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# Fractional integro-differentiation in harmonic mixed norm spaces on a half-space

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*Abstract.* In this paper some embedding theorems related to fractional integration and differentiation in harmonic mixed norm spaces $h(p,q,\alpha)$ on the half-space are established. We prove that mixed norm is equivalent to a "fractional derivative norm" and that harmonic conjugation is bounded in $h(p,q,\alpha)$ for the range $0 < p \le \infty$, $0 < q \le \infty$. As an application of the above, we give a characterization of $h(p,q,\alpha)$ by means of an integral representation with the use of Besov spaces.

*Keywords:* embedding theorems, integral representations, conjugation, projections

*Classification:* Primary 31B05; Secondary 31B10, 26A33

## 0. Introduction

**0.1.** Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x|^2 = x_1^2 + \dots + x_n^2$, $dx = dx_1 \cdots dx_n$. Let $\mathbb{R}^{n+1}_+$ denote the upper half-space $\mathbb{R}^n \times (0,\infty)$. A point of this half-space will be represented by $(x,y) = (x_1, \dots, x_n, y)$, $x \in \mathbb{R}^n$, $y > 0$. It will be frequently convenient to set $x_0 = y$. If $f(x,y)$ is a measurable function in $\mathbb{R}^{n+1}_+$ then we write

$$M_p(f;y) = \|f\|_{L^p(\mathbb{R}^n,dx)}, \qquad y > 0, \quad 0 < p \le \infty.$$

The collection of all harmonic (complex-valued) functions $u(x,y)$ for which

$$\|u\|_{h^p} = \sup_{y>0} M_p(u;y) < +\infty$$

is the class $h^p(\mathbb{R}^{n+1}_+)$.

The quasi-normed space $L(p,q,\alpha)$ $\quad (0 < p,q \le \infty, \alpha > 0)$ is the set of those functions $f(x,y)$ measurable in the half-space $\mathbb{R}^{n+1}_+$, for which the quasi-norm

$$\|f\|_{p,q,\alpha} = \begin{cases} \left( \displaystyle\int_0^{+\infty} y^{\alpha q - 1} M_p^q(f;y)\, dy \right)^{1/q}, & 0 < q < \infty, \\ \operatorname*{ess\,sup}_{y>0} \; y^\alpha M_p(f;y), & q = \infty, \end{cases}$$

is finite. Let $h(p,q,\alpha)$ be the subspace of $L(p,q,\alpha)$ consisting of harmonic functions. Harmonic mixed norm spaces $h(p,q,\alpha)$ were investigated by several authors: Taibleson [23], Flett [13]–[15], Bui Huy Qui [4], Ricci and Taibleson [18], A.E. Djrbashian [5], Ramey and Yi [17]. When $p = q < \infty$ the spaces $h(p,q,\alpha)$ are called weighted Bergman spaces, although Bergman ([2], [3]) himself considered since 1929 only functions whose squares are integrable without weight, i.e. the Hilbert space $h(2,2,1/2)$. Weighted classes $h(p,p,\alpha)$, $p \geq 1$, for functions holomorphic in the unit disk were introduced by M.M. Djrbashian ([8], [9]). However, many important theorems concerning holomorphic subspaces of $h(p,q,\alpha)$ are contained in classical works of Hardy and Littlewood. See [12]–[15] for references.

M.M. Djrbashian ([8], [9]) found as well some integral representations for $h(p,p,\alpha)$. Later Ricci and Taibleson ([18]) obtained a family of integral representations for $h(p,q,\alpha)$ on the half-plane (see also [10]). The integral in all the mentioned representations is taken over whole domain. The present paper establishes some other integral representations for $h(p,q,\alpha)$ on the half-space, where the integral is taken over the boundary of $\mathbb{R}^{n+1}_+$ and Besov functions on $\mathbb{R}^n$ are used (Section 4). Our proofs are essentially based on the techniques of fractional integro-differentiation in $h(p,q,\alpha)$. The latter subject was raised in Hardy's and Littlewood's works and can be formulated as follows: How does the fractional integro-differentiation act as a bounded operator in the spaces $h(p,q,\alpha)$? Flett ([12]–[15]) studied in detail this question especially for functions holomorphic in the unit disk.

In Section 3 we generalize his results to functions harmonic on the half-space. The case of small $p$ causes some difficulties because $|\nabla f|^p$ ($f$ harmonic) need not be subharmonic for $p < (n-1)/n$ and $M_p(f;y)$ in general not necessarily monotonic by $y > 0$. Applying the Whitney expansion of $\mathbb{R}^{n+1}_+$ we prove a Hardy-Littlewood type max-theorem (Theorem 6) for $h(p,p,\alpha)$, $0 < p < \infty$, that allows us to overcome the mentioned difficulties. As an easy consequence we obtain that harmonic conjugation (Riesz transform) is bounded for all $p$ and $q$, $0 < p \leq \infty$, $0 < q \leq \infty$ (Corollary 3), which is a generalization of a result from [5], [17]. More information about harmonic (pluriharmonic) conjugation on various domains of $\mathbb{R}^n$ and $\mathbb{C}^n$, especially for $p \leq 1$, can be found in [15], [19], [18], [5], [6], [7], [21], [17].

If $T$ is a bounded operator mapping $X$ to $Y$, i.e. $\|Tf\|_Y \leq C\|f\|_X$, $\forall\, f \in X$, then we shall write $T : X \longrightarrow Y$. Main results obtained on fractional differentiation and integration can be presented by the following table ordered by growth $\beta$:

$$
\begin{aligned}
&\mathcal{D}^{-\beta}: h(p,q,\alpha) \longrightarrow h(p,q,\alpha-\beta), && -\infty<\beta<\alpha,\ 0<p,q\le\infty, && \text{(Th.7)}\\
&\mathcal{D}^{-\beta}: h(p,q,\alpha) \longrightarrow h^p, && \beta=\alpha,\ 0<p<\infty,\ 0<q\le\min\{2,p\}, && \text{(Cor.2)}\\
&\mathcal{D}^{-\beta}: h(p,q,\alpha) \longrightarrow h^{p_0}, && \alpha<\beta<\alpha+n/p,\ 0<p<\infty,\ q\le p_0, && \text{(Cor.2)}\\
&\mathcal{D}^{-\beta}: h(p,q,\alpha) \longrightarrow h(p_0,q_0), && \alpha<\beta<\alpha+n/p,\ 1\le p<\infty, && \\
& && \quad 0<q\le q_0\le\infty,\ 1<q_0\le\infty, && \text{(Th.5)}\\
&\mathcal{D}^{-\beta}: h(p,q,\alpha) \longrightarrow \mathcal{B}, && \beta=\alpha+n/p,\ p=\infty,\ 0<q\le\infty, && \text{(Cor.4)}\\
&\mathcal{D}^{-\beta}: h(p,q,\alpha) \longrightarrow \mathrm{BMOh}, && \beta=\alpha+n/p,\ 0<p<\infty,\ 0<q\le\infty, && \text{(Th.5)}\\
&\mathcal{D}^{-\beta}: h(p,q,\alpha) \longrightarrow h^\infty, && \beta=\alpha+n/p,\ 0<p\le\infty,\ 0<q\le 1. && \text{(Cor.2)}
\end{aligned}
$$

Here $p_0=\frac{n}{\alpha+n/p-\beta}$, $h(p,q)$ denotes the harmonic Lorentz space, $\mathcal{B}$ the harmonic Bloch space and BMOh the space of harmonic functions in $\mathbb{R}^{n+1}_+$ having BMO boundary values on $\mathbb{R}^n$.

**0.2.** We shall use some natural notations. For functions $f(x,y)$ defined in $\mathbb{R}^{n+1}_+$, we shall use the Riemann-Liouville integro-differential operator $\mathcal{D}^{-\alpha}\equiv\mathcal{D}_y^{-\alpha}$ (Riesz potential) with respect to the variable $y$:

$$
\mathcal{D}^{-\alpha}f(x,y)=\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\sigma^{\alpha-1}f(x,y+\sigma)\,d\sigma,
$$
$$
\mathcal{D}^0 f=f,\quad \mathcal{D}^{\alpha}f(x,y)=(-1)^m\mathcal{D}^{-(m-\alpha)}\frac{\partial^m}{\partial y^m}f(x,y),
$$

where $\alpha>0$ and $m$ is the integer deduced from $m-1<\alpha\le m$. For details on this operator see, for example, [4].

In the half-space $\mathbb{R}^{n+1}_+$, the Poisson kernel $P\equiv P_0$ and the conjugate Poisson kernels $P_j$ $(1\le j\le n)$ are given by

$$
P(x,y)=\Gamma\left(\frac{n+1}{2}\right)\pi^{-(n+1)/2}\frac{y}{(|x|^2+y^2)^{(n+1)/2}},
$$
$$
P_j(x,y)=\Gamma\left(\frac{n+1}{2}\right)\pi^{-(n+1)/2}\frac{x_j}{(|x|^2+y^2)^{(n+1)/2}},\qquad 1\le j\le n.
$$

Throughout the paper, the letters $C(\alpha,\beta,\dots), c_\alpha$ etc. will denote positive constants possibly different at different places and depending only on the parameters $\alpha,\beta,\dots$ . Any inequality $A\le B$ quoted or proved is to be interpreted as meaning 'if $B$ is finite, then $A$ is finite, and $A\le B$'. For $A,B>0$ the notation $A\asymp B$ denotes the two-sided estimate $c_1A\le B\le c_2A$ with some positive constants $c_1$ and $c_2$ independent of the variables involved.

For any $p$, $1 \le p \le \infty$, we define the conjugate index $p' = p/(p-1)$ (we interpret $1/+\infty = 0$ and $1/0 = +\infty$). Let $\mathbb{Z}_+^{n+1}$ be the set of all ordered $(n+1)$-tuples of nonnegative integers, and for each $\lambda = (\lambda_1, \dots, \lambda_n, \lambda_{n+1}) \in \mathbb{Z}_+^{n+1}$ $(\lambda_j \in \mathbb{Z}_+)$ let $|\lambda| = \lambda_1 + \dots + \lambda_n + \lambda_{n+1}$ and $\partial^\lambda = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\lambda_n} \left(\frac{\partial}{\partial y}\right)^{\lambda_{n+1}}$. When a function $f(x,y)$ is complex-valued we use the $\mathbb{C}^{n+1}$-norm to calculate $|\nabla f|$.

## 1. Preliminaries. Littlewood-Paley type inequalities

The most of this section extends to $\mathbb{R}_+^{n+1}$ the results of Flett [12, Theorems 1–5]. For $\alpha > 0$ and $0 < q \le \infty$ we shall consider the Littlewood-Paley type $g$-function (cf. [12], [22, Chapter IV])

$$g_{q,\alpha}(x) \equiv g_{q,\alpha}(f)(x) = \begin{cases} \left( \displaystyle\int_0^{+\infty} y^{\alpha q - 1} |\mathcal{D}^\alpha f(x,y)|^q \, dy \right)^{1/q}, & 0 < q < \infty, \\ \operatorname*{ess\,sup}\limits_{y>0} \; y^\alpha |\mathcal{D}^\alpha f(x,y)|, & q = \infty. \end{cases}$$

We gather some auxiliary lemmas and a Littlewood-Paley type theorem. The proofs are very standard, so we omit the details.

**Lemma 1.** *If $\alpha > 0$, $\lambda \in \mathbb{Z}_+^{n+1}$, $\frac{n}{n+\alpha} < p \le \infty$, then for each $j \in [0,n]$, $x \in \mathbb{R}^n$ and $y > 0$*

$$|\mathcal{D}^\alpha P_j(x,y)| \le C(\alpha,n) \frac{1}{(|x|+y)^{\alpha+n}}, \quad |\partial^\lambda P_j(x,y)| \le C(\lambda,n) \frac{1}{(|x|+y)^{|\lambda|+n}},$$
$$M_p(\mathcal{D}^\alpha P_j; y) \le C(\alpha,n,p) \frac{1}{y^{\alpha+n-n/p}}, \quad M_p(\partial^\lambda P_j; y) \le C(\lambda,n,p) \frac{1}{y^{|\lambda|+n-n/p}}.$$

**Lemma 2.** *Let $f(x,y)$ be a harmonic function in $\mathbb{R}_+^{n+1}$ and $0 < p, q \le \infty$, $\alpha > 0$. Then*

$$|\mathcal{D}^\alpha f(x,y)| \le C(p,q,\alpha,n) y^{-\alpha-n/p} \|g_{q,\alpha}(f)\|_{L^p}, \qquad x \in \mathbb{R}^n,\ y > 0.$$

**Lemma 3.** *Let $\beta > 0$ and $f(x,y)$ be a harmonic function in $\mathbb{R}_+^{n+1}$ such that $\mathcal{D}^\beta f(x,y)$ vanishes as $y \to +\infty$, uniformly for $x \in \mathbb{R}^n$. If either $1 \le p \le q < \infty$, $\alpha > 1/p - 1/q$, or $1 < p \le q < \infty$, $\alpha = 1/p - 1/q$, then*

$$g_{q,\beta}(f)(x) \le C(\alpha,\beta,p,q)\, g_{p,\beta+\alpha}(f)(x), \qquad x \in \mathbb{R}^n.$$

**Lemma 4.** *Let $f(x,y)$ be a harmonic function in $\mathbb{R}^{n+1}_+$, $\alpha > 0$, $\delta > 0$ and let $\Gamma_\delta(x) = \{(\xi,\eta) \in \mathbb{R}^{n+1}_+;\ |\xi - x| < \delta\eta\}$ be the Lusin cone with the vertex at $x \in \mathbb{R}^n$. If $f^*_\delta(x) = \sup\{|f(\xi,\eta)|;\ (\xi,\eta) \in \Gamma_\delta(x)\}$ is the nontangential maximal function of $f$, then*

$$|\mathcal{D}^\alpha f(x,y)| \le C(\alpha,\delta)\, y^{-\alpha} f^*_\delta(x), \qquad x \in \mathbb{R}^n,\ y > 0. \tag{1.1}$$

**Theorem 1.** *Let $\alpha > 0$ and $1 < p < \infty$.*

(i) *If $2 \le q < \infty$ and $f(x,y)$ is the Poisson integral of $f(x) \in L^p(\mathbb{R}^n)$, then*

$$\|g_{q,\alpha}(f)\|_{L^p} \le C(p,q,\alpha,n)\|f\|_{L^p}. \tag{1.2}$$

(ii) *If $0 < q \le 2$ and $f(x,y)$ is harmonic in $\mathbb{R}^{n+1}_+$, vanishes as $y \to +\infty$, uniformly for $x \in \mathbb{R}^n$, and $g_{q,\alpha}(f) \in L^p$, then $f(x,y)$ is the Poisson integral of a function $f(x) \in L^p$ and*

$$\|f\|_{L^p} \le C(p,q,\alpha,n)\|g_{q,\alpha}(f)\|_{L^p}. \tag{1.3}$$

## 2. Harmonic mixed norm spaces and projections on them

The following lemma is an $n$-dimensional extension of [18, Proposition 2.2] and it can be proved by similar arguments with the use of interpolation theorems ([1], [16]).

**Lemma 5.** *If $0 < p \le p_0 \le \infty$, $0 < q \le q_0 \le \infty$, $\alpha + n/p = \alpha_0 + n/p_0$, then the following inclusion is valid and continuous:*

$$h(p,q,\alpha) \subset h(p_0,q_0,\alpha_0).$$

*Moreover, if $u(x,y) \in h(p,q,\alpha)$ with $q < \infty$, then $y^\alpha M_p(u;y) = o(1)$ as $y \to +0$ and $y \to +\infty$.*

The inclusion $h(p,q,\alpha) \subset h(p,\infty,\alpha)$ of this lemma implies a useful property of spaces $h(p,q,\alpha)$: If $u_\eta(x,y) = u(x,y+\eta)$, then the quasi-norm $\|u_\eta\|_{p,q,\alpha}$ $(0 < p,q \le \infty, \alpha > 0)$ is effectively decreasing by $\eta \ge 0$, i.e.

$$\|u_{\eta_1}\|_{p,q,\alpha} \le C(p,q,\alpha,n)\|u_{\eta_2}\|_{p,q,\alpha}, \qquad \eta_1 > \eta_2 \ge 0. \tag{2.1}$$

For a function $u(x,y)$ harmonic in $\mathbb{R}^{n+1}_+$ and satisfying the condition $u(x,y) = O(y^{-\delta})$, $y \to +\infty, \delta > 0$, the Riesz transforms of $u$ are defined by

$$u_j(x,y) = (R_j u)(x,y) = -\int_y^{+\infty} \frac{\partial u(x,\eta)}{\partial x_j}\, d\eta, \qquad 1 \le j \le n.$$

The vector function $F = (u_0, u_1, \ldots, u_n)$, $u = u_0$, is a system of conjugate harmonic functions, i.e. the functions $u_j$ satisfy the generalized Cauchy-Riemann equations

$$\sum_{j=0}^n \frac{\partial u_j}{\partial x_j} = 0, \qquad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \qquad 0 \le j,\ k \le n.$$

**Theorem 2.** *Let $\alpha > 0$ and $u \equiv u_0 \in h(p,q,\alpha)$. If either $0 < p, q \leq \infty$, $\beta > \max\{\alpha + n/p - n, \alpha\}$, or $p = 1$, $0 < q \leq 1$, $\beta \geq \alpha$, then for each $j \in [0,n]$, $x \in \mathbb{R}^n$ and $y > 0$*

$$u_j(x,y) = \frac{2^\beta}{\Gamma(\beta)} \iint_{\mathbb{R}^{n+1}_+} u(\xi,\eta)\, \mathcal{D}^\beta P_j(x-\xi, y+\eta)\, \eta^{\beta-1}\, d\xi\, d\eta, \tag{2.2}$$

$$u_j(x,y) = \frac{2^\beta}{\Gamma(\beta)} \iint_{\mathbb{R}^{n+1}_+} u_j(\xi,\eta)\, \mathcal{D}^\beta P(x-\xi, y+\eta)\, \eta^{\beta-1}\, d\xi\, d\eta. \tag{2.3}$$

Proof: The representation (2.2) with $j = 0$ is due to Ricci and Taibleson ([18]) for integral $\beta$ and $n = 1$ (see also [5]). For $j \in [1,n]$ and $0 < p < \infty$ the representation (2.2) follow from a semigroup formula involving conjugate Poisson kernels:

$$u_j(x,y) = \int_{\mathbb{R}^n} u(\xi, y/2) P_j(x-\xi, y/2)\, d\xi.$$

We postpone the proof of (2.3) until Subsection 3.4. The representation (2.3) will follow immediately from Corollary 3 of Theorem 7. $\square$

Now consider the operator

$$T_{\alpha,j}(f)(x,y) = \iint_{\mathbb{R}^{n+1}_+} f(\xi,\eta)\, \mathcal{D}^\alpha P_j(x-\xi, y+\eta)\, \eta^{\alpha-1}\, d\xi\, d\eta, \quad \alpha > 0,\ 0 \leq j \leq n.$$

The next theorem is a partial converse to Theorem 2.

**Theorem 3.** *If $1 \leq p, q \leq \infty$, $\beta > \alpha > 0$, $0 \leq j \leq n$, then the operator $T_{\beta,j}$ is a bounded projection of $L(p,q,\alpha)$ onto $h(p,q,\alpha)$.*

Proof: Let $f(x,y) \in L(p,q,\alpha)$ and $q$ be finite. By Minkowski's inequality and Lemma 1

$$M_p(T_{\beta,j} f; y) \leq C \int_0^{+\infty} \frac{\eta^{\beta-1}}{(y+\eta)^\beta} M_p(f;\eta)\, d\eta.$$

A further application of Hardy's inequality (see, e.g., [22]) shows that

$$\|T_{\beta,j} f\|_{p,q,\alpha} \leq C \|f\|_{p,q,\alpha}.$$

Note that the assertion of Theorem 3 with $j = 0$ is proved in [5] for $p = q$ and integral $\beta$. $\square$

The following question now arises: Does the finiteness of $\|u\|_{p,q,\alpha}$ imply the finiteness of $\|u_j\|_{p,q,\alpha}$? An affirmative answer involving all values $p, q \in (0,\infty]$ is given in Corollary 3 of Theorem 7.

## 3. Fractional differentiation and integration in $h(p,q,\alpha)$

**3.1.** For each measurable function $f$ on $\mathbb{R}^n$, let $\lambda_f$ be its distribution function, i.e. $\lambda_f(t) = |\{x \in \mathbb{R}^n; |f(x)| > t\}|$, $t > 0$, where $|E| = \operatorname{mes} E$ is the Lebesgue measure of the set $E \subset \mathbb{R}^n$. The decreasing rearrangement of $f$ is the function $f^*$ given by

$$f^*(s) = \inf\{t > 0; \lambda_f(t) \leq s\}.$$

The Lorentz space $L(p,q)$ is defined to be the collection of all functions $f$ such that $\|f\|_{L(p,q)} < +\infty$, where

$$\|f\|_{L(p,q)} = \begin{cases} \left(\displaystyle\int_0^{+\infty} \left[t^{1/p} f^*(t)\right]^q \frac{dt}{t}\right)^{1/q}, & 0 < p, q < \infty, \\ \displaystyle\sup_{t>0}\, t^{1/p} f^*(t), & 0 < p \leq \infty,\ q = \infty. \end{cases} \tag{3.1}$$

It is well known that

$$L(p,q_1) \subset L(p,p) = L^p \subset L(p,q_2) \subset L(p,\infty) \subset L^1\left(\frac{dt}{1+|t|^{n+1}}\right)$$

whenever $1 \leq p \leq \infty, 0 < q_1 \leq p \leq q_2 \leq \infty$. The harmonic Lorentz space $h(p,q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ (see [14], [4]) is defined to be the collection of all functions $u(x,y)$ harmonic in $\mathbb{R}^{n+1}_+$ such that $\|u\|_{h(p,q)} = \sup_{y>0} \|u(x,y)\|_{L(p,q)}$ is finite. So that $h(p,p) = h^p$, $1 < p < \infty$.

**Theorem 4.** *Let $\alpha > 0$ and $1 < p \leq q \leq \infty$. Then*

$$\mathcal{D}^\alpha : h^p \longrightarrow h(p,q,\alpha), \qquad 2 \leq q \leq \infty, \tag{3.2}$$

$$\mathcal{D}^\alpha : h^p \longrightarrow h(p_0,q,\alpha + n/p - n/p_0), \qquad 1 < p < p_0 \leq \infty. \tag{3.3}$$

Proof: The relation (3.2) follows from Theorem 1 and a corollary

$$\Big\| \|F(\xi,\eta)\|_{L^p(d\xi)} \Big\|_{L^q(d\eta)} \leq \Big\| \|F(\xi,\eta)\|_{L^q(d\eta)} \Big\|_{L^p(d\xi)}, \qquad 0 < p \leq q, \tag{3.4}$$

of Minkowski's inequality. Indeed, let $u(x,y)$ be a function of $h^p$ ($p < \infty$). Then

$$\begin{aligned} \|\mathcal{D}^\alpha u\|_{p,q,\alpha} &\leq \Big\| \|y^\alpha \mathcal{D}^\alpha u\|_{L^q(dy/y)} \Big\|_{L^p(dx)} \\ &= \|g_{q,\alpha}(u)\|_{L^p} \leq C\|u\|_{h^p}. \end{aligned}$$

By combining with (3.2) and Lemma 5 one obtains the relation (3.3). □

**3.2 Harmonic BMO and Lorentz spaces.** We proceed to the fractional integration involving BMO and Lorentz spaces. A function $u(x,y)$ harmonic in $\mathbb{R}^{n+1}_+$ and having BMO boundary values on $\mathbb{R}^n$ is said to belong to the class BMOh.

**Theorem 5.** (i) *If* $0<p<\infty$, $0<q\le\infty$, $\alpha>0$, $\beta=\alpha+n/p$, *then*

$$\mathcal{D}^{-\beta} : h(p,q,\alpha) \longrightarrow \mathrm{BMOh}. \tag{3.5}$$

(ii) *If* $1\le p<\infty$, $0<q\le q_0\le\infty$, $1<q_0\le\infty$, $0<\alpha<\beta<\alpha+\frac{n}{p}$, $p_0=\frac{n}{\alpha+n/p-\beta}$, *then*

$$\mathcal{D}^{-\beta} : h(p,q,\alpha) \longrightarrow h(p_0,q_0). \tag{3.6}$$

Proof: (i) It is enough to prove (3.5) only for $q=\infty$, i.e. for the widest (by $q$) space $h(p,\infty,\alpha)$. Let $u(x,y)\in h(p,\infty,\alpha)$ be arbitrary. For any $y>0$, consider the following linear functional on the real Hardy space $H^1(\mathbb{R}^n)$, generated by $\varphi(x,y)=\mathcal{D}^{-\beta}u(x,y)$:

$$F_\varphi(g)=\int_{\mathbb{R}^n}\varphi(x,y)g(x)\,dx, \tag{3.7}$$

where $g\in H^1_0(\mathbb{R}^n)\subset H^1(\mathbb{R}^n)$ (see [11], [22, Section 7.3]). If $v(x,y)$ is the Poisson integral of $g$, then

$$F_\varphi(g)=\frac{1}{\Gamma(\beta)}\int_0^{+\infty}\sigma^{\beta-1}\left[\int_{\mathbb{R}^n}u\Big(x,\frac{\sigma}{2}\Big)v\Big(x,y+\frac{\sigma}{2}\Big)\,dx\right]d\sigma. \tag{3.8}$$

Assuming $0<p<1$ and applying Hölder's inequality for any fixed $k_0$, $1\le k_0<\infty$, one can evaluate

$$\begin{aligned}|F_\varphi(g)|&\le C\int_0^{+\infty}\sigma^{\beta-1}M_{k_0}\Big(u;\frac{\sigma}{2}\Big)M_{k_0'}\Big(v;y+\frac{\sigma}{2}\Big)\,d\sigma\\&\le C\|u\|_{k_0,\infty,\alpha+n/p-n/k_0}\|v\|_{k_0',1,n/k_0}.\end{aligned}$$

By Lemma 5 and the continuous inclusion $h^1\subset h(k_0',1,n/k_0)$ of Flett ([14, Theorem 3]) we get

$$|F_\varphi(g)|\le C\|u\|_{p,\infty,\alpha}\|v\|_{h^1}\le C\|u\|_{p,\infty,\alpha}\|g\|_{H^1(\mathbb{R}^n)}.$$

Since the subclass $H^1_0$ is dense in $H^1(\mathbb{R}^n)$, $F_\varphi$ induces a bounded linear functional on $H^1(\mathbb{R}^n)$. Besides, Fefferman's duality $\big(H^1(\mathbb{R}^n)\big)^*=\mathrm{BMO}(\mathbb{R}^n)$ (see [11]) implies

$$\|\varphi\|_{\mathrm{BMO}}\le C\sup\Big\{|F_\varphi(g)|; g\in H^1_0, \|g\|_{H^1}=1\Big\}\le C\|u\|_{p,\infty,\alpha}. \tag{3.9}$$

Assuming now $1 \le p < \infty$ and applying again Hölder's inequality with indices $p$ and $p'$ we obtain from (3.8)

$$|F_\varphi(g)| \le C\|u\|_{p,\infty,\alpha}\|v\|_{p',1,\beta-\alpha}.$$

Further, the same arguments together with the inclusion $h^1 \subset h(p',1,n/p)$ lead to (3.9) for $1 \le p < \infty$.

(ii) The relation (3.6) follows by similar arguments after applying the inclusion $h(p_0',q') \subset h(p',q',\beta-\alpha)$ (see [14, Theorem 9]) and duality $\big(L(p_0',q')\big)^* = L(p_0,q)$. Thus the proof of the theorem is complete. □

**3.3 Max-theorem.** We shall need the following two auxiliary lemmas. The first of them is the well-known Whitney expansion.

**Lemma A.** *There exists a collection $\{\Delta_k\}_{k=1}^\infty$ of closed cubes $\Delta_k \subset \mathbb{R}^{n+1}_+$ with sides parallel to coordinate axes, such that*

(i) $\bigcup_{k=1}^\infty \Delta_k = \mathbb{R}^{n+1}_+$ *and* diam $\Delta_k \asymp$ dist $\left(\Delta_k, \partial\mathbb{R}^{n+1}_+\right)$.

(ii) *The interiors of all $\Delta_k$ are pairwise disjoint.*

(iii) *If $\Delta_k^*$ is a cube with the same centre as $\Delta_k$, but extended $5/4$ times, then the system $\{\Delta_k^*\}_{k=1}^\infty$ forms a finitely multiple covering of $\mathbb{R}^{n+1}_+$. More precisely, each cube $\Delta_k^*$ intersects at most $12^{n+1}$ cubes $\Delta_k$.*

**Lemma B.** *Let $\Delta_k$ and $\Delta_k^*$ be some cubes from the previous lemma, and let $(\xi_k,\eta_k)$ be the centre of $\Delta_k$. If a function $u$ is harmonic in $\mathbb{R}^{n+1}_+$, then for any $0<p<\infty$ and $\alpha>0$*

$$\eta_k^{\alpha p-1} \max_{(\xi,\eta)\in\Delta_k} |u(\xi,\eta)|^p \le \frac{C}{|\Delta_k^*|} \iint_{\Delta_k^*} \eta^{\alpha p-1}|u(\xi,\eta)|^p \, d\xi\, d\eta.$$

For a proof of Lemma A see [22], and of Lemma B see [5]. Observe that $|\Delta_k| \asymp |\Delta_k^*| \asymp \eta_k^{n+1}$.

The following key result is an analogue of classical max-theorems of Hardy and Littlewood and of Lemma 14 from [13].

**Theorem 6.** *Let $\alpha>0$, $0<p<\infty$, $u(x,y) \in h(p,p,\alpha)$. Then the maximal function*

$$u^*(x,y) = \sup\Big\{|u(\xi,\eta)|; |\xi-x|^2 + (\eta-y)^2 \le y^2/4\Big\}, \qquad x\in\mathbb{R}^n,\ y>0$$

*satisfies the inequality*

$$\|u^*\|_{p,p,\alpha} \le C(\alpha,p,n)\|u\|_{p,p,\alpha}. \tag{3.10}$$

Proof: For $p \geq 1$ the inequality (3.10) is obtained immediately from Lemma 14 of [13]. For smaller $p$ the non-subharmonicity of $|\nabla f|^p$ ($f$ harmonic) leads to difficulties in estimation. Let $0 < p < 1$. We have now by using the representation (2.2) with $j = 0$ and $\beta > \alpha + n/p - n$:

$$\|u^*\|^p_{p,p,\alpha} = \frac{2^{\beta p}}{\Gamma^p(\beta)} \iint\limits_{\mathbb{R}^{n+1}_+} y^{\alpha p-1} \sup_{\xi,\eta} \left| \iint\limits_{\mathbb{R}^{n+1}_+} u(t,\theta)\mathcal{D}^\beta P(\xi - t, \eta + \theta)\theta^{\beta-1}\, dt\, d\theta \right|^p dx\, dy$$

$$\leq C \iint\limits_{\mathbb{R}^{n+1}_+} y^{\alpha p-1} \sup_{\xi,\eta} \sum_{k=1}^{\infty} \left( \iint\limits_{\Delta_k} |u(t,\theta)|\ |\mathcal{D}^\beta P(\xi - t, \eta + \theta)|\ \theta^{\beta-1}\, dt\, d\theta \right)^p dx\, dy.$$

It is easy to verify that $\max\limits_{(t,\theta)\in\Delta_k} |\mathcal{D}^\beta P(\xi - t, \eta + \theta)| \leq C(n,\beta)|\mathcal{D}^\beta P(\xi - \xi_k, \eta + \eta_k)|$. Consequently,

(3.11)

$$\|u^*\|^p_{p,p,\alpha}$$

$$\leq C \iint\limits_{\mathbb{R}^{n+1}_+} y^{\alpha p-1} \sup_{\xi,\eta} \sum_{k=1}^{\infty} \max_{\Delta_k} |u(t,\theta)|^p\ |\mathcal{D}^\beta P(\xi - \xi_k, \eta + \eta_k)|^p\ \eta_k^{p(\beta-1)} |\Delta_k|^p\, dx\, dy$$

$$\leq C \sum_{k=1}^{\infty} |\Delta_k|^p \eta_k^{p(\beta-1)} \max_{\Delta_k} |u(t,\theta)|^p \iint\limits_{\mathbb{R}^{n+1}_+} y^{\alpha p-1} \sup_{\xi,\eta} |\mathcal{D}^\beta P(\xi - \xi_k, \eta + \eta_k)|^p\, dx\, dy.$$

Denoting the last integral by $J$ and choosing $\beta$ large enough we estimate $J$:

$$J \leq \int_0^{+\infty} y^{\alpha p-1} \left[ \int\limits_{\mathbb{R}^n} \sup_{\substack{|\xi - x| \leq y/2 \\ |\eta - y| \leq y/2}} |\mathcal{D}^\beta P(\xi - \xi_k, \eta + \eta_k)|^p\, dx \right] dy$$

$$\leq C \int_0^{+\infty} y^{\alpha p-1} \Bigg[ \int\limits_{|x-\xi_k| \leq y/2} \frac{dx}{(y/2 + \eta_k)^{p(\beta+n)}} +$$

$$+ \int\limits_{|x-\xi_k| > y/2} \frac{dx}{(|x - \xi_k| + \eta_k)^{p(\beta+n)}} \Bigg] dy \leq C \frac{1}{\eta_k^{p(\beta+n)-n-\alpha p}}.$$

Substituting this in (3.11) and applying Lemma B we can continue the estimate and get

$$\begin{aligned}\|u^*\|^p_{p,p,\alpha} &\le C\sum_{k=1}^{\infty}|\Delta_k|^p\eta_k^{\alpha p+n-pn-p}\max_{\Delta_k}|u(\xi,\eta)|^p\\ &\le C\sum_{k=1}^{\infty}|\Delta_k|\eta_k^{\alpha p-1}\max_{\Delta_k}|u(\xi,\eta)|^p\\ &\le C\sum_{k=1}^{\infty}|\Delta_k|\frac{1}{|\Delta_k^*|}\iint\limits_{\Delta_k^*}\eta^{\alpha p-1}|u(\xi,\eta)|^p\,d\xi\,d\eta \le C\|u\|^p_{p,p,\alpha},\end{aligned}$$

and this is the required result. □

Applying Theorem 6 we deduce

**Corollary 1.** *Let $u \in h(p,p,\alpha)$ and $\alpha > 0$.*

(i) *If $0 < p < \infty$ then there exists a function $f \in L^1(\mathbb{R}^n)$ such that*

$$\begin{aligned}\|f\|_{L^1} &\le C(\alpha,n,p)\|u\|^p_{p,p,\alpha}\,,\\ |u(x,y)|^p &\le C(\alpha,n,p)\,y^{-\alpha p}f(x), \qquad x\in\mathbb{R}^n,\ y>0.\end{aligned}$$

(ii) *If $0 < p \le 1$ then additionally $\mathcal{D}^{-\alpha} : h(p,p,\alpha) \longrightarrow h^p$.*

**Corollary 2.** *Let $0 < p,q \le \infty$, $0 < \alpha \le \beta \le \alpha + n/p$, $p_0 = \frac{n}{\alpha+n/p-\beta}$. Then:*

$$\begin{aligned}&\mathcal{D}^{-\beta} : h(p,q,\alpha) \longrightarrow h^p, && \beta=\alpha,\ 0<p<\infty,\ 0<q\le\min\{2,p\},\\ &\mathcal{D}^{-\beta} : h(p,q,\alpha) \longrightarrow h^{p_0}, && \alpha<\beta<\alpha+n/p,\ 0<p<\infty,\ 0<q\le p_0,\\ &\mathcal{D}^{-\beta} : h(p,q,\alpha) \longrightarrow h^{\infty}, && \beta=\alpha+n/p,\ 0<p\le\infty,\ 0<q\le 1.\end{aligned}$$

Proof of Corollary 1: (i) By an inequality of Hardy-Littlewood-Fefferman-Stein [11], for each point $(x,y) \in \mathbb{R}^{n+1}_+$ we have

$$\begin{aligned}|u(x,y)|^p &\le \frac{C(p,\alpha,n)}{y^{\alpha p}}\int_{3y/4}^{5y/4}\eta^{\alpha p-1}\big(u^*(x,\eta)\big)^p\,d\eta\\ &\le \frac{C(p,\alpha,n)}{y^{\alpha p}}f(x),\end{aligned}$$

where $f(x)$ is defined as follows:

$$f(x) = \int_0^{+\infty}\eta^{\alpha p-1}\big(u^*(x,\eta)\big)^p\,d\eta, \qquad x\in\mathbb{R}^n.$$

It is easy to see in view of Theorem 6 that

$$\|f\|_{L^1} = \|u^*\|^p_{p,p,\alpha} \le C(\alpha,n,p)\|u\|^p_{p,p,\alpha}\,.$$

(ii) Suppose $p<1$. Then by part (i)

$$|\mathcal{D}^{-\alpha}u(x,y)| \le C(\alpha,n,p)\big(f(x)\big)^{(1-p)/p}\int_0^{+\infty}\sigma^{\alpha p-1}|u(x,y+\sigma)|^p\,d\sigma.$$

After integrating and applying Hölder's inequality with indices $\frac{1}{p-1}$, $\frac{1}{p}$ and property (2.1), we get

$$\begin{aligned}\int_{\mathbb{R}^n}|\mathcal{D}^{-\alpha}u(x,y)|^p\,dx &\le C(\alpha,n,p)\|f\|^{1-p}_{L^1}\|u\|^{p^2}_{p,p,\alpha}\\ &\le C(\alpha,n,p)\|u\|^p_{p,p,\alpha}\,.\end{aligned}$$

□

Proof of Corollary 2: It suffices to prove the following assertions:

| | | |
|---|---|---|
| (a) | $\mathcal{D}^{-\alpha}: h(p,p,\alpha)\longrightarrow h^p,$ | $0<p\le 2,$ |
| (b) | $\mathcal{D}^{-\alpha}: h(p,2,\alpha)\longrightarrow h^p,$ | $2\le p<\infty,$ |
| (c) | $\mathcal{D}^{-\beta}: h(p,p_0,\alpha)\longrightarrow h^{p_0},$ | $\alpha<\beta<\alpha+n/p,\ 0<p<\infty,$ |
| (d) | $\mathcal{D}^{-\alpha-n/p}: h(p,1,\alpha)\longrightarrow h^{\infty},$ | $0<p\le\infty.$ |

Here (a) is contained in Corollary 1 and Theorem 1(ii). To prove (b) we apply (3.4) and Theorem 1(ii). The assertion (c) for $1\le p<\infty$ is the case $q_0=p_0$ in Theorem 5(ii). For $0<p<1$ we shall distinguish two cases.

*Case* $0<p<1$, $p_0\ge 1$. Then the previous case of (c) and Lemma 5 give

$$\|\mathcal{D}^{-\beta}u\|_{h^{p_0}} \le C\|u\|_{p_0,p_0,\alpha+n/p-n/p_0} \le C\|u\|_{p,p_0,\alpha}\,.$$

*Case* $0<p<1, 0<p_0<1$. Then by Corollary 1 and Lemma 5

$$\|\mathcal{D}^{-\beta}u\|_{h^{p_0}} \le C\|u\|_{p_0,p_0,\beta} \le C\|u\|_{p,p_0,\alpha}\,.$$

The case $p=\infty$ in (d) is obvious. The general case follows from this and Lemma 5.

□

**3.4 "Fractional derivative norm" characterization.** The following auxiliary lemma extends to smaller $p$ a result of Flett [13, Theorem 7].

**Lemma 6.** *Let $m$ be a nonnegative integer, let $0 < p < \infty$, and let $u(x,y)$ be a harmonic function in $\mathbb{R}^{n+1}_+$. Then*

$$\int_{\mathbb{R}^n} |\nabla^m u(x,y)|^p \, dx \le C(m,n,p) \frac{1}{y^{mp+1}} \int_{y/2}^{3y/2} M_p^p(u;t)\, dt, \qquad y > 0,$$

*where $\nabla^m u$ is the gradient of $u$ of order $m$.*

This follows immediately from a corollary

$$|\nabla^m u(x,y)|^p \le \frac{C(m,n,p)}{y^{n+1+mp}} \iint\limits_{|\xi - x|^2 + (\eta - y)^2 < y^2/4} |u(\xi,\eta)|^p \, d\xi\, d\eta, \quad x \in \mathbb{R}^n, y > 0$$

of an inequality of Hardy-Littlewood-Fefferman-Stein ([11]).

**Theorem 7.** *Let $0 < p, q \le \infty$.*

(i) *If $0 < \beta < \alpha$, then $\mathcal{D}^{-\beta} : h(p,q,\alpha) \longrightarrow h(p,q,\alpha - \beta)$.*

(ii) *If $\alpha > 0$, $\beta > 0$, then $\mathcal{D}^{\beta} : h(p,q,\alpha) \longrightarrow h(p,q,\alpha + \beta)$.*

(iii) *If $\alpha > 0$, $\alpha > \beta > -\infty$, $q < \infty$ and $u \in h(p,q,\alpha)$, then $y^{\alpha-\beta} M_p(\mathcal{D}^{-\beta} u; y) = o(1)$ as $y \to +0$ and $y \to +\infty$.*

(iv) *If $\alpha > 0$, $\alpha > \beta > -\infty$ and $u \in h(p,\infty,\alpha)$, then the condition $y^{\alpha} M_p(u;y) = o(1)$ as $y \to +0$ $(y \to +\infty)$ implies $y^{\alpha-\beta} M_p(\mathcal{D}^{-\beta} u; y) = o(1)$ as $y \to +0$ ($y \to +\infty$, respectively).*

(v) *The assertions* (ii), (iii), (iv) *for the derivative $\mathcal{D}^{\beta}$ $(\beta > 0)$ hold with $\partial^{\lambda}(\lambda \in \mathbb{Z}^{n+1}_+)$ instead of $\mathcal{D}^{\beta}$, and $|\lambda|$ instead of $\beta$.*

Proof: Note that (i)–(iv) are proved by Bui Huy Qui [4, Theorem 3.5] for $1 \le p, q \le \infty$. Corollaries 1, 2 and Lemma 6 enable us to extend the assertions (i)–(iv) to all $p, q \in (0, \infty]$. Here we prove only (ii) and (v) when $0 < q \le p < 1$. The relation

$$\partial^{\lambda} : h(q,q,\alpha) \longrightarrow h(q,q,\alpha + |\lambda|) \tag{3.12}$$

is clear in view of Lemma 6. Besides, the relation

$$\partial^{\lambda} : h(1,q,\alpha) \longrightarrow h(1,q,\alpha + |\lambda|) \tag{3.13}$$

is also valid. By a version of Riesz-Thorin interpolation theorem for quasi-normed spaces (see [16]) the relations (3.12) and (3.13) lead to $\partial^{\lambda} : h(p,q,\alpha) \longrightarrow h(p,q,\alpha + |\lambda|)$ for any $p \in [q,1]$. For nonintegral $\beta$ $(m-1 < \beta < m,\ m \in \mathbb{Z}_+)$, assertion (ii) follows from (i) and above:

$$\|\mathcal{D}^{\beta} u\|_{p,q,\alpha+\beta} = \|\mathcal{D}^{-(m-\beta)} \mathcal{D}^m u\|_{p,q,\alpha+\beta} \le C \|\mathcal{D}^m u\|_{p,q,\alpha+m} \le C \|u\|_{p,q,\alpha}.$$

□

**Corollary 3.** *Let $0 < p, q \leq \infty$, $\alpha > 0$ and $u \equiv u_0 \in h(p,q,\alpha)$. Let $F = (u_0, u_1, \dots, u_n)$ be a system of harmonic conjugates. Then:*

(i) $\|F\|_{p,q,\alpha} \leq C\|u\|_{p,q,\alpha}$.

(ii) *The condition $y^\alpha M_p(u;y) = o(1)$ as $y \to +0$ ($y \to +\infty$) is equivalent to $y^\alpha M_p(F;y) = o(1)$ as $y \to +0$ ($y \to +\infty$, respectively).*

**3.5 Bloch functions.** The "fractional derivative norm" characterization and harmonic conjugation results are easily applicable to Bloch functions. This corresponds to the case $p = q = \infty$ in Theorem 7 and Corollary 3.

A function $u$ harmonic on $\mathbb{R}^{n+1}_+$ is said to be harmonic Bloch (we write $u \in \mathcal{B}$) if

$$\|u\|_{\mathcal{B}} = \sup y|\nabla u(x,y)| < +\infty, \tag{3.14}$$

where the supremum is taken over all $(x,y) \in \mathbb{R}^{n+1}_+$. A harmonic Bloch function $u$ is called harmonic little Bloch if it satisfies the following vanishing condition:

$$y|\nabla u(x,y)| = o(1) \qquad \text{as} \quad (x,y) \to \partial^\infty \mathbb{R}^{n+1}_+, \tag{3.15}$$

where $\partial^\infty \mathbb{R}^{n+1}_+ = \mathbb{R}^n \cup \{\infty\}$ (see [24]). The space of all harmonic little Bloch functions is denoted by $\mathcal{B}_0$. Let $\widetilde{\mathcal{B}}$ (resp. $\widetilde{\mathcal{B}}_0$) denote the subspace of functions in $\mathcal{B}$ (resp. $\mathcal{B}_0$) that vanish at $(x_0, y_0) = (0,1)$. The gradient in (3.14) may be replaced by $\mathcal{D}^1$, and Bloch $\|\cdot\|_{\mathcal{B}}$-norm may be characterized by the equivalent "derivative norm" condition

$$\sup_{(x,y)} y^m |\mathcal{D}^m u(x,y)| < +\infty, \qquad m \in \mathbb{Z}_+, m \geq 1 \tag{3.16}$$

as $u$ ranges over $\widetilde{\mathcal{B}}$ (see [17]). Moreover, as follows from Corollary 3 and the case $p = q = \infty$ of Theorem 7, (3.16) is true for fractional derivatives $\mathcal{D}^\beta (\beta > 0)$ as well.

**Corollary 4** (see [17]). *Suppose that $u$ is in $\widetilde{\mathcal{B}}$. Then:*

(i) *For each $\beta > 0$,*

$$\|u\|_{\mathcal{B}} \asymp \|\mathcal{D}^\beta u\|_{\infty,\infty,\beta}\,.$$

(ii) *For any $j \in [1,n]$,*

$$\|u_j\|_{\mathcal{B}} \leq C(n)\|u\|_{\mathcal{B}}.$$

**Corollary 5.** (i) *Suppose that $u$ is in $\widetilde{\mathcal{B}}_0$. Then for each $\beta > 0$ the condition*

$$y|\nabla u(x,y)| = o(1)$$

*is equivalent to $y^\beta |\mathcal{D}^\beta u(x,y)| = o(1)$ as $(x,y) \to \partial^\infty \mathbb{R}^{n+1}_+$.*

(ii) *If $u \in \widetilde{\mathcal{B}}_0$, then $u_j \in \widetilde{\mathcal{B}}_0$ for any $j \in [1,n]$.*

## 4. Integral representations in $h(p,q,\alpha)$

In this section we present some applications of Theorems 4–7. We characterize $h(p,q,\alpha)$ by means of an integral representation with the use of Besov spaces $\Lambda_\alpha^{p,q}$ on $\mathbb{R}^n$. Let $1 \le p,q \le \infty$, $\alpha > 0$ and let $f(x)$ be a measurable function on $\mathbb{R}^n$. The Besov's seminorm is defined as follows:

$$\|f\|_{\Lambda_\alpha^{p,q}} = \begin{cases} \left( \displaystyle\int_{\mathbb{R}^n} |t|^{-n-\alpha q} \|\Delta_t^k f(x)\|_{L^p(dx)}^q \, dt \right)^{1/q}, & 1 \le q < \infty, \\ \displaystyle\sup_{|t|>0} |t|^{-\alpha} \|\Delta_t^k f(x)\|_{L^p(dx)}, & q = \infty, \end{cases} \tag{4.1}$$

where $\Delta_t^1 f(x) = f(x+t) - f(x)$ and $\Delta_t^k f(x) = \Delta_t^1 \Delta_t^{k-1} f(x)$, $k$ is an integer, $k > \alpha$. There is an equivalent definition (see [23])

$$\|f\|_{\Lambda_\alpha^{p,q}} = \|\mathcal{D}^k v\|_{p,q,k-\alpha}\,, \tag{4.2}$$

where $v = v(x,y)$ is the Poisson integral of $f$ in $\mathbb{R}_+^{n+1}$. Observe that the definition (4.2) is suitable as well for any $q$, $0 < q \le \infty$.

For any real number $b$ let $\mathcal{H}_b$ be the linear space [4, p. 254], consisting of all harmonic functions $v(x,y)$ in $\mathbb{R}_+^{n+1}$ such that if $\lambda \in \mathbb{Z}_+^{n+1}$, $\rho > 0$ and $K$ is any compact subset of $\mathbb{R}^n$, then there exists a positive constant $C = C(\lambda,\rho,K)$ such that

$$|\partial^\lambda v(x,y)| \le C y^{-b-|\lambda|}, \qquad x \in K,\ y \ge \rho.$$

We shall also write $f(x) \in \mathcal{H}_b$ when its harmonic extension to $\mathbb{R}_+^{n+1}$ belongs to $\mathcal{H}_b$.

The following result is a slight improvement of Lemma 4.5 from [4].

**Lemma C.** *Let $1 \le p,q \le \infty$, $\alpha > 0$ and let $f(x)$ be a measurable function on $\mathbb{R}^n$ whose Poisson integral $v(x,y)$ exists, and $v(x,y) \in \bigcap_{b>0} \mathcal{H}_{(-b)}$. Then* (4.1) *and $\|\mathcal{D}^\gamma v\|_{p,q,\gamma-\alpha}$ are equivalent for each $\gamma > \alpha$.*

Now we need the following

**Lemma 7.** (a) *Suppose that $f$ is in* BMO$(\mathbb{R}^n)$*. Then $f$ belongs to $L^p\left(\frac{dt}{1+|t|^{n+1}}\right)$ for each $p$, $0 < p < \infty$, and hence to $L^1\left(\frac{dt}{1+|t|^{n+\gamma}}\right)$ and $\mathcal{H}_{(-\gamma)}$ for each $\gamma$, $0 < \gamma < 1$.*

(b) *Suppose that $f$ is in $L(p,\infty)$ for some $p, 1 < p < \infty$. Then $f$ belongs to $L^1\left(\dfrac{dt}{1+|t|^n}\right)$ and hence to $\mathcal{H}_0$.*

Proof: The case $p = 1$ of the first inclusion in (a) is a well-known result of Fefferman and Stein [11]. The general case in (a) can be proved by similar methods making use of the inequality

$$\frac{1}{|B|}\int_B |f-f_B|^p\,dx \le C_p\|f\|^p_{\mathrm{BMO}}, \quad \text{for any ball } \ B\subset\mathbb{R}^n, \quad f_B=\frac{1}{|B|}\int_B f\,dx,$$

which is a consequence of the John-Nirenberg inequality. The last inclusion in (a) follows from

$$|\partial^\lambda v(x,y)|\le C(\lambda,n)\frac{1}{y^{-\gamma+|\lambda|}}\max\left\{1,\frac{1+|x|}{y}\right\}^{n+\gamma}\int_{\mathbb{R}^n}\frac{|f(t)|dt}{1+|t|^{n+\gamma}}, \qquad \lambda\in\mathbb{Z}^{n+1}_+,$$

where $v(x,y)$ is the Poisson integral of $f$. The first inclusion in (b) follows from

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{|f(t)|dt}{1+|t|^n} &\le \int_0^{+\infty} f^*(s)\left(\frac{1}{1+|t|^n}\right)^* ds\\ &\le \|f\|_{p,\infty}\int_0^{+\infty}\frac{ds}{s^{1/p}(1+s/\omega_n)},\end{aligned}$$

where it is assumed that $g^*(s)$ is the decreasing rearrangement of $g(t)$ and $\omega_n=\frac{\pi^{n/2}}{\Gamma(1+n/2)}$. □

Now we are ready to formulate and prove the main result of this section.

**Theorem 8.** *Let $1\le p<\infty$, $0<q\le\infty$ and $\alpha>0$ be any numbers. Then:*

(i) *The space $h(p,q,\alpha)$ coincides with the set of functions $u(x,y)$ representable in the form*

$$u(x,y)=\int_{\mathbb{R}^n}\mathcal{D}^\beta P(x-t,y)\,\varphi(t)\,dt, \qquad x\in\mathbb{R}^n,\ y>0, \tag{4.3}$$

*where $\beta$ $(\alpha<\beta<\alpha+n/p)$ is any number and*

$$\varphi(t)\in\Lambda^{p,q}_{\beta-\alpha}\bigcap L^1\left(\frac{dt}{1+|t|^n}\right). \tag{4.4}$$

*At the same time,*

$$\|u\|_{p,q,\alpha}\asymp\|\varphi\|_{\Lambda^{p,q}_{\beta-\alpha}}. \tag{4.5}$$

(ii) *The function $\varphi$ in (4.3) can be deduced from the following inversion formula*

$$\varphi(x)=\lim_{y\to+0}\mathcal{D}^{-\beta}u(x,y), \qquad \text{a.e.}\ \ x\in\mathbb{R}^n. \tag{4.6}$$

(iii) *The space* $h(p,q,\alpha)$ *coincides with the set of functions* $u(x,y)$ *representable in the form* (4.3), *where* $\beta$ $(\alpha<\beta\le\alpha+n/p)$ *is any number and*

$$\varphi(t)\in\Lambda^{p,q}_{\beta-\alpha}\bigcap\left(\bigcap_{0<\gamma<1}L^1\left(\frac{dt}{1+|t|^{n+\gamma}}\right)\right).$$

*At the same time,* (4.5) *and* (4.6) *are valid.*

PROOF: (i) Let $u(x,y)\in h(p,q,\alpha)$ be any function and $\beta$ $(\alpha<\beta<\alpha+n/p)$ is any number. Denote $\varphi(x,y)=\mathcal{D}^{-\beta}u(x,y)$ and let $\varphi(x)$ be its boundary values on $\mathbb{R}^n$. By virtue of Theorem 5 (3.6), the function $\varphi(x)$ belongs to $L(p_0,\infty)$ with $p_0=n/(\alpha+n/p-\beta)$. Hence, by Lemma 7(b) $\varphi(x)\in L^1\left(\frac{dx}{1+|x|^n}\right)$ and so $\varphi(x,y)$ is representable by its Poisson integral:

$$\varphi(x,y)=\int_{\mathbb{R}^n}P(x-t,y)\,\varphi(t)\,dt,\qquad x\in\mathbb{R}^n,\ y>0.$$

Therefore,

$$u(x,y)=\mathcal{D}^{\beta}\varphi(x,y)=\int_{\mathbb{R}^n}\mathcal{D}^{\beta}P(x-t,y)\,\varphi(t)\,dt,$$

where the integral is absolutely convergent. At the same time, by Lemma C

$$\|\varphi\|_{\Lambda^{p,q}_{\beta-\alpha}}\le C\|\mathcal{D}^{\beta}\varphi\|_{p,q,\beta-(\beta-\alpha)}=C\|u\|_{p,q,\alpha}\,.$$

Conversely, suppose $u(x,y)$ is representable in the form (4.3)–(4.4). Let $\varphi(x,y)$ be the Poisson integral of $\varphi(t)$. Differentiation by means of $\mathcal{D}^{\beta}$ yields

$$\mathcal{D}^{\beta}\varphi(x,y)=\int_{\mathbb{R}^n}\mathcal{D}^{\beta}P(x-t,y)\,\varphi(t)\,dt=u(x,y).$$

Since, by Lemma 7 (b) $\varphi\in\mathcal{H}_0$, in view of Lemma C we have

$$\|u\|_{p,q,\alpha}=\|\mathcal{D}^{\beta}\varphi\|_{p,q,\beta-(\beta-\alpha)}\le C\|\varphi\|_{\Lambda^{p,q}_{\beta-\alpha}}\,.$$

(ii) To prove (4.6) it suffices to integrate the representation (4.3) by means of $\mathcal{D}^{-\beta}$, then to use the invertibility of $\mathcal{D}^{-\beta}$ and to let $y\to+0$. The assertion (iii) can be proved in the same way with the use of Lemmas C and 7(a). □

**Remark.** The connection between Besov spaces and weighted classes $A^*_\alpha$ of Nevanlinna-Djrbashian ([8], [9]) of functions holomorphic in the unit disk was established by Shamoyan [20].

Finally, we present a simpler integral formula for the space $h(2,2,\alpha)$.

**Theorem 9.** *The space $h(2,2,\alpha)$ $(\alpha > 0)$ coincides with the set of functions $u(x,y)$ representable in the form*

$$u(x,y) = \int_{\mathbb{R}^n} \mathcal{D}^{\alpha} P(x-t,y)\ \varphi(t)\,dt, \qquad x \in \mathbb{R}^n, \quad y > 0, \tag{4.7}$$

*where $\varphi(t) \in L^2(\mathbb{R}^n)$.*

*Here the function $\varphi$ can be deduced by the following inversion formula*

$$\varphi(x) = \lim_{y\to+0} \mathcal{D}^{-\alpha} u(x,y), \qquad \text{a.e.} \quad x \in \mathbb{R}^n.$$

Proof: $h(2,2,\alpha) = \mathcal{D}^{\alpha}(h^2)$ (see Corollary 2 and Theorem 4 (3.2)). □

A corresponding formula for functions holomorphic in the unit disk was established by M.M. Djrbashian [9, Theorems V–VI].

**Remark.** In a recent paper [25] of the author some analogues of Theorems 5(i), 8 and Corollary 4 for the unit disk are contained.

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