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Admissible maps, intersection results, coincidence theorems

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Abstract. We obtain generalizations of the Fan's matching theorem for an open (or closed) covering related to an admissible map. Each of these is restated as a KKM theorem. Finally, applications concerning coincidence theorems and section results are given.

Keywords: acyclic map, convex space, matching theorem, coincidence theorem

Classification: 47H10, 54H25, 54C60

1. Introduction

The KKM principle provides the foundations for many of the modern essential results in diverse areas of mathematical sciences (see [23]). In 1987, the "open" version of the KKM principle was presented by Kim [14] and Shih and Tan [27], and later Lassonde [16] showed that the classical (closed) and open versions of the KKM principle can be derived from each other. Each of the two versions of the KKM principle may be restated in its contraposition form and in terms of the complements of the covering members obtaining in this manner the two versions, open and closed, of Fan's matching theorem (see [10] and [16]). In this paper, using a fixed point theorem due to Gorniewicz [12] we obtain a matching theorem involving an admissible map (in the sense of Gorniewicz). Further on we establish new KKM theorems related to an admissible map, mutually equivalent with another matching theorems. In the last section we give two versions of a coincidence theorem and some applications. Our results include, as particular cases, a large number of known theorems, specified in the paper.

2. Preliminaries

A *convex space* X ([15]) is a nonempty convex set X in a vector space with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. In fact, we may regard that X has the relative finite topology.

A subset A of a topological space Y is said to be *compactly open* (respectively *closed*) in Y if for every compact set $K \subset Y$ the set $A \cap K$ is open (respectively closed) in K.

A nonempty topological space is *acyclic* if all its reduced Čech homology groups over rationals vanish. In particular, any contractible space is acyclic, and thus any convex or star-shaped set is acyclic.

A map $F : X \to Y$ is a function from a set X into the power set 2^Y of Y, that is, a function with the values $Fx \subset Y$ for $x \in X$ and the fibers $F^{-1}y = \{x \in X : y \in Fx\}$ for $y \in Y$. If $A \subset X$, let $F(A) = \bigcup\{Fx : x \in A\}$.

For topological spaces X and Y, a map $F : X \to Y$ is *upper semicontinuous* (u.s.c.) if: (i) Fx is compact for each $x \in X$ and (ii) for each open set $U \subset Y$ the set $\{x \in X : Fx \subset U\}$ is open in X. Note that the image of a compact set under an upper semicontinuous map is compact. A map $F : X \to Y$ is said to be *compact* if the range $F(X)$ is contained in a compact subset of the topological space Y.

Let X and Y be two Hausdorff topological spaces. A function $p : X \to Y$ is said to be a *Vietoris function* provided the following conditions are satisfied:

(i) for any compact $K \subset Y$, the counter image $p^{-1}(K)$ is also compact;
(ii) for each $y \in Y$ the set $p^{-1}(y)$ is acyclic.

A map $F : X \to Y$ is called *admissible* (in the sense of Gorniewicz, see [11] and [12]) if there exists a diagram $X \xleftarrow{p} Z \xrightarrow{q} Y$ such that:

(i) Z is a Hausdorff topological space and p, q are continous functions;
(ii) p is a Vietoris function;
(iii) $q(p^{-1}(x)) \subset Fx$ for each $x \in X$.

Observe that an acyclic map (i.e. an upper semicontinous map with acyclic values) or, in particular, a continous function is an admissible map. It is worth noticing that if $F : X \to Y$ and $T : Y \to Z$ are two admissible maps, then the composition $T \circ F$ is an admissible map (see [12, Theorem 2.7]).

Throughout this paper the topological spaces will be supposed Hausdorff. For a set D, let $\langle D\rangle$ denote the set of all nonempty finite subsets of D.

3. Matching theorems and KKM theorems

The following lemma is an immediate consequence of Corollary 3.7 in [12].

Lemma 1. *Let X be a compact convex set in a Euclidian space and $F : X \to X$ be an admissible map. Then there exits a point $x_0 \in X$ such that $x_0 \in Fx_0$.*

The following result generalizes Theorem 1 in [18] which in turn extends the open version of Fan's matching theorem ([10]).

Theorem 2. *Let D be a nonempty subset of a convex space, Y a topological space and $G : D \to Y$ a map such that:*

(i) *for each $x \in D$, Gx is compactly open in Y;*
(ii) *$G(D) = Y$.*

Then for each admissible compact map $F : \operatorname{co} D \to Y$ there exists $A \in \langle D \rangle$ such that $F(\operatorname{co} A) \cap \bigcap\{Gx : x \in A\} \neq \emptyset$.

Proof: Since F is a compact map, we may and shall assume that Y is a compact space, and for each $x \in D$, Gx is an open subset of Y. Consequently there is a finite subset $D_1 = \{x_1, x_2, \dots, x_n\}$ of D such that $Y = \bigcup_{i=1}^n Gx_i$. Let $\{\alpha_i\}_{i=1}^n$ be a continuous partition of unity subordinated to this covering of Y. Define a continuous function $g : Y \to \operatorname{co} D_1$ by

$$g(y) = \alpha_1(y) \cdot x_1 + \cdots + \alpha_n(y) \cdot x_n, \quad y \in Y.$$

Since $g \circ F$ is an admissible map, by Lemma 1, it has a fixed point. Hence there exist $x_0 \in \operatorname{co} D_1$ and $y_0 \in Y$ such that $x_0 = g(y_0)$ and $y_0 \in Fx_0$. Denote by $I = \{i \in \{1, 2, \dots, n\} : \alpha_i(y_0) > 0\}$. Clearly $I \neq \emptyset$. If $i \in I$, then y_0 is in the support of α_i and therefore in Gx_i. Thus $y_0 \in \bigcap\{Gx_i : i \in I\}$. On the other side $x_0 = g(y_0) \in \operatorname{co}\{x_i : i \in I\}$, whence $y_0 \in Fx_0 \subset F(\operatorname{co}\{x_i : i \in I\})$.

Taking $A = \{x_i : i \in I\}$ we get $y_0 \in F(\operatorname{co} A) \cap \bigcap\{Gx : x \in A\}$. □

Theorem 2 can be restated in its contraposition form and in terms of the complement Sx of Gx in Y as follows.

Theorem 3. *Let D be a nonempty subset of a convex space, Y a topological space and $S : D \to Y$ a map with compactly closed values. If there exists an admissible compact map $F : \operatorname{co} D \to Y$ such that*

$$F(\operatorname{co} A) \subset S(A) \quad \textit{for each} \quad A \in \langle D \rangle\,, \tag{1}$$

then $\bigcap\{Sx : x \in D\} \neq \emptyset$.

Proof: Suppose that $\bigcap\{Sx : x \in D\} = \emptyset$. Then $Y = G(D)$, where $G(x) = Y \backslash Sx$, for each $x \in D$. By Theorem 2 there exists $A \in \langle D \rangle$ such that

$$F(\operatorname{co} A) \cap \bigcap\{Gx : x \in A\} \neq \emptyset, \text{ that is, } \; F(\operatorname{co} A) \not\subset S(A).$$

This contradicts (1). □

The above KKM theorem includes earlier results of Lassonde [15], Chang [5], Sehgal, Singh and Whitfield [25], Shioji [29]. The compactness condition imposed to the map F can be relaxed as in the next theorem. The relaxing method used is not new. Its origin goes back to Lassonde [15] and it appeared in many papers (see for instance [6], [10], [18], [22]).

Theorem 4. *Let D be a nonempty subset of a convex space, Y a topological space, $S : D \to Y$ a map and $F : \operatorname{co} D \to Y$ an admissible u.s.c. map such that*

(i) *for each $x \in D$, Sx is compactly closed in Y;*

(ii) *for each* $A \in \langle D\rangle$, $F(\operatorname{co} A) \subset S(A)$;

(iii) *there exists a nonempty compact subset* K *of* Y *such that either*

(a) $\bigcap\{Sx : x \in A_0\} \subset K$ *for some* $A_0 \in \langle D\rangle$; *or*

(b) *for each* $A \in \langle D\rangle$ *there exists a compact convex subset* L_A *of* $\operatorname{co} D$ *containing* A *such that*
$F(L_A) \cap \bigcap\{Sx : x \in L_A \cap D\} \subset K$.

Then $\overline{F(\operatorname{co} D)} \cap K \cap \bigcap\{Sx : x \in D\} \neq \emptyset$.

Proof: Suppose the conclusion does not hold and put $Gx = Y \setminus Sx$, $x \in D$. Since $\overline{F(\operatorname{co} D)} \cap K$ is compact and Gx is compactly open for each $x \in X$, there exists $A_1 \in \langle D\rangle$ such that

$$\overline{F(\operatorname{co} D)} \cap K \subset G(A_1). \tag{2}$$

We examine successively the two cases looking every time for obtaining a contradiction.

Case (a). In this case

$$\overline{F(\operatorname{co} D)} \setminus K \subset Y \setminus K \subset G(A_0), \tag{3}$$

hence, by (2) and (3) $\overline{F(\operatorname{co} D)} \subset G(A)$, where $A = A_0 \cup A_1$. Since $\operatorname{co} A$ is compact and F is upper semicontinuous, $F(\operatorname{co} A)$ is a compact set and $F(\operatorname{co} A) \subset G(A)$.

By Theorem 2 there exists a nonempty set $B \subset A$ such that

$$F(\operatorname{co} B) \cap \bigcap\{Gx : x \in B\} \neq \emptyset, \text{ that is, } \ F(\operatorname{co} B) \not\subset S(B).$$

This contradicts (ii).

Case (b). By hypothesis there exists a compact convex set L such that $A_1 \subset L \subset \operatorname{co} D$ and

$$F(L) \cap \bigcap\{Sx : x \in L \cap D\} \subset K. \tag{4}$$

We claim that $F(L) \subset G(L \cap D)$. Taking into account (2) we have

$$F(L) \cap K \subset F(\operatorname{co} D) \cap K \subset G(A_1) \subset G(L \cap D).$$

Taking into account (4) we have $F(L) \setminus K \subset G(L \cap D)$. Hence, we have $F(L) \subset G(L \cap D)$. Since $F(L)$ is compact, there exists $B \in \langle L \cap D\rangle$ such that $F(\operatorname{co} B) \subset F(L) \subset G(B)$. For the remainder of the proof we can just follow that of Case (a). $\square$

Theorem 4 is a slight generalization of Theorem 3 in [21] which in turn generalizes earlier results of Fan [9], [10], Lassonde [15], Chang [5], Park [20].

Theorem 4 can be also stated in its contraposition form and in terms of the complement $G(x)$ of $S(x)$ obtaining in this way a generalization of Theorem 2, namely:

Theorem 5. *Let D be a nonempty subset of a convex space, Y a topological space, $G : D \to Y$ a map and $F : \operatorname{co} D \to Y$ an admissible u.s.c. map. Suppose that*

(i) *for each $x \in D$, Gx is compactly open in Y;*
(ii) *there exists a nonempty compact subset K of Y such that $\overline{F(\operatorname{co} D)} \cap K \subset G(D)$; and*
(iii) *either*
  (a) *$Y \backslash K \subset G(A_0)$ for some $A_0 \in \langle D \rangle$; or*
  (b) *for each $A \in \langle D \rangle$, there exists a compact convex subset L_A of $\operatorname{co} D$ containing A such that $F(L_A) \backslash K \subset G(L_A \cap D)$.*

Then there exists an $A \in \langle D \rangle$ such that $F(\operatorname{co} A) \cap \bigcap\{Gx : x \in A\} \neq \emptyset$.

Proof: Suppose, on contrary, that for each $A \in \langle D \rangle$, $F(\operatorname{co} A) \cap \bigcap\{Gx : x \in A\} = \emptyset$. Consider the map $S : D \to Y$, defined by $Sx = Y \backslash Gx$. It can be easily verified that all the conditions of Theorem 4 are satisfied. Therefore $\overline{F(\operatorname{co} D)} \cap K \cap \bigcap\{Sx : x \in D\} \neq \emptyset$. But this contradicts (ii). □

The following lemma is necessary in order to obtain an open-valued version of Theorem 3. Its proof uses the machinery developed by Shih in the proof of Theorem 1 in [26] and Park and Kim in the proof of Theorem 5 in [24].

Lemma 6. *Let D be a nonempty finite subset of a convex space, Y a compact space, $G : D \to Y$ an open-valued map and $F : \operatorname{co} D \to Y$ an admissible u.s.c. map such that*

$$F(\operatorname{co} A) \subset G(A) \quad \text{for each nonempty set} \quad A \subset D.$$

Then there is a closed-valued map $S : D \to Y$ such that $Sx \subset Gx$ for all $x \in D$ and $F(\operatorname{co} A) \subset S(A)$ for each nonempty $A \subset D$.

Proof: For any $y \in G(D)$, let $Hy = \bigcap\{Gx : x \in D\}$. Then Hy is an open set containing y. As Y is regular, there exists an open set U_y in Y such that $y \in U_y \subset \overline{U_y} \subset Hy$.

Now for any $A \in \langle D \rangle$ we have

$$F(\operatorname{co} A) \subset G(A) \subset \bigcup\{U_y : y \in G(A)\}.$$

Since $F(\operatorname{co} A)$ is compact, there exists $B_A \in \langle G(A) \rangle$ such that

$$F(\operatorname{co} A) \subset \bigcup\{U_y : y \in B_A\}.$$

Let $B = \bigcup\{B_A : A \in \langle D \rangle\}$. Define $S : D \to Y$ by

$$Sx = \bigcup\{\overline{U_y} : y \in B \cap Gx\}, \quad x \in D.$$

Then Sx is closed in Y for each $x \in D$ and $Sx \subset Gx$, since $\overline{U_y} \subset Hy \subset Gx$ if $y \in Gx$. For each $A \in \langle D \rangle$ and any $z \in F(\operatorname{co} A)$, we have $z \in U_y$ for some $y \in B_A \subset G(A) \cap B$; that is $y \in Gx \cap B$ for some $x \in A$. Hence $F(\operatorname{co} A) \subset S(A)$. □

Theorem 7. *Let D be a nonempty subset of a convex space, Y a topological space and $G : D \to Y$ be a map with compactly open values. If there exists an admissible u.s.c. map $F : \operatorname{co} D \to Y$ such that $F(\operatorname{co} A) \subset G(A)$ for each $A \in \langle D\rangle$, then $\{Gx : x \in D\}$ has the finite intersection property.*

Proof: Let $D_1 \in \langle D\rangle$. Since F is an upper semicontinuous map, $Y_1 = F(\operatorname{co} D_1)$ is a compact set. By Lemma 6 there exists a closed-valued map $S : D_1 \to Y_1$ such that $Sx \subset Gx \cap Y_1$ for all $x \in D_1$ and $F(\operatorname{co} A) \subset S(A)$ for each $A \in \langle D_1\rangle$. According to Theorem 3 we have $\bigcap\{Gx \cap Y_1 : x \in D_1\} \supset \bigcap\{Sx : x \in D_1\} \neq \emptyset$. □

The origin of Theorem 7 is due to Kim [14, Theorem 1]. Our theorem includes earlier results of Lassonde [16] and Park [19], [22].

In turn Theorem 7 can be easily reformulated obtaining the following matching theorem which is a closed-valued version of Theorem 2.

Theorem 8. *Let D be a nonempty finite subset of a convex space, Y a topological space and $S : D \to Y$ a map such that:*

(i) *for each $x \in D$, Sx is compactly closed in Y;*
(ii) *$S(D) = Y$.*

Then for each admissible u.s.c. map $F : \operatorname{co} D \to Y$ there exists $A \in \langle D\rangle$ such that $F(\operatorname{co} A) \cap \bigcap\{Sx : x \in A\} \neq \emptyset$.

4. Coincidence theorems and applications

As an application of Theorem 5 we give the following coincidence theorem.

Theorem 9. *Let D be a nonempty subset of a convex space, Y a topological space, $G : D \to Y$, $T : \operatorname{co} D \to Y$ maps and $F : \operatorname{co} D \to Y$ an admissible u.s.c. map. Suppose that the conditions* (i)–(iii) *in Theorem* 5 *are satisfied and moreover assume that:*

(iv) *for each $y \in F(\operatorname{co} D)$, $\operatorname{co}(G^{-1}y) \subset T^{-1}y$.*

Then there exists $x_0 \in \operatorname{co} D$ such that $F(x_0) \cap T(x_0) \neq \emptyset$.

Proof: By Theorem 5 there exist $A \in \langle D\rangle$ and

$$y_0 \in F(\operatorname{co} A) \cap \bigcap\{Gx : x \in A\}.$$

Therefore $y_0 \in F(x_0)$ for some $x_0 \in \operatorname{co} A$.

On the other hand, from $y_0 \in \bigcap\{Gx : x \in A\}$, taking into account (iv) we get $x_0 \in \operatorname{co} A \subset \operatorname{co}(G^{-1}y_0) \subset T^{-1}y_0$. Consequently $y_0 \in F(x_0) \cap T(x_0)$. □

Theorem 9 extends results of Tarafdar [31], [32], Ben-El-Mechaiekh and others [3], Park [18] on fixed points and coincidences for multivalued maps, these results being themselves generalizations of the well known Fan-Browder fixed point theorem [4], [7].

Similarly, using as argument Theorem 8 instead of Theorem 5 we can readily prove the following theorem:

Theorem 10. *Let D be a nonempty finite subset of a convex space, Y a topological space, $S : D \to Y$, $T : \operatorname{co} D \to Y$ maps and $F : \operatorname{co} D \to Y$ an admissible u.s.c. map. Suppose that conditions* (i), (ii) *in Theorem* 8 *are satisfied and moreover assume that:*

(iii) *for each $y \in F(\operatorname{co} D)$, $\operatorname{co}(S^{-1}y) \subset T^{-1}y$.*

Then there exists $x_0 \in \operatorname{co} D$ such that $F(x_0) \cap T(x_0) \neq \emptyset$.

Using his infinite version of the KKM theorem, Fan proved in [7] a section lemma leading to a proof of Tychonoff's fixed point theorem. Using Theorems 9 and 10 we obtain two section theorems including results previously given by Takahashi [30], Ha [13], Shioji [26], Lin [17], Balaj [2].

Theorem 11. *Let D be a nonempty subset of a convex space, Y a topological space, $F : \operatorname{co} D \to Y$ an admissible u.s.c. map, $\Omega \subset \operatorname{co} D \times Y$, $\Gamma \subset D \times Y$ sets. Suppose that:*

(i) *$\Gamma \subset \Omega$;*
(ii) *for each $x \in \operatorname{co} D$, $\{x\} \times Fx \subset \Omega$;*
(iii) *for each $x \in D$, $\{y \in Y : (x, y) \in \Gamma\}$ is compactly closed in Y;*
(iv) *for each $y \in F(\operatorname{co} D)$, $\{x \in \operatorname{co} D : (x, y) \notin \Omega\}$ is convex;*
(v) *there exists a nonempty compact subset K of Y such that either*
  (a) *for each $y \in Y \backslash K$, $A_0 \times \{y\} \not\subset \Gamma$, for some $A_0 \in \langle D\rangle$; or*
  (b) *for each $A \in \langle D\rangle$, there exists a compact convex subset L of $\operatorname{co} D$ containing A such that for each $y \in F(L_A)\backslash K$, $(L_A \cap D) \times \{y\} \not\subset \Gamma$.*

Then there exists $y_0 \in \overline{F(\operatorname{co} D)} \cap K$ such that $D \times \{y_0\} \subset \Gamma$.

Proof: Consider the maps $G : D \to Y$ and $T : \operatorname{co} D \to Y$ given by

$$Gx = \{y \in Y : (x, y) \notin \Gamma\} \text{ for } x \in D, \text{ and}$$
$$Tx = \{y \in Y : (x, y) \notin \Omega\} \text{ for } x \in \operatorname{co} D.$$

Suppose that the conclusion is false. Then $\overline{F(\operatorname{co} D)} \cap K \subset G(D)$. By (iii), for each $x \in D$, Gx is compactly open. The conditions (va), (vb) are clearly equivalent with the conditions (iiia), respectively (iiib) in Theorem 5. By (iv), for each $y \in F(\operatorname{co} D)$, $T^{-1}y$ is convex, and taking into account (i) we infer that $\operatorname{co}(G^{-1}y) \subset T^{-1}y$.

Therefore all hypothesis of Theorem 9 are satisfied, hence T and F have a coincidence point $x_0 \in \operatorname{co} D$. For $y \in T(x_0) \cap F(x_0)$, we have $(x_0, y) \notin \Omega$. But this contradicts (ii). $\square$

In similar manner, from Theorem 10 we can obtain

Theorem 12. *Let D be a finite nonempty subset of a convex space, Y a topological space, $F : \operatorname{co} D \to Y$ an admissible u.s.c. map, $\Omega \subset \operatorname{co} D \times Y$, $\Gamma \subset D \times Y$ sets. Suppose that conditions* (i), (ii), (iv) *in Theorem* 11 *hold and moreover assume that*

(iii′) *for each $x \in D$, $\{y \in Y : (x, y) \in \Gamma\}$ is compactly open in Y.*

Then there exists $y_0 \in Y$ such that $D \times \{y_0\} \subset \Gamma$.

As direct consequences of Theorems 11 and 12 we have the next dual corollaries. The first one generalizes earlier results of Fan [8], Allen [1], Lin [17], Shih and Tan [28].

Corollary 13. *Let D be a nonempty subset of a convex space, Y a topological space, $F : \operatorname{co} D \to Y$ an admissible u.s.c. map. If $f : \operatorname{co} D \times Y \to \mathbb{R}$, $g : D \times Y \to \mathbb{R}$ are two real-valued functions satisfying:*

(i) *for each $(x, y) \in D \times Y$, $g(x, y) \leq 0$ implies $f(x, y) \leq 0$;*
(ii) *for each $x \in \operatorname{co} D$ and any $y \in Fx$, $f(x, y) \leq 0$;*
(iii) *for each $x \in D$, the function $y \to g(x, y)$ is lower semicontinuous on each compact subset of Y;*
(iv) *for each $y \in F(\operatorname{co} D)$, $\{x \in \operatorname{co} D : f(x, y) > 0\}$ is convex;*
(v) *there exists a nonempty subset K of Y such that either*
  (a) *there exists an $A_0 \in \langle D\rangle$ such that for each $y \in Y \backslash K$, $g(x, y) > 0$ for some $x \in A_0$; or*
  (b) *for each $A \in \langle D\rangle$, there exists a compact convex subset L of $\operatorname{co} D$ containing A such that, for each $y \in F(L_A)\backslash K$, $g(x, y) > 0$ for some $x \in L_A \cap D$.*

Then there is $y_0 \in \overline{F(\operatorname{co} D)} \cap K$ such that $g(x, y_0) \leq 0$ for all $x \in D$.

Proof: Put $\Omega = \{(x, y) \in \operatorname{co} D \times Y : f(x, y) \leq 0\}$, $\Gamma = \{(x, y) \in D \times Y : g(x, y) \leq 0\}$ and apply Theorem 11. □

Corollary 14. *Let D be a nonempty finite subset of a convex space, Y a topological space, $F : \operatorname{co} D \to Y$ an admissible u.s.c. map. If $f : \operatorname{co} D \times Y \to \mathbb{R}$, $g : D \times Y \to \mathbb{R}$ are two real-valued functions satisfying:*

(i) *for each $(x, y) \in D \times Y$, $g(x, y) < 0$ implies $f(x, y) < 0$;*
(ii) *for each $x \in \operatorname{co} D$ and any $y \in Fx$, $f(x, y) < 0$;*
(iii) *for each $x \in D$, the function $y \to g(x, y)$ is upper semicontinuous on each compact subset of Y;*
(iv) *for each $y \in F(\operatorname{co} D)$, $\{x \in \operatorname{co} D : f(x, y) \geq 0\}$ is convex.*

Then there is $y_0 \in Y$ such that $g(x, y_0) < 0$ for all $x \in D$.

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