, \Omega)$ and

$$\left(\int_{\{|u^\gamma|>k\}} |f_0^\gamma|^{(p^*)'} \, dx \right)^{\frac{p^*-1}{p-1}} \le c_2 |\{|u^\gamma| > k\}|^{\frac{(p^*-1)(q_*-(p^*)')}{q_*(p-1)}}, \tag{3.14}$$

where $c_2 = c_2(f_0^\gamma, n, p, q, \Omega)$.
It is easy to see that the exponents at the right hand side of (3.13) and (3.14) coincide; we set

$$\beta = \frac{p^*(q-p')}{pq} , \tag{3.15}$$

so that, by (3.12) we obtain

$$\int_{\{|u^\gamma|>k\}} (|u^\gamma| - k)^{p^*} \, dx \le c |\{|u^\gamma| > k\}|^\beta \tag{3.16}$$

where $c = c(n, p, q, \nu, f^\gamma, f_0^\gamma, \Omega)$.

Since for every $h > k$ the inclusion $\{|u^\gamma| > h\} \subset \{|u^\gamma| > k\}$ holds true, we have

$$\int_{\{|u^\gamma|>k\}} (|u^\gamma| - k)^{p^*} \, dx \ge \int_{\{|u^\gamma|>h\}} (|u^\gamma| - k)^{p^*} \, dx \ge (h-k)^{p^*} |\{|u^\gamma| > h\}|.$$

Thus (3.16) becomes

$$|\{|u^\gamma| > h\}| \le \frac{c}{(h-k)^{p^*}} |\{|u^\gamma| > k\}|^\beta, \tag{3.17}$$

where $h > k \geq s_0 \geq 0$.

We use Lemma 2.2 with $\varphi(h) = |\{|u^\gamma| > h\}|$ and $\alpha = p^*$ and we see that

$$\beta > 1 \qquad \Longleftrightarrow \qquad q > \frac{n}{p-1}\,,$$

so, in this case, (3.17) and (i) of Lemma 2.2 guarantee that there exists a positive constant c such that

$$\|u^\gamma\|_{L^\infty} \leq c.$$

On the other hand

$$\beta < 1 \qquad \Longleftrightarrow \qquad q < \frac{n}{p-1}$$

and then, by (ii) of Lemma 2.2 we obtain a positive constant c such that

$$|\{|u^\gamma| > h\}| \leq \frac{c}{h^\mu}\,,$$

where $\mu = \frac{\alpha}{1-\beta} = [q(p-1)]^*$. This concludes the proof of Theorem 2.3. □

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Dipartimento di Matematica Pura ed Applicata, Università di L'Aquila,
67100 L'Aquila, Italy

E-mail: leonetti@univaq.it

Dipartimento di Matematica e Applicazioni per l'Architettura,
Università di Firenze, Piazza Ghiberti 27, 50122 Firenze, Italy

E-mail: siepe@math.unifi.it

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