A generalization of the Schauder fixed point theorem via multivalued contractions

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Abstract. We establish a fixed point theorem for a continuous function $f : X \to E$, where E is a Banach space and $X \subseteq E$. Our result, which involves multivalued contractions, contains the classical Schauder fixed point theorem as a special case. An application is presented.

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1. Introduction

The aim of this short note is to point out the following result.

Theorem 1. *Let E be a Banach space, X a nonempty closed convex subset of E, $f : X \to E$ a continuous function, $G : X \times X \to 2^E$ a multifunction with non-empty values. Moreover, assume that:*

(i) $f(f(X) \cap X) \subseteq X$;
(ii) $f(f^{-1}(X))$ *is relatively compact;*
(iii) *for every $x \in X$, one has $G(x, x) = \{0_E\}$ and the multifunction $G(x, \cdot\,)$ is upper semicontinuous and with convex graph;*
(iv) *the multifunction $F_G : X \to 2^X$ defined by putting*

$$F_G(x) = \Big\{ y \in X : \big[G(x, y) + f(x)\big] \cap X \neq \emptyset \Big\}$$

for all $x \in X$, is a multivalued contraction.

Then, f admits at least one fixed point.

The proof of Theorem 1 will be given in Section 2. When X is compact and $f(X) \subseteq X$, each assumption of Theorem 1 is satisfied. In particular, it suffices to take $G(x, y) \equiv \{0_E\}$. Hence, the classical Schauder fixed point theorem is a particular case of Theorem 1.

As an application of Theorem 1, in Section 2 we shall also prove the following result.

Theorem 2. *Let E be a Banach space, and let $X = B(x_0, R)$ be the closed ball centered at $x_0 \in E$ with radius $R > 0$. Let $f : X \to E$ be a continuous function satisfying conditions* (i) *and* (ii) *of Theorem* 1*. Moreover, assume that:*

(iii)′ $\alpha := \sup_{x\in X} \|x - f(x)\| < 2R$;

(iv)′ *the function $x \in X \to x - f(x)$ is a contraction with constant $L < \psi(\alpha)$, where*

$$\psi(t) := \begin{cases} \frac{1}{2} & \text{if } t \in [0, R] \\ 1 - \frac{t}{2R} & \text{if } t \in]R, 2R[\,. \end{cases}$$

Then, f admits at least one fixed point.

2. The proofs

This section is devoted to the proofs of Theorems 1 and 2. For the basic facts and definitions about multifunctions, we refer to [1], [4].

Proof of Theorem 1: If we put $\mathrm{Fix}(F_G) := \{x \in X : x \in F_G(x)\}$, by (iii) we have

$$\mathrm{Fix}(F_G) = f^{-1}(X).$$

On the other hand, for every $x \in X$ we have

$$F_G(x) = \Big\{y \in X : G(x, y) \cap (X - f(x)) \neq \emptyset\Big\}.$$

Hence, by (iii), it follows that the set $F_G(x)$ is closed and convex. Consequently, by (iv) and Theorem 1 of [5], the set $\mathrm{Fix}(F_G)$, endowed with the relative norm topology, is a non-empty absolute extensor for paracompact spaces. Hence, in particular, it is an absolute retract (see [2, p. 92]). On the other hand, (i) is equivalent to the fact that $f(f^{-1}(X)) \subseteq f^{-1}(X)$. At this point, our conclusion follows from Theorem 10.8 at page 94 of [2]. □

If A and D are nonempty subsets of the Banach space E and $x \in E$, we put

$$d(x, D) := \inf_{v\in D} \|x - v\|, \quad d^*(A, D) := \sup_{u\in A} d(u, D).$$

Moreover, we denote by $d_H(A, D)$ the Hausdorff distance between A and D, namely we put

$$d_H(A, D) := \max\{d^*(A, D), d^*(D, A)\}.$$

Proof of Theorem 2: We want to apply Theorem 1 by taking $G(x, y) = \{y-x\}$. Of course, condition (iii) of Theorem 1 is satisfied. We now prove that assumption (iv) is also satisfied. To this aim, we first observe that for each $x \in X$ one has

$$F_G(x) = X \cap B(x - f(x) + x_0, R). \tag{1}$$

Now we claim that, for each $z \in X$ and each $v \in E$, with $\|v\| < 2R$, one has

(2) $$d(z, X \cap B(v + x_0, R)) \le \psi(\|v\|)^{-1}\, d(z, B(v + x_0, R)).$$

To prove (2), fix z and v as above. We distinguish two cases.

(a) $\|v\| \le R$. Since $x_0 \in X \cap B(v + x_0, R)$, we have

$$d^*(X \cap B(v + x_0, R), E \setminus X) \ge d(x_0, E \setminus X) = R.$$

Consequently, by Lemma 1 of [3] we have that the inequality

$$d(z, X \cap B(v + x_0, R)) \le \frac{2R}{\rho}\, d(z, B(v + x_0, R))$$

holds for all $\rho \in]0, R[$. Of course, this implies

$$d(z, X \cap B(v + x_0, R)) \le 2\, d(z, B(v + x_0, R)),$$

as desired.

(b) $\|v\| > R$. Since

$$u := x_0 + v\Big(1 - \frac{R}{\|v\|}\Big) \in X \cap B(v + x_0, R)$$

and $B(u, 2R - \|v\|) \subseteq X$, we get

$$d^*(X \cap B(v + x_0, R), E \setminus X) \ge d(u, E \setminus X) \ge 2R - \|v\|.$$

Again by Lemma 1 of [3], the inequality

$$d(z, X \cap B(v + x_0, R)) \le \frac{2R}{\rho}\, d(z, B(v + x_0, R))$$

holds for all $\rho \in]0, 2R - \|v\|[$. This implies

$$d(z, X \cap B(v + x_0, R)) \le \frac{2R}{2R - \|v\|}\, d(z, B(v + x_0, R)),$$

as desired. Hence, (2) holds.

At this point, fix $x, y \in X$. By (1) and (2) we have

$$\begin{aligned} d^*(F_G(y), F_G(x)) &\le \psi(\|x - f(x)\|)^{-1}\, d^*(F_G(y), B(x - f(x) + x_0, R)) \\ &\le \psi(\|x - f(x)\|)^{-1}\, d_H(B(y - f(y) + x_0, R), B(x - f(x) + x_0, R)). \end{aligned}$$

Since ψ decreases in $[0, \alpha]$, by assumption (iv)$'$ and by the previous inequality we get

$$d_H(F_G(y), F_G(x)) \le \frac{L}{\psi(\alpha)}\, \|x - y\|,$$

hence F_G is a multivalued contraction. By Theorem 1 our claim follows. $\square$

Remark. When E is an Hilbert space, the more precise estimation given in Lemma 1 of [3] allows us to take the function $\psi(t)$ in the statement of Theorem 2 in the following better way:

$$\psi(t) := \begin{cases} 1 & \text{if } \ t \in [0, R] \\ \frac{2R}{t} - 1 & \text{if } \ t \in]R, 2R[\,. \end{cases}$$

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