# Characterizations of almost transitive superreflexive Banach spaces

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*Abstract.* Almost transitive superreflexive Banach spaces have been considered in [7] (see also [4] and [6]), where it is shown that such spaces are uniformly convex and uniformly smooth. We prove that convex transitive Banach spaces are either almost transitive and superreflexive (hence uniformly smooth) or extremely rough. The extreme roughness of a Banach space $X$ means that, for every element $u$ in the unit sphere of $X$, we have

$$\limsup_{\|h\|\to 0} \frac{\|u+h\|+\|u-h\|-2}{\|h\|} = 2.$$

We note that, in general, the property of convex transitivity for a Banach space is weaker than that of almost transitivity.

*Keywords:* convex transitive, almost transitive, superreflexive, uniformly smooth, rough norm

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## 1. Introduction

Throughout this paper $X$ will denote a Banach space over the field $\mathbb{K}$ of real or complex numbers, $S = S(X)$ and $B = B(X)$ will be the unit sphere and the closed unit ball of $X$, respectively, and $\mathcal{G} = \mathcal{G}(X)$ will stand for the group of all surjective linear isometries on $X$. The Banach space $X$ is said to be *almost transitive* whenever, for every (equivalently, some) element $u$ in $S$, $\mathcal{G}(u)$ is dense in $S$. We denote by $\mathcal{J}$ the class of almost transitive superreflexive Banach spaces. This class has been first considered by C. Finet [7] (see also [6; Corollary IV.5.7]) and, very recently, has been revisited by F. Cabello [4] and the authors [2].

According to [7], every member of $\mathcal{J}$ is uniformly smooth and uniformly convex. By his part, F. Cabello shows that, for an almost transitive Banach space, superreflexivity is equivalent to reflexivity (and even to either enjoy the Radon-Nikodym property or be Asplund). He also proves, that, for a superreflexive Banach space, the notion of almost transitivity is equivalent to that (in general weaker) of convex transitivity. We recall that a Banach space $X$ is said to be *convex transitive* if, for every $u$ in $S$, we have $\overline{\mathrm{co}}\,\mathcal{G}(u) = B$, where $\overline{\mathrm{co}}$ means closed convex hull. In [2], we show that members of $\mathcal{J}$ can be characterized as those convex transitive Banach spaces which either have the Radon-Nikodym property or are Asplund.

Actually, the result just reviewed follows from a more general fact involving the concept of a rough space. For $u$ in $S$, we put

$$\eta(X,u) \ := \ \limsup_{\|h\|\to 0} \frac{\|u+h\|+\|u-h\|-2}{\|h\|} \ .$$

Given $\epsilon > 0$, the Banach space $X$ is said to be *$\epsilon$-rough* if, for every $u$ in $S$, we have $\eta(X,u) \geq \epsilon$. We say that $X$ is *rough* whenever it is $\epsilon$-rough for some $\epsilon > 0$, and *extremely rough* whenever it is 2-rough. Since, for $u$ in $S$, the Fréchet differentiability of the norm of $X$ at $u$ can be characterized by the equality $\eta(X,u) = 0$ ([6; Lemma I.1.3]), it follows that the roughness of $X$ can be seen as a uniform non Fréchet-differentiability of the norm, and hence becomes the extremely opposite situation to that of the uniform smoothness. We proved in [2] that a Banach space $X$ is a member of $\mathcal{J}$ if (and only if) it is convex transitive and either $X$ or $X^*$ is non rough.

As main result, we show in the present paper that the Banach space $X$ is a member of $\mathcal{J}$ if (and only if) it is convex transitive and either $X$ or $X^*$ is not extremely rough. Through a technical lemma, namely Lemma 1, the main tool in the proof is a theorem, essentially due to R.C. James, establishing that uniformly non-square Banach spaces are superreflexive ([5; Theorem VII.4.4]). We also find another remarkable characterization of members of $\mathcal{J}$ involving the notion of a big point. Let us say that an element $u$ of $X$ is a *big point* of $X$ if $u$ belongs to $S$ and $\overline{\mathrm{co}}\mathcal{G}(u) = B$ (so that $X$ is convex transitive precisely when all elements in $S$ are big points of $X$). We prove that $X$ lies in $\mathcal{J}$ if (and only if) there exists a big point $u$ in $X$ such that the norm of $X$ is Fréchet differentiable at $u$.

## 2. The results

A Banach space $X$ is said to be *uniformly non-square* if there exists $0 < a < 1$ such that $\|x-y\| < 2a$ whenever $x, y$ are in $B$ with $\|x+y\| \geq 2a$.

**Lemma 1.** *Assume that there exists a big point $u$ in $X$ such that $\eta(X,u) < 2$. Then $X$ is uniformly non-square.*

Proof: Let us fix $\epsilon$ satisfying $\eta(X,u) < \epsilon < 2$. Then there is $0 < \delta < 1$ such that

$$\frac{\|u+h\|+\|u-h\|-2}{\|h\|} \ \leq \ \epsilon$$

whenever $h$ is in $X\backslash\{0\}$ and $\|h\| \leq \delta$. Now

$$\left\{ v \in X : \frac{\|v+h\|+\|v-h\|-2}{\|h\|} \ \leq \ \epsilon \ \text{ whenever } h \text{ is in } X\backslash\{0\} \text{ with } \ \|h\| \leq \delta \right\}$$

is a closed, convex, and $\mathcal{G}$-invariant subset of $X$ containing $u$. It follows from the bigness of $u$ that

$$(*)\qquad \frac{\|v+h\|+\|v-h\|-2}{\|h\|} \le \epsilon \text{ whenever } v \text{ is in } B \text{ and } h \text{ is in } X\backslash\{0\} \text{ with } \|h\| \le \delta.$$

Take $\sigma$ with $\epsilon < \sigma < 2$, and $a$ with

$$\frac{1}{2}\max\{2-(\sigma-\epsilon)\delta, 2-(2-\sigma)\delta\} \; < \; a \; < \; 1.$$

Let $x, y$ be in $B$ such that $\|x+y\| \ge 2a$. Then we have

$$\|x+y\| \;\ge\; 2-(\sigma-\epsilon)\delta,$$

and hence

$$\|x+\delta y\| \;\ge\; 2-(\sigma-\epsilon)\delta-(1-\delta).$$

Since, on the other hand, the equality

$$\|x-\delta y\| \;\ge\; \|x-y\|-(1-\delta)$$

holds, we obtain

$$\|x+\delta y\| \;+\; \|x-\delta y\| \;\ge\; \|x-y\| \;+\; (2-\sigma+\epsilon)\delta.$$

It follows from $(*)$ that

$$\|x-y\| \;+\; (2-\sigma+\epsilon)\delta \;\le\; 2+\epsilon\delta,$$

and therefore

$$\|x-y\| \;\le\; 2-(2-\sigma)\delta \;<\; 2a. \qquad \square$$

We say that an element $f$ of $X^*$ is a $w^*$-big point of $X$ if $f$ belongs to $S(X^*)$ and the convex hull of $\mathcal{G}(X^*)(f)$ is $w^*$-dense in $B(X^*)$. By keeping in mind that the norm of $X^*$ is lower $w^*$-semicontinuous, the proof of Lemma 2 below is similar to that of Lemma 1.

**Lemma 2.** *Assume that there exists a $w^*$-big point $f$ in $X^*$ such that $\eta(X^*, f) < 2$. Then $X^*$ is uniformly non-square.*

Let $u$ be in $S$. For $x$ in $X$, the number $\lim_{\alpha\to 0^+}\frac{\|u+\alpha x\|-1}{\alpha}$ (which always exists because the mapping $\alpha \to \|u+\alpha x\|$ from $\mathbb{R}$ to $\mathbb{R}$ is convex) is usually denoted by $\tau(u, x)$. We say that the norm of $X$ is strongly subdifferentiable at $u$ if

$$\lim_{\alpha\to 0^+}\frac{\|u+\alpha x\|-1}{\alpha} \;=\; \tau(u,x) \;\text{ uniformly for } x \text{ in } B.$$

The reader is referred to [1] and [8] for a comprehensive view of the usefulness of the strong subdifferentiability of the norm in the theory of Banach spaces.

For $u$ in $S$, we put

$$D(X,u) := \{g \in X^* \;:\; \|g\| = g(u) = 1\}.$$

**Lemma 3.** *Let $u$ be a big point of $X$ such that the norm of $X$ is strongly subdifferentiable at $u$. Then the set*

$$\{T^*(f)\ :\ f \in D(X,u),\ T \in \mathcal{G}\}$$

*is norm-dense in $S(X^*)$.*

Proof: Let $\epsilon$ be a positive number. Since the norm of $X$ is strongly subdifferentiable at $u$, we can apply ([8; Theorem 1.2(iv)⇒(i)]) to find $0 < \delta$ such that $d(g, D(X,u)) < \epsilon$ whenever $g$ belongs to $B(X^*)$ and $|g(u) - 1| < \delta$. Now, let $h$ be in $S(X^*)$. Since $u$ is a big point of $X$, there exists $T$ in $\mathcal{G}$ satisfying $|h(T(u)) - 1| < \delta$. Now $T^*(h)$ lies in $B(X^*)$ and satisfies $|T^*(h)(u) - 1| < \delta$, and hence there is $f$ in $D(X,u)$ such that $\|T^*(h) - f\| < \epsilon$. For such an $f$, we have $\|h - T^{*-1}(f)\| < \epsilon$. □

The dual $X^*$ of the Banach space $X$ is said to be convex $w^*$-transitive if every element of $S(X^*)$ is a $w^*$-big point of $X^*$. An easy and well-known consequence of the Hahn-Banach theorem is that convex transitivity of $X$ implies convex $w^*$-transitivity of $X^*$. Recall that the symbol $\mathcal{J}$ stands for the class of almost transitive superreflexive Banach spaces.

**Theorem 1.** *The following assertions are equivalent:*

1. *$X$ is a member of $\mathcal{J}$.*
2. *There exists a big point $u$ in $X$ such that the norm of $X$ is Fréchet differentiable at $u$.*
3. *There exists a $w^*$-big point $f$ in $X^*$ such that the norm of $X^*$ is Fréchet differentiable at $f$.*
4. *$X$ is convex transitive and the norm of $X$ is not extremely rough.*
5. *$X^*$ is convex $w^*$-transitive and the norm of $X^*$ is not extremely rough.*

Proof: Certainly the implications $1 \Rightarrow 4$ and $1 \Rightarrow 5$ are true.

$2 \Rightarrow 1$. Since the norm of $X$ is Fréchet differentiable at $u$, we have $\eta(X,u) = 0 < 2$, so that, since $u$ is a big point of $X$, we can apply Lemma 1 and the already quoted James' theorem ([5; Theorem VII.4.4]) to obtain that $X$ is superreflexive. On the other hand, the Fréchet differentiability of the norm of $X$ at $u$ implies that the norm of $X$ is strongly subdifferentiable at $u$ and that $D(X,u)$ reduces to a singleton, so that, by Lemma 3, $X^*$ is almost transitive. Now, surely, there exits in the unit sphere of the reflexive Banach space $X^*$ some point $g$ such that the norm of $X^*$ is Fréchet differentiable at $g$, and such a point is a big point of $X^*$ (because $X^*$ is almost transitive). Repeating the argument with $(X^*, g)$ instead of $(X,u)$, we obtain that $X$ is almost transitive.

$3 \Rightarrow 1$. With $X^*$ instead of $X$, and Lemma 2 instead of Lemma 1, we can argue as in the proof of $2 \Rightarrow 1$ above to obtain that $X^*$ (and hence also $X$) is superreflexive, and that $X$ is almost transitive.

$4 \Rightarrow 2$. Since the norm of $X$ is not extremely rough, there exists $v$ in $S$ such that $\eta(X, v) < 2$. Since $X$ is convex transitive, such an $v$ is a big point of $X$. By Lemma 1, $X$ is reflexive, so that there is some $u$ in $S$ such that the norm of $X$ is Fréchet differentiable at $u$. Applying again that $X$ is convex transitive, we obtain that $u$ is a big point of $X$.

$5 \Rightarrow 3$. With $X^*$ instead of $X$ and Lemma 2 instead of Lemma 1, the proof is similar to that of $4 \Rightarrow 2$ above. □

It follows from Theorem 1 (or even from its forerunner [2; Theorem 3.2]) that any of the following two assertions is sufficient (and of course necessary) to convert a Banach space $X$ into a member of $\mathcal{J}$:

(i) $X$ is convex transitive and either has the Radon-Nikodym property or is Asplund.

(ii) $X^*$ is convex $w^*$-transitive and either $X$ has the Radon-Nikodym property or $X$ is Asplund.

Now, recall that a subset $R$ of a topological space $T$ is said to be nowhere dense in $T$ if the interior of the closure of $R$ in $T$ is empty. Actually, Theorem 3.2 in [2] contains enough information to derive other characterizations of members $X$ of $\mathcal{J}$, like the following:

(iii) There exists a non nowhere dense subset of $S$ consisting of big points of $X$, and $X$ has the Radon-Nikodym property.

(iv) There exists a non nowhere dense subset of $S(X^*)$ consisting of $w^*$-big points of $X^*$, and $X$ is Asplund.

Now, we can complete the situation by proving the next corollary.

**Corollary 1.** *The following assertions are equivalent:*

1. *$X$ lies in $\mathcal{J}$.*
2. *There exists a non nowhere dense subset of $S$ consisting of big points of $X$, and $X$ is Asplund.*
3. *There exists a non nowhere dense subset of $S(X^*)$ consisting of $w^*$-big points of $X^*$, and $X$ has the Radon-Nikodym property.*

Proof: By the Hahn-Banach theorem, an element $u$ in $S$ is a big point of $X$ if and only if, for every $g$ in $S(X^*)$, we have

$$\sup\{|g(T(u))| \ : \ T \in \mathcal{G}\} = 1.$$

Analogously, an element $f$ in $S(X^*)$ is a $w^*$-big point of $X^*$ if and only if, for every $x$ in $S$, we have

$$\sup\{|F(f)(x)| \ : \ F \in \mathcal{G}(X^*)\} = 1.$$

Therefore the set of all big points of $X$ is closed in $S$, and the set of all $w^*$-big points of $X^*$ is norm-closed in $S(X^*)$. Assume that Assertion 2 holds. Then, by the first requirement, there is a non-empty open subset of $S$ consisting of big points of $X$. By the second requirement, there must exist a point $u$ in such an open set such that the norm of $X$ is Fréchet differentiable at $u$. By the implication $2 \Rightarrow 1$ in Theorem 1, $X$ is a member of $\mathcal{J}$. Now, assume that Assertion 3 holds. Then there is a non-empty open subset (say $A$) of $S(X^*)$ consisting of $w^*$-big points of $X^*$. Since $X$ has the Radon-Nikodym property, we can apply [3; Theorem 5.7.4] to find some $f$ in $A$ such that the norm of $X^*$ is Fréchet differentiable at $f$. Then $X$ lies in $\mathcal{J}$ by $3 \Rightarrow 1$ in Theorem 1. $\square$

Given $1 \leq p \leq \infty$, a subspace $M$ of the Banach space $X$ is said to be an *$L^p$-summand* of $X$ if there is a linear projection $\pi$ from $X$ onto $M$ such that, for every $x$ in $X$, we have

$$\|x\|^p = \|\pi(x)\|^p + \|x - \pi(x)\|^p \ (1 \leq p < \infty),$$
$$\|x\| = \max\{\|\pi(x)\|, \|x - \pi(x)\|\} \ (p = \infty).$$

If $M$ is an $L^p$-summand of $X$, then the projection $\pi$ above is uniquely determined by $M$, and is called the *$L^p$-projection* from $X$ onto $M$.

**Corollary 2.** *Assume that there exists a big point $u$ in $X$ such that $\mathbb{K}u$ is an $L^p$-summand of $X$ for some $1 < p \leq \infty$. Then $X$ is a Hilbert space. If in addition $p \neq 2$, then $X$ is one-dimensional.*

Proof: First of all, note that a Hilbert space of dimension $\geq 2$ cannot have one-dimensional $L^p$-summands for $p \neq 2$, so that it is enough to show that $X$ is a Hilbert space. Since $1 < p \leq \infty$, and $\mathbb{K}u$ is an $L^p$-summand of $X$, the norm of $X$ is Fréchet differentiable at $u$. It follows from the bigness of $u$ and the implication $2 \Rightarrow 1$ in Theorem 1 that $X$ is almost transitive. Assume that $\mathbb{K} = \mathbb{C}$. Then, since $L^p$-projections on complex Banach spaces are hermitian operators, the result follows from [10; Theorem 6.4]. Now assume that $\mathbb{K} = \mathbb{R}$. Then, denoting by $\pi$ the $L^p$-projection from $X$ onto $\mathbb{K}u$, $\mathbf{1} - 2\pi$ becomes an isometric reflexion on $X$. It follows from [11; Theorem 2.a] that $X$ is a Hilbert space. $\square$

Corollary 2 above does not remain true for $p = 1$. Indeed, for $X$ equal to either $\ell_1$ or $\ell_1^n$ ($n \in \mathbb{N}$), every element $u$ in the natural basis of $X$ is a big point of $X$ such that $\mathbb{K}u$ is an $L^1$-summand of $X$. In any case, if $X$ is convex transitive and has a one-dimensional $L^1$-summand, then $X$ is one-dimensional ([2; Corollary 3.5]).

We conclude this paper with two remarks related to the matter we have developed.

**Remark 1.** Concerning Lemma 1, it is worth mentioning that, if the Banach space $X$ is uniformly non-square, then we have $\eta(X, u) < 2$ for every $u$ in $S$. To verify this assertion, assume that there exists some $u$ in $S$ satisfying $\eta(X,u)=2$. By the proof of [6; Proposition I.1.11], for every $n$ in $\mathbb{N}$, there are $f_n,\ g_n$ in $B(X^*)$ satisfying $Re(f_n(u)) > 1-\frac{1}{n}$, $Re(g_n(u)) > 1-\frac{1}{n}$, and $\|f_n - g_n\| \geq 2-\frac{1}{n}$. Now, assume additionally that $X$ is uniformly non-square. Then so is $X^*$ ([5; p. 173]), and hence there is $0<a<1$ such that $\|f+g\| < 2a$ whenever $f,g$ are in $B(X^*)$ with $\|f-g\| \geq 2a$. Taking $n$ big enough to have $\|f_n-g_n\| \geq 2a$, $Re(f_n(u)) > a$, and $Re(g_n(u)) > a$, it follows

$$2a \;<\; Re(f_n(u)+g_n(u)) \;\leq\; \|f_n+g_n\| \;<\; 2a,$$

a contradiction. □

**Remark 2.** We say that the Banach space $X$ has the *Mazur's intersection property* whenever every bounded closed convex subset of $X$ can be represented as an intersection of closed balls in $X$. Analogously, we say that $X^*$ has the *Mazur's $w^*$-intersection property* whenever every bounded $w^*$-closed convex subset of $X^*$ can be expressed as an intersection of closed balls in $X^*$. We proved in [2; Theorem 3.4] that $X$ lies in $\mathcal{J}$ if and only if there exists a big point in $X$, and the set of all denoting points of $B$ is dense in $S$. Applying [9; Theorem 3.1], we have:

(i) $X$ is a member of $\mathcal{J}$ if and only if $X^*$ has the Mazur's $w^*$-intersection property and there is a big point in $X$.

We also proved in [2; Theorem 3.4] that $X$ lies in $\mathcal{J}$ if and only if there exists an $w^*$-big point in $X^*$, and the set of all $w^*$-denoting points of $B(X^*)$ is norm-dense in $S(X^*)$. With [9; Theorem 2.1] in the mind, this result reads as follows:

(ii) $X$ is a member of $\mathcal{J}$ if and only if $X$ has the Mazur's intersection property and there is an $w^*$-big point in $X^*$.

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