Relative exact covers

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Abstract. Recently Rim and Teply [11] found a necessary condition for the existence of σ-torsionfree covers with respect to a given hereditary torsion theory for the category R-mod. This condition uses the class of σ-exact modules; i.e. the σ-torsionfree modules for which every its σ-torsionfree homomorphic image is σ-injective. In this note we shall show that the existence of σ-torsionfree covers implies the existence of σ-exact covers, and we shall investigate some sufficient conditions for the converse.

Keywords: precover, cover, hereditary torsion theory σ, σ-injective module, σ-exact module, σ-pure submodule

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In what follows, R stands for an associative ring with identity and R-mod denotes the category of all left unital R-modules. A class $\mathcal{G}$ of modules is called *abstract* if it is closed under isomorphic copies. Recall that a homomorphism $\varphi : G \to M$ is called a $\mathcal{G}$*-precover* of the module M if $G \in \mathcal{G}$ and every homomorphism $f : F \to M$, $F \in \mathcal{G}$, factors through φ, i.e. there exists $g : F \to G$ such that $\varphi g = f$. Moreover, a $\mathcal{G}$-precover φ of M is said to be a $\mathcal{G}$*-cover*, if each endomorphism f of G such that $\varphi f = \varphi$ is the automorphism of the module G.

As usual, a *hereditary torsion theory* $\sigma = (\mathcal{T}, \mathcal{F})$ for the category R-mod consists of two abstract classes $\mathcal{T}$ and $\mathcal{F}$, the σ*-torsion class* and the σ*-torsionfree class*, respectively, such that $\mathrm{Hom}(T, F) = 0$ whenever $T \in \mathcal{T}$ and $F \in \mathcal{F}$. The class $\mathcal{T}$ is closed under submodules, factor modules, extensions and direct sums. The class $\mathcal{F}$ is closed under submodules, extensions and direct products. For each module M there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathcal{T}$ and $F \in \mathcal{F}$. To each hereditary torsion theory σ it is associated the *Gabriel filter* $\mathcal{L}$ of left ideals consisting of all left ideals $I \leq R$ with $R/I \in \mathcal{T}$. Recall that σ is said to be *of finite type* if $\mathcal{L}$ contains a cofinal subset $\mathcal{L}'$ of finitely generated left ideals. A module F is called σ*-injective* if it is injective with respect to all exact sequences $E : 0 \to A \to B \to C \to 0$, where $C \in \mathcal{T}$. Baer Test Lemma states that it suffices to consider the sequence E where $B = R$ or, equivalently, the sequences

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where $B = R$ and $A \in \mathcal{L}$. The class of all σ-torsionfree σ-injective modules will be denoted by $\mathcal{J}$. Following [11], we say that a σ-torsionfree module is *σ-exact* if any its σ-torsionfree homomorphic image is σ-injective. The class of all such modules will be denoted by $\mathcal{E}$.

The investigations of this paper are motivated by several sources. First, if σ is an *exact torsion theory* for R-mod, i.e. a hereditary torsion theory such that $E(M)/M \in \mathcal{J}$ whenever $M \in \mathcal{F}$ and $M \in \mathcal{J}$, then $\mathcal{J} = \mathcal{E}$ and it is well-known (see e.g. [12] or [6]) that every module has a $\mathcal{J}$-cover whenever σ is of finite type, i.e. if σ is *perfect*. Further, it is known that a sufficient condition for the existence of $\mathcal{F}$-covers (σ-torsionfree covers) is equivalent to the condition that the directed unions of σ-torsionfree σ-injective modules are σ-injective (see [12] and [9; Proposition 43.9]). On the other hand, recently [11] presented a necessary condition saying that the directed union of σ-exact modules is σ-injective. Some other examples having similar motivating character are mentioned at the end of the paper.

So, the purpose of this note is to investigate and unify the methods and relations between σ-torsionfree and σ-exact covers under several hypotheses on the torsion theories or classes of modules, respectively.

Recall from [4] that a submodule N of a module M is said to be *$\mathcal{G}$-pure*, $\mathcal{G}$ being an abstract class of modules, if the factor module M/N belongs to the class $\mathcal{G}$. Further, it is well-known (and not too hard to verify) that a σ-torsionfree precover $\varphi : G \to M$ of a module M is the σ-torsionfree cover of M if and only if $\operatorname{Ker} \varphi$ contains no non-zero submodule σ-pure (i.e. $\mathcal{F}$-pure) in G. The following results are well-known, we include these assertions here for the sake of completeness.

1. Lemma. *Let $E : 0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be an exact sequence. Then*

(i) *if $B \in \mathcal{E}$ and $C \in \mathcal{F}$ then $A, C \in \mathcal{E}$;*
(ii) *if $A, C \in \mathcal{E}$ then $B \in \mathcal{E}$.*

□

2. Remark. If $F \in \mathcal{E}$, then a submodule K of F is $\mathcal{E}$-pure in F if and only if it is σ-pure (i.e. σ-closed) in F. This simple fact follows immediately from the definitions but, since it will be frequently used in the sequel, it seems to be convenient to formulate it here explicitly. A similar situation occurs with the fact that if an abstract class $\mathcal{G}$ is closed under finite direct sums (or, more generally, under extensions) and under directed unions, then it is closed under arbitrary direct sums. Clearly, let $F = \bigoplus_{\lambda \in \Lambda} F_\lambda, F_\lambda \in \mathcal{G}$, and let $\{K_\omega | \omega \in \Omega\}$ be the collection of all finite subset of the set Λ. Setting $F_\omega = \bigoplus_{\lambda \in K_\omega} F_\lambda$, we see that $F_\omega \in \mathcal{E}$ and the union $F = \bigcup_{\omega \in \Omega} F_\omega$ is directed in the natural way.

Now we are going to show that the existence of σ-torsionfree covers implies that the class $\mathcal{E}$ is closed under directed unions and that for each $F \in \mathcal{E}$ the set of all $\mathcal{E}$-pure submodules is closed under directed unions, too. The first assertion slightly generalizes [11; Theorem 1].

3. Proposition. *Let $\sigma = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category* R-mod *such that every module has a σ-torsionfree cover. Then*

(i) *the class $\mathcal{E}$ is closed under directed unions;*
(ii) *for an arbitrary module $F \in \mathcal{E}$ the set of all $\mathcal{E}$-pure submodules of F is closed under directed unions.*

Proof: (i) Let $F = \bigcup_{\lambda\in\Lambda} F_\lambda$ be a directed union of σ-exact modules. We first show that $F \in \mathcal{J}$. If $\varphi : G \to E_\sigma(F)/F$ is an $\mathcal{F}$-cover of $E_\sigma(F)/F = \sigma(E(F)/F)$, then for each $\lambda \in \Lambda$ we have the commutative diagram

$$\begin{tikzcd} E_\sigma(F)/F_\lambda \arrow[r, equal] \arrow[d, "f_\lambda"'] & E_\sigma(F)/F_\lambda \arrow[d, "\pi_\lambda"] \\ G \arrow[r, "\varphi"] & E_\sigma(F)/F \end{tikzcd}$$

where π_λ is the canonical projection and f_λ exists by the definition of a precover in view of the fact that $E_\sigma(F)/F_\lambda$ is σ-torsionfree, F_λ being σ-injective. Clearly, $\operatorname{Ker} f_\lambda \subseteq F/F_\lambda$ and we show that the equality $\operatorname{Ker} f_\lambda = F/F_\lambda$ holds for each $\lambda \in \Lambda$. Assuming that $\operatorname{Ker} f_\lambda \neq F/F_\lambda$ for some $\lambda \in \Lambda$, we can find an index $\mu \in \Lambda$ such that $f_\lambda(\frac{F\mu+F_\lambda}{F_\lambda}) \neq 0$. Now $f_\lambda(\frac{F\mu+F_\lambda}{F_\lambda})$ is σ-exact as the σ-torsionfree homomorphic image of $\frac{F\mu}{F_\lambda\cap F\mu}$ and so it is σ-pure in G. Obviously, $f_\lambda(\frac{F\mu+F_\lambda}{F_\lambda}) \subseteq \operatorname{Ker}\varphi$, which contradicts by Remark 2 the fact that φ is an $\mathcal{F}$-cover of $E_\sigma(F)/F$. Thus $\operatorname{Ker} f_\lambda = F/F_\lambda$; hence $\operatorname{Im} f_\lambda \cong E_\sigma(F)/F$, $E_\sigma(F)/F$ is σ-torsionfree and $F = E_\sigma(F)$ is σ-torsionfree σ-injective.

Now let $K \leq F$ be any σ-closed submodule. Then $F/K = \bigcup_{\lambda\in\Lambda} \frac{F_\lambda+K}{K}$ is a directed union of σ-exact submodules $\frac{F_\lambda+K}{K} \cong \frac{F_\lambda}{F_\lambda\cap K}$ and so it is σ-injective by the first part of the proof.

(ii) Let $K = \bigcup_{\lambda\in\Lambda} K_\lambda$ be a directed union of $\mathcal{E}$-pure submodules of the module $F \in \mathcal{E}$. The only property we need to show is that K is σ-closed in F. However, K_λ is, as a σ-closed submodule of F, σ-exact for each $\lambda \in \Lambda$ by Lemma 1 and, consequently, K is σ-exact by (i). Hence K is σ-closed in F by [9; Proposition 10.1]. □

Similarly to the case of σ-torsionfree covers (Remark 2), we have the following result for $\mathcal{E}$-covers.

4. Proposition. *Let $\sigma = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category* R-mod. *An $\mathcal{E}$-precover $\varphi : G \to M$ of a module M is an $\mathcal{E}$-cover of M if and only if* $\operatorname{Ker}\varphi$ *contains no non-zero submodule $\mathcal{E}$-pure in G.*

Proof: Assume first that φ is an $\mathcal{E}$-cover of M, and let $K \subseteq \operatorname{Ker}\varphi$ be an $\mathcal{E}$-pure submodule of G. Consider the following diagram

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\varphi"'] & G/K \arrow[r, "\varrho"] \arrow[d, "\bar\varphi"] & G \arrow[d, "\varphi"] \\ M \arrow[r, equal] & M \arrow[r, equal] & M \end{tikzcd}$$

where π is the canonical projection and $\bar{\varphi}$ is the corresponding natural map. Since $G/K \in \mathcal{E}$, the definition of a precover yields the existence of a homomorphism $\varrho : G/K \to G$ such that $\varphi\varrho = \bar{\varphi}$. So, $\varphi(\varrho\pi) = \bar{\varphi}\pi = \varphi$ gives that $\varrho\pi$ is an automorphism of G; hence π is injective and $K = 0$.

Conversely, let the condition be satisfied and let f be an endomorphism of the module G such that $\varphi f = \varphi$. Obviously, $\operatorname{Ker} f \subseteq \operatorname{Ker} \varphi$ and so $G/\operatorname{Ker} f \cong \operatorname{Im} f \in \mathcal{F}$ yields by Remark 2 that f is injective. Further, the submodule $H = \{u - f(u) \mid u \in G\}$ of G is clearly contained in $\operatorname{Ker} \varphi$ and as an epimorphic image of G it is σ-pure in G by [9; Proposition 10.1]. Thus $H = 0$ and f is surjective. □

Following [4], we say that an abstract class $\mathcal{G}$ of modules satisfies condition (P), if to each infinite cardinal λ there exists a cardinal $\kappa > \lambda$ such that for every $F \in \mathcal{G}$ with $|F| \geq \kappa$ and every $K \leq F$ with $|F/K| \leq \lambda$, the submodule K contains a non-zero submodule L such that $F/L \in \mathcal{G}$. In [4; Theorem 2 and Corollary 3] it has been proved that if an abstract class $\mathcal{G}$ of modules is closed under direct sums, satisfies condition (P) and, for each $F \in \mathcal{G}$, the set of all $\mathcal{G}$-pure submodules of F is inductive, then every module has a $\mathcal{G}$-precover. If, in addition, the class $\mathcal{G}$ is closed under direct limits, then every module has a $\mathcal{G}$-cover.

5. Theorem. *Let $\sigma = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category R-mod. If every module has a σ-torsionfree cover, then every module has an $\mathcal{E}$-cover.*

Proof: The class $\mathcal{E}$ is closed under arbitrary direct sums by Proposition 3(i) and Remark 2. The set of all $\mathcal{E}$-pure submodules of each member of $\mathcal{E}$ is inductive by Proposition 3(ii) and $\mathcal{E}$ satisfies condition (P) by [6; Theorem 11]. Thus every module has an $\mathcal{E}$-precover by [4; Theorem 2]. Moreover, in view of Proposition 3(ii) we can find an $\mathcal{E}$-precover $\varphi : G \to M$ of the module M such that $\operatorname{Ker} \varphi$ contains no non-zero $\mathcal{E}$-pure submodule of G. Thus φ is an $\mathcal{E}$-cover of M by Proposition 4. □

6. Corollary. *Let $\sigma = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory of finite type for the category R-mod. Then*

(i) *the class $\mathcal{E}$ is closed under directed unions;*
(ii) *the set of all $\mathcal{E}$-pure submodules of any member of $\mathcal{E}$ is closed under directed unions;*
(iii) *every module has an $\mathcal{E}$-cover.*

Proof: By [12; Theorem] and [5; Corollary 4.1], every module has a σ-torsionfree cover and it suffices to apply Proposition 3 and Theorem 5. $\square$

Now we proceed to formulate some conditions that are sufficient to prove the converse statement to Theorem 5. Note that the hypotheses of Proposition 8 are motivated by the ordinary torsion theory on the category Ab of all abelian groups.

7. Proposition. *Let $\sigma = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category R-mod.*

(i) *If $0 \neq F = \bigcup_{\lambda \in \Lambda} F_\lambda$ is a directed union of members of the class $\mathcal{E}$ and if F has an $\mathcal{E}$-cover, then F belongs to the class $\mathcal{E}$.*

(ii) *If $K = \bigcup_{\lambda \in \Lambda} K_\lambda$ is a directed union of $\mathcal{E}$-pure submodules of a module $F \in \mathcal{E}$ such that K has an $\mathcal{E}$-cover, then K is $\mathcal{E}$-pure in F.*

Proof: (i) For an $\mathcal{E}$-cover $\varphi : G \to F$ of the module F, consider the diagram

$$\begin{array}{ccccc} F_\lambda & \xrightarrow{\iota_{\mu\lambda}} & F_\mu & \xrightarrow{\varrho_\mu} & G \\ \| & & \| & & \downarrow{\scriptstyle\varphi} \\ F_\lambda & \xrightarrow{\iota_{\mu\lambda}} & F_\mu & \xrightarrow{\iota_\mu} & F \end{array}$$

where $0 \neq F_\lambda \subseteq F_\mu$ and $\iota_{\mu\lambda}, \iota_\mu$ are the inclusion maps. By hypothesis, for each $\lambda \in \Lambda$ there exists $\varrho_\lambda : F_\lambda \to G$ such that $\varphi\varrho_\lambda = \iota_\lambda$. From this we infer that $\varphi \neq 0$ and since F is obviously σ-torsionfree, $\operatorname{Ker}\varphi$ is σ-pure in G. So, by Remark 2, it is $\mathcal{E}$-pure in G. Thus φ is injective by Proposition 4 and so $\varphi(\varrho_\lambda - \varrho_\mu\iota_{\mu\lambda}) = \iota_\lambda - \iota_\mu\iota_{\mu\lambda} = 0$ yields $\varrho_\lambda = \varrho_\mu\iota_{\mu\lambda}$. This means that ϱ_μ extends ϱ_λ whenever $F_\lambda \subseteq F_\mu$ and, consequently, there naturally exists a homomorphism $\psi : F \to G$ such that $\psi\iota_\lambda = \varrho_\lambda$ for each $\lambda \in \Lambda$. By the same argument $\varphi\psi\iota_\lambda = \varphi\varrho_\lambda = \iota_\lambda$ gives $\varphi\psi = 1$. Thus φ is an isomorphism and hence $F \in \mathcal{E}$.

(ii) Each K_λ, $\lambda \in \Lambda$, is σ-pure in F by [9; Proposition 10.1] and so it lies in the class $\mathcal{E}$ by Lemma 1. By (i) we have that $K \in \mathcal{E}$; hence K is σ-pure in F by [9; Proposition 10.1] again. Remark 2 finishes the proof. $\square$

8. Proposition. *Let $\sigma = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category R-mod. If F is a σ-torsionfree σ-injective module such that every non-zero σ-torsionfree factor module and every submodule of F that is expressible as a directed union of members of $\mathcal{E}$ has a non-zero $\mathcal{E}$-cover, then F is σ-exact.*

Proof: Let $\varphi_0 : G_0 \to F$ be an $\mathcal{E}$-cover of the module F. We may assume $F \neq 0$, the case $F = 0$ being trivial. By the hypothesis and the unicity of covers we have $\varphi_0 \neq 0$. Since $G_0/\operatorname{Ker}\varphi_0 \cong \operatorname{Im}\varphi_0 \in \mathcal{F}$, φ_0 is a monomorphism by Proposition 4 and Remark 2. Hence $K_0 = \varphi_0(G_0)$ is a non-zero submodule of F lying in the class $\mathcal{E}$. For $K_0 \neq F$ assume that for some ordinal α the submodules $K_\beta, \beta < \alpha$, of F have already been constructed in such a way that $K_\gamma \subset K_{\gamma+1}$

for each $\gamma + 1 < \alpha$ and $K_\beta \in \mathcal{E}$ for each $\beta < \alpha$. For a limit ordinal α we simply set $K_\alpha = \bigcup_{\beta<\alpha} K_\beta$, and we have $K_\alpha \in \mathcal{E}$ by the hypothesis and Proposition 7(i). If $\alpha = \beta + 1$ is a successor ordinal then, similarly to the beginning, we take an $\mathcal{E}$-cover $\varphi_\beta : G_\beta \to F/K_\beta$ and we denote $\varphi_\beta(G_\beta) = \frac{K_{\beta+1}}{K_\beta} = \frac{K_\alpha}{K_\beta}$. Then $K_\beta \subset K_\alpha$ and $K_\alpha/K_\beta \in \mathcal{E}$, which together with Lemma 1 yields that $K_\alpha \in \mathcal{E}$. Thus $F = \bigcup_{\alpha<\lambda} K_\alpha$ for some ordinal λ and $F \in \mathcal{E}$ by Proposition 7(i), again. □

9. Theorem. *Let $\sigma = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category R-mod such that $Q_\sigma(R) \in \mathcal{E}$. The following conditions are equivalent:*

(i) *σ is of finite type;*
(ii) *every module has a σ-torsionfree cover;*
(iii) *every module has an $\mathcal{E}$-cover.*

Proof: (ii) follows from (i) by [12; Theorem] in the faithful case and by [5; Corollary 4.1] in the general case. Further, (iii) follows from (ii) by Theorem 5 and we are going to show that (i) follows from (iii). So, let $J = \sigma(R)$, $I \in \mathcal{L}$ be arbitrary and $I = \bigcup_{\lambda\in\Lambda} I_\lambda$ be a directed union of a finitely generated left ideal of R. For each $\lambda \in \Lambda$ we take the σ-closure K_λ of $\frac{I_\lambda+J}{J}$ in $E_\sigma(R/J) = Q_\sigma(R)$. It is easy to see that for $I_\lambda \subseteq I_\mu$ we have $K_\lambda \subseteq K_\mu$ and so, taking the directed union $K = \bigcup_{\lambda\in\Lambda} K_\lambda$ we get that $\frac{I+J}{J} \subseteq K$. However, from the isomorphism $(R/J)/((I+J)/J) \cong R/(I+J)$ we infer that $(I+J)/J$ is σ-dense in K, $R/(I+J)$ being σ-torsion as the homomorphic image of R/I. Consequently, $(I + J)/J$ is σ-dense in $E_\sigma(R/J)$. On the other hand, K_λ is $\mathcal{E}$-pure in $E_\sigma(R/J)$ for each $\lambda \in \Lambda$ by the hypothesis and so Proposition 7(ii) yields that K is $\mathcal{E}$-pure in $E_\sigma(R/J)$. Thus $K = E_\sigma(R/J)$ and hence there is an index $\lambda \in \Lambda$ such that $1 + J \in K_\lambda$, which means that $R/J \subseteq K_\lambda$ and so $\frac{I_\lambda+J}{J}$ is σ-dense in R/J. Finally, considering the exact sequence $0 \to \frac{I_\lambda+J}{I_\lambda} \to \frac{R}{I_\lambda} \to \frac{R}{I_\lambda+J} \to 0$, we see that $R/(I_\lambda + J) \in \mathcal{T}$ by the above part. Since $\frac{I_\lambda+J}{I_\lambda} \cong \frac{J}{J\cap I_\lambda} \in \mathcal{T}$ obviously, we have $I_\lambda \in \mathcal{L}$, as we had to show. □

10. Remarks. We conclude by some simple applications of the above theory and we also present some examples motivating this theory, as promised at the beginning.

It follows easily from Corollary 6 that if $\sigma = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory of finite type for the category R-mod over a left semihereditary ring R, then the classes $\mathcal{J}$ and $\mathcal{E}$ coincide and, consequently, every module has a σ-torsionfree σ-injective cover. Further, as a consequence of Theorem 9 one can obtain that an exact torsion theory $\sigma = (\mathcal{T}, \mathcal{F})$ is perfect if and only if every module has an $\mathcal{J}$-cover. Since $\mathcal{J} = \mathcal{E}$ in this case, σ is perfect if and only if every module has an $\mathcal{E}$-cover.

If σ is the ordinary torsion theory on the category Ab of all abelian groups, then $E_\sigma(Z) = Q_\sigma(Z) = Q$ is σ-cocritical and, consequently, it belongs to the class

$\mathcal{E}$ trivially. In this case $\mathcal{J} = \mathcal{E}$ is the class of all torsionfree divisible groups and so every module has an $\mathcal{E}$-cover by Theorem 9, σ being obviously of finite type.

Further, if σ is any hereditary torsion theory for the category Ab of all abelian groups, then it is well-known that σ is the P-torsion for a suitable subset P of the set of all primes Π. Then $E_\sigma(Z)$ consist of all rationals with denominators divisible by the primes from P only, and it is an easy exercise to verify that any σ-torsionfree homomorphic image of $E_\sigma(Z)$ is either $E_\sigma(Z)$ itself, or it is a finite groups of order relatively prime to each $p \in P$. Such groups are obviously σ-injective and so $E_\sigma(Z)$ belongs to the class $\mathcal{E}$.

Finally, let R be an arbitrary commutative domain and σ be the ordinary torsion theory on the category R-mod. By [9; Corollary 44.3] σ is exact as the Goldie's torsion theory. Since σ is obviously of finite type, it is perfect and, consequently, every module has a σ-torsionfree injective cover in view of the simple fact that for each $F \in \mathcal{F}$ the factor module $E(F)/F$ is σ-torsion and, consequently, the class $\mathcal{J}$ consists just of σ-torsionfree injective modules.

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