Equation with residuated functions

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Abstract. The structure of solution-sets for the equation $F(x) = G(y)$ is discussed, where F, G are given residuated functions mapping between partially-ordered sets. An algorithm is proposed which produces a solution in the event of finite termination: this solution is maximal relative to initial trial values of x, y. Properties are defined which are sufficient for finite termination. The particular case of max-based linear algebra is discussed, with application to the synchronisation problem for discrete-event systems; here, if data are rational, finite termination is assured. Numerical examples are given. For more general residuated real functions, lower semicontinuity is sufficient for convergence to a solution, if one exists.

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1. Motivation and overview

Various non-classical algebraic structures, such as (max, +) and (max, min) are used in the theory and modelling of discrete dynamic processes ([1], [5]). These structures are semirings in which the role of addition is played by the binary operator max and that of multiplication is played as appropriate by the binary operator +, or min, etc. The elements of the algebra are drawn from the real numbers $\mathbb{R}$ or one its substructures. In this context, the equation

$$F(x) = G(y), \tag{1.1}$$

may be seen as concerned with the question of *synchronisation*: whether, and how, two disparate systems can arrive at the same state.

Little is known structurally about solution-sets for (1.1), even though the theory of linear and rational algebra over these algebraic structures is now well-developed. However, there has been some success in devising algorithms for finding particular solutions when e.g. F, G are linear mappings over (max, +) ([3], [6]).

The effectiveness of one such algorithm — the Alternating Algorithm — depends crucially on the fact that these mappings are *residuated* ([2], [3]). This

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raises the question whether this algorithm can be generalised to other residuated structures, and in particular to (max, min). We now present such a generalisation, in a self-contained treatment independent of [3].

After a discussion of residuation in Section 2, the structure of solution sets for (1.1) when F, G are residuated is studied in Section 3. The generalised algorithm is introduced in Section 4 and the consequences of finite termination are considered. In Section 5, two particular properties — *discreteness* and *pervasiveness* are shown to be sufficient for finite termination.

Max-based linear algebras, covering in particular the important practical cases of $(\max, +)$ and $(\max, \min)$, are defined in Section 6. These structures are residuated relative to the natural partial order of real n-tuples and have the property of pervasiveness. Discreteness also holds in the case of integer data and may be extended to problems with rational data. It follows that finite termination occurs for most practical use of the algorithm with these structures. Section 7 gives numerical illustrations.

For more general functions F and G, residuated relative to the natural partial order of real n-tuples, finite termination may not occur. In Section 8, we show that lower-semicontinuity of F and G is sufficient to ensure convergence to a solution whenever one exists below the starting-point of the algorithm.

Assumptions are stated formally as needed, and then remain binding for the remainder of the paper. For brevity, we shall assume for the most part that elements introduced in the argument belong to sets, and have dimensions, which are obvious from the context and need not always be stated explicitly.

2. Relevant results from residuation theory

Assume given partially-ordered sets $(X, \leq)$, $(Z, \leq)$. The notation $x < u$ will mean $x \leq u$ but $x \neq u$.

Definition 2.1. A function $f : X \mapsto Z$ is *residuated* if there exists a function $f^* : Z \mapsto X$ such that

(i) f and f^* are isotone relative to the given partial orders;
(ii) $f \circ f^* \leq i_Z$, where i_Z is the identity operator on Z;
(iii) $i_X \leq f^* \circ f$, where i_X is the identity operator on X.

Function f^* is called the *residual* of f and is unique.

For a full account of residuation theory, see [2]. A standard and easily proved property of residuation is its ability to invert inequalities:

Proposition 2.1. $f(x) \leq a$ iff $x \leq f^*(a)$. □

Definition 2.2. A set $X = (X, \wedge)$ is an (*inf-*)*semilattice* if

(i) X is closed under the commutative, associative binary operation $\wedge$;
(ii) $x \wedge x = x$.

Given $x_1, \ldots, x_k$, the notation $\wedge_j x_j$ will denote $x_1 \wedge \cdots \wedge x_k$.

It is well-known that the relation $\leq$ defined by $x \leq y$ iff $x = x \wedge y$ defines a partial order. The binary operation $\wedge$ then defines the infimum of two elements with respect to this partial order:

Proposition 2.2. *($w \leq x$ AND $w \leq y$) iff $w \leq x \wedge y$.* □

Furthermore, the binary operation *wedge* is isotone with respect to this partial order:

Proposition 2.3. *$(x, y) \leq (w, z) \Rightarrow x \wedge y \leq w \wedge z$.* □

Assumption 1. All partial orders will be assumed to be derivable from a semilattice structure.

We remark that Assumption 1 is certainly true for the natural partial order of n-tuples over $\mathbb{R}$.

3. The solution set $M(u, v)$

Given partially-ordered sets $(X, \leq)$, $(Y, \leq)$, $(Z, \leq)$ and functions $F : X \mapsto Z$, $G : Y \mapsto Z$, our aim is to construct an algorithm which will find a pair $(x, y) \in X \times Y$ satisfying (1.1). Such a pair will be called simply a *solution* and the set of all solutions will be denoted by M.

Recall that an (*order*) *ideal* is a set I such that $u \in I \Rightarrow x \in I$ if $x \leq u$. The *principal ideal* $N(u)$ is the set $\{x : x \leq u\}$. A set S will be said to have a *maximum element* u if $u \in S$ AND $(x \in S \Rightarrow x \leq u)$. Clearly, any principal ideal is such a set.

Because the algorithm introduced in Section 4 constructs monotonically non-increasing sequences of trial solutions and functions values, and proceeds from an initial trial solution (u, v), the computations in effect take place within certain ideals $X^0 \subset X$, $Y^0 \subset Y$, $Z^0 \subset Z$ such that $u \in X^0$, $v \in Y^0$, and $F(u), G(v) \in Z^0$. The precise interpretation of this notation depends upon the details of particular applications, as illustrated later.

Assumption 2. The restrictions $F : X^0 \mapsto Z^0$, $G : Y^0 \mapsto Z^0$ are residuated with residuals $F^* : Z^0 \mapsto X^0$, $G^* : Z^0 \mapsto Y^0$.

We define the mappings:

$$\pi : y \mapsto (F^* \circ G)(y); \qquad \psi : x \mapsto (G^* \circ F)(x). \tag{3.1}$$

These are compositions of isotone mappings and therefore isotone.

Lemma 3.1. *If* $(x, y) \in X^0 \times Y^0$, *then* $(x, y) \in M$ *iff*

$$(x, y) = (\pi(y) \wedge x, \psi(x) \wedge y). \tag{3.2}$$

Proof: (3.2) is equivalent to $(x, y) \leq (\pi(y), \psi(x))$, which from (3.1) and Proposition 2.1 is equivalent to

$$(F(x), G(y)) \leq (G(y), F(x)),$$

which is equivalent to $F(x) = G(y)$. □

Lemma 3.2. *If* $(x, y) \in X^0 \times Y^0$, *then* $(x, y) \notin M$ *iff*

$$(\pi(y) \wedge x, \psi(x) \wedge y) < (x, y). \tag{3.3}$$

Proof: Trivially,

$$(\pi(y) \wedge x, \psi(x) \wedge y) \leq (x, y) \quad \text{for all} \quad (x, y), \tag{3.4}$$

so the result follows from Lemma 3.1. □

For given $(u, v) \in X^0 \times Y^0$, define:

$$N(u, v) = N(u) \times N(v), \tag{3.5}$$
$$M(u, v) = N(u, v) \cap M. \tag{3.6}$$

In other words, $M(u, v)$ is the set of all solutions $(x, y) \leq (u, v)$.

The following theorem underlies the algorithm presented below.

Theorem 3.1. $M(\pi(v) \wedge u, \psi(u) \wedge v) = M(u, v)$ *for all* $(u, v) \in X^0 \times Y^0$.

Proof: If $(x, y) \in M$ with $(x, y) \leq (u, v)$, then by Lemma 3.1 and isotonicity of π, ψ and $\wedge$,

$$(x, y) = (\pi(y) \wedge x, \psi(x) \wedge y) \leq (\pi(v) \wedge u, \psi(u) \wedge v).$$

Hence $M(u, v) \subset M(\pi(v) \wedge u, \psi(u) \wedge v)$. The converse follows trivially from (3.4). □

4. The basic algorithm

We take an initial trial solution $(u, v) \in X^0 \times Y^0$. Setting

$$(x(0), y(0)) = (u, v); \qquad k = 0,$$

the basic algorithm step is:

REPEAT

(4.1) $k := k + 1;\ \ (x(k), y(k)) := (\pi(y(k-1)) \wedge x(k-1), \psi(x(k-1)) \wedge y(k-1))$

UNTIL *Termination Criterion*

From (4.1) and Theorem 3.1, we have:

Lemma 4.1. *The sets $M(x(k), y(k))$, $k = 0, 1, \ldots$ are all identical, so every $(x(k), y(k))$ upper-bounds $M(u, v) = M(x(0), y(0))$.* □

Relations (3.4), (4.1) show that the sequence $\{(x(k), y(k))\}$ is monotonically non-increasing and all terms of the sequence therefore lie in $N(u, v)$. Generally speaking, if the sets X, Y, Z are continua, some kind of monotone convergence property is needed to take the argument further, as in Section 8 below. However, the sequence may *terminate finitely (at iteration r)* in the sense that *Termination Criterion* holds for one or more values of k, and r is the least such value. Consider:

Termination Criterion $TC1$: $(x(k), y(k))$ is a solution.

Termination Criterion $TC2$: $(x(k), y(k)) = (x(k-1), y(k-1))$.

Theorem 4.1. *If the sequence terminates finitely (at iteration r) under $TC1$ or $TC2$, then $(x(r), y(r)) \in M(u, v)$; in fact $(x(r), y(r))$ is the greatest element of $M(u, v)$.*

Proof: It is clear, using Lemma 3.1 and the algorithm definition, that $(x(r), y(r))$ is a solution. Now, all terms of the sequence lie in $N(u, v)$, so $(x(r), y(r)) \in M(u, v)$. Since every $(x(k), y(k))$ upper-bounds $M(u, v)$, so does $(x(r), y(r))$. □

5. Discreteness and pervasiveness

In this section, we consider the effect of two properties — *discreteness* and *pervasiveness* — on the behaviour of the algorithm. These conditions both hold in a range of important applications, as we discuss in Section 6 below.

For every $(s, t), (u, v) \in X^0 \times Y^0$ with $(s, t) \leq (u, v)$, define the *interval*

$$\Delta\left[(s, t), (u, v)\right] = \{(x, y) : (s, t) \leq (x, y) \leq (u, v)\}.$$

Definition 5.1. $X^0 \times Y^0$ has the *discreteness property* if every interval is of finite cardinality.

Theorem 5.1. *If $X^0 \times Y^0$ has the discreteness property, then the sequence $\{(x(k), y(k))\}$ terminates finitely (at some iteration r) under $TC1$ or $TC2$ iff $M(u,v)$ is non-empty; and then $(x(r), y(r))$ is the greatest element of $M(u,v)$.*

PROOF: Suppose $M(u,v)$ is non-empty — say $(s,t) \in M(u,v)$. Every $(x(k), y(k))$ upper-bounds $M(u,v)$, so every $(x(k), y(k))$ lies in the interval $\Delta[(s,t),(u,v)]$. Hence the sequence can have only a finite number of distinct terms. If finite termination does not occur, some term recurs, say $(x(r), y(r)) = (x(j), y(j))$ with $r > j$. Since the sequence is non-increasing,

$$(x(r), y(r)) \le (x(r-1), y(r-1)) \le \cdots \le (x(j), y(j)) = (x(r), y(r)).$$

So all these terms must be equal: $(x(r), y(r)) = (x(r-1), y(r-1))$. Thus $TC2$ (and $TC1$) will be satisfied and finite termination *must* occur. Conversely, by Theorem 4.1, finite termination implies that $M(u,v)$ is non-empty and $(x(r), y(r))$ is the greatest element of $M(u,v)$. □

Since we may elect to start the algorithm at any given pair (u,v), Theorems 4.1 and 5.1 together imply the following structural result.

Corollary 5.1. *If $X^0 \times Y^0$ has the discreteness property, then every non-empty $M(u,v)$ has a greatest element.*

In general, there may be nothing to guide this initial choice to ensure that $M(u,v)$ is non-empty, but this ceases to be a problem if the outcome is essentially independent of the choice, or if the choice is dictated by other considerations. Addressing the first case:

Definition 5.2. The solution set M is *pervasive* if $M \neq \emptyset \Rightarrow M(u,v) \neq \emptyset$ $\forall\,(u,v) \in X^0 \times Y^0$.

Corollary 5.2. *If $X^0 \times Y^0$ has the discreteness property, and M is non-empty and pervasive, the sequence $\{(x(k), y(k))\}$ terminates finitely under $TC1$ or $TC2$ for any (u,v).* □

Addressing the second case, it may be given or known *ab initio* that a solution lies in a certain *interval of constraint* $\Delta[(s,t),(u,v)]$, or alternatively that only a solution in that interval is acceptable, if one exists. We refer then to the *constrained case*.

Termination Criterion $TC3$: $(x(k), y(k)) \in M$ OR $(x(k), y(k)) \notin \Delta[(s,t),(u,v)]$.

Theorem 5.2. *In the constrained case, if $X^0 \times Y^0$ has the discreteness property, then the sequence $\{(x(k), y(k))\}$ terminates finitely (at some iteration r) under $TC3$. Furthermore:*

(i) *if $(x(r), y(r)) \in \Delta[(s,t),(u,v)]$, then $(x(r), y(r))$ is the greatest element of $M(u,v)$;*
(ii) *otherwise there is no solution in $\Delta[(s,t),(u,v)]$.*

Proof: Argue as for Theorem 5.1: if $\Delta[(s,t),(u,v)]$ contains a solution, the terms of the algorithm are confined to (a subinterval of) $\Delta[(s,t),(u,v)]$; conversely if the algorithm produces only terms in $\Delta[(s,t),(u,v)]$, then for some r, $(x(r), y(r))$ is the greatest element of $M(u,v)$ and $TC3$ is satisfied.

The only remaining possibility is that $\Delta[(s,t),(u,v)]$ contains no solution and for some k, $(x(k), y(k)) \notin \Delta[(s,t),(u,v)]$. $\square$

6. Max-based linear algebras

We focus now on certain structures based on the real numbers.

Assumption 3. $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$, $Z \subseteq \mathbb{R}^p$, and $\leq$ denotes the natural order of real numbers, extended to the natural partial order of real tuples.

Lemma 6.1. *If $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ is isotone, then*

$$\varphi(x, \max(y,z)) = \max(\varphi(x,y), \varphi(x,z)). \tag{6.1}$$

Proof: From isotonicity, $\varphi(x,y) \leq \varphi(x, \max(y,z))$ and $\varphi(x,z) \leq \varphi(x, \max(y,z))$. However, $\max(y,z)$ must equal one of y, z so $\varphi(x, \max(y,z))$ must equal one of $\varphi(x,y)$, $\varphi(x,z)$. $\square$

If we now write $y \oplus z$ for $\max(y,z)$ and $x \otimes w$ for $\varphi(x,w)$, (6.1) assumes the form

$$x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z, \tag{6.2}$$

showing the 'left multiplication' $x\otimes$ to be distributive over the 'addition' $\oplus$. Now, $\oplus$ is commutative and associative, so the basic structure for linear algebra is present provided

$$x \otimes (y \otimes z) = (x \otimes y) \otimes z. \tag{6.3}$$

Of several known examples, perhaps the two of greatest practical importance are given by $\varphi(x,y) = x + y$ and $\varphi(x,y) = x \wedge y$, giving rise to $(\max, +)$ algebra and $(\max, \min)$ algebra respectively ([5], [7]).

Definition 6.1. According to context, the symbol $\Im$ will represent the real numbers $\mathbb{R}$, the rational numbers $\Re$, the integers $\mathbb{I}$, or appropriate subrings, ideals or other substructures thereof.

Lemma 6.2. *If $\Im$ is $\mathbb{R}$, $\Re$ or $\mathbb{I}$, then for fixed parameter $a \in \Im$, the mapping $m_a : x \mapsto a \otimes x$ in $(\Im, \max, +)$ algebra is residuated.*

PROOF: Since $m_a(x) = a + x$, the isotone mapping $m_a^* : y \mapsto -a + y$ is the inverse and therefore the residual of m_a. □

Lemma 6.3. *If $\Im$ is a principal ideal of $\mathbb{R}$, $\Re$ or $\mathbb{I}$, then for fixed parameter $a \in \Im$, the mapping $m_a : x \mapsto a \otimes x$ in $(\Im, \max, \min)$ algebra is residuated.*

PROOF: Here $m_a(x) = a \wedge x$. Let u be the maximum element of $\Im$. The residual of m_a is m_a^* where $m_a^*(y) = u$ if $a \leq y$; $m_a^*(y) = y$ otherwise. This is easily verified and is shown (for $u = 1$) in e.g [4]. □

In these algebraic structures, we may write linear functions. For example,

$$3 \otimes x_1 \oplus 1 \otimes x_2 \oplus 4 \otimes x_3 \tag{6.4}$$

denotes $\max(3+x_1, 1+x_2, 4+x_3)$ over $(\max, +)$, and denotes $\max(3 \wedge x_1, 1 \wedge x_2, 4 \wedge x_3)$ over $(\max, \min)$. A set of such linear functions is represented by a matrix A of coefficients inducing a linear transformation $F : x \mapsto A \otimes x$ of real tuples in the obvious way. See Section 7 for examples, and [1], [5], [7] for further background and application to discrete dynamic processes, with various interpretations of $\Im$.

In the present section, therefore, F, G will represent such linear transformations, implemented by $(p \times m), (p \times n)$ matrices A, B respectively, and

$$X = \Im^m \subseteq \mathbb{R}^m \quad Y = \Im^n \subseteq \mathbb{R}^n \quad Z = \Im^p \subseteq \mathbb{R}^p.$$

That such F, G are residuated follows from:

Theorem 6.1. *Let $m_{ij} : \Im \mapsto \Im$ $(i = 1, \dots, p,\ j = 1, \dots, m)$ be given residuated functions and let*

$$F : \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \mapsto \begin{bmatrix} \max(m_{11}(x_1), \dots, m_{1m}(x_m)) \\ \vdots \\ \max(m_{p1}(x_1), \dots, m_{pm}(x_m)) \end{bmatrix}. \tag{6.5}$$

F is residuated with residual

$$F^* : \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \mapsto \begin{bmatrix} m_{11}^*(y_1) \wedge \cdots \wedge m_{p1}^*(y_p) \\ \vdots \\ m_{1m}^*(y_1) \wedge \cdots \wedge m_{pm}^*(y_p) \end{bmatrix}. \tag{6.6}$$

PROOF: F^* is a composition of isotone functions and hence isotone. Using isotonicity,

$$m_{ij}\left(\wedge_h m^*_{hj}(y_h)\right) \le m_{ij} \circ m^*_{ij}(y_i) \le y_i \qquad (\forall\, i, j)$$

whence

$$\max_j \left(m_{ij}\left(\wedge_h m^*_{hj}(y_h)\right)\right) \le y_i \qquad (\forall\, i)$$

which is $F \circ F^*(y) \le y$. Similarly, $x \le F^* \circ F(x)$. □

Remark 6.1. From the proofs of Lemma 6.2 and Theorem 6.1, it is evident that for $(\max, +)$ algebra, the residual of $F : x \mapsto A \otimes x$ can be implemented as a linear transformation $F^* : y \mapsto A^* \otimes' y$ over $(\otimes', \oplus)$, i.e. $(\min, +)$, algebra, where A^* is the negated transposed of A. (See [5] for further background on this duality.) We make use of this in Section 7.

Theorem 6.2. *Pervasiveness holds in linear algebra over* $(\mathbb{R}, \max, +)$ *and* $(\mathbb{R}, \max, \min)$.

PROOF: Scalar multiplication $a \otimes x$ commutes with the linear operators F, G. Hence, if (x, y) is a solution, so is $(a \otimes x, a \otimes y)$ and in both $(\max, +)$ and $(\max, \min)$ algebra, we have only to choose a sufficiently small to ensure that $(a \otimes x, a \otimes y) \le (u, v)$. □

Evidently, moreover

Lemma 6.4. *If* $\Im \subseteq \mathbb{I}$, $X^0 \times Y^0$ *has the discreteness property.* □

We infer our principal result for these algebraic structures, with application to the synchronisation problem discussed in Section 1.

Theorem 6.3. *For linear algebra over* $(\Re, \max, +)$ *and* $(\Re, \max, \min)$, *finite termination of the algorithm at a solution is guaranteed under* $TC1$ *or* $TC2$ *for any choice of* (u, v) *provided only that a solution exists.*

PROOF: By Corollary 5.2, the preceding two results establish the conclusion for linear algebra over $(\mathbb{I}, \max, +)$ and $(\mathbb{I}, \max, \min)$. We can extend this to linear algebra over $(\Re, \max, +)$ and $(\Re, \max, \min)$, because we can regard the arithmetic as set in the domain of integer multiples of δ^{-1} where δ is the least common multiple of the denominators of all matrix elements and elements of $x(0)$, $y(0)$. □

Theorem 6.3 applies widely, because in numerical work, it is rational numbers which are used in practice, since infinite precision is not usually available.

7. Numerical examples

In the following three examples, we take $x(0) = (10, 10, 10)^T$; $y(0) = (10, 10, 10)^T$,

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 3 & 7 & 8 \\ 0 & 8 & 0 \end{bmatrix}; \qquad B = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 6 & 7 \\ 1 & 0 & 7 \end{bmatrix},$$

and thus

$$A^* = \begin{bmatrix} -4 & -3 & 0 \\ -2 & -7 & -8 \\ -6 & -8 & 0 \end{bmatrix}; \qquad B^* = \begin{bmatrix} -3 & -2 & -1 \\ -1 & -6 & 0 \\ -5 & -7 & -7 \end{bmatrix}.$$

In the case of $(\max, +)$, we work in $\mathbb{I}$ or $\Re$; in the case of $(\max, \min)$, we work in the principal ideal $N(20)$ within $\mathbb{I}$.

7.1 (Max, +) algebra

$$F(x(0)) = A \otimes x(0) = (16, 18, 18)^T; \qquad G(y(0)) = B \otimes y(0) = (15, 17, 17)^T.$$

Hence, in the light of Remark 6.1,

$$\begin{aligned} x(1) &= (F^* \circ G)(y(0)) \wedge x(0) = A^* \otimes' (15, 17, 17)^T \wedge x(0) \\ &= (11, 9, 9)^T \wedge (10, 10, 10)^T = (10, 9, 9)^T; \\ y(1) &= (G^* \circ F)(x(0)) \wedge y(0) = B^* \otimes' (16, 18, 18)^T \wedge y(0) \\ &= (13, 12, 11)^T \wedge (10, 10, 10)^T = (10, 10, 10)^T. \end{aligned}$$

Now $F(x(1)) = A \otimes x(1) = (15, 17, 17)^T = B \otimes y(1) = G(y(1))$ and $TC1$ is satisfied.

(Max, Min) algebra

$$F(x(0)) = A \otimes x(0) = (6, 8, 8)^T; \qquad G(y(0)) = B \otimes y(0) = (5, 7, 7)^T.$$

Hence

$$\begin{aligned} x(1) &= (F^* \circ G)(y(0)) \wedge x(0) = (20, 7, 5)^T \wedge (10, 10, 10)^T = (10, 7, 5)^T; \\ y(1) &= (G^* \circ F)(x(0)) \wedge y(0) = (20, 20, 20)^T \wedge (10, 10, 10)^T = (10, 10, 10)^T. \end{aligned}$$

Now $F(x(1)) = A \otimes x(1) = (5, 7, 7)^T = B \otimes y(1) = G(y(1))$ and $TC1$ is satisfied.

A mixed example

To illustrate the fact that there is no assumption that the algebraic structures underlying F and G are the same, we conclude by taking F, G over $(\max, +)$, $(\max, \min)$ respectively.

As before,

$$F(x(0)) = (16, 18, 18)^T; \qquad G(y(0)) = (5, 7, 7)^T.$$

Hence

$$x(1) = (F^* \circ G)(y(0)) \wedge x(0) = (1, -1, -1)^T \wedge (10, 10, 10)^T = (1, -1, -1)^T;$$
$$y(1) = (G^* \circ F)(x(0)) \wedge y(0) = (20, 20, 20)^T \wedge (10, 10, 10)^T = (10, 10, 10)^T.$$

Now $F(x(1)) = A \otimes x(1) = (5, 7, 7)^T = B \otimes y(1) = G(y(1))$ and $TC1$ is satisfied.

In the above examples, had we worked with termination criterion $TC2$, one more iteration would have been required. Had we stipulated an interval of constraint, say

$$\Delta = ((5, 5, 5)^T, (10, 10, 10)^T),$$

then termination criterion $TC3$ would have indicated that no solution was possible.

8. Convergence

We turn now to the general case of a pair of functions F, G satisfying Assumptions 2 and 3, i.e. any pair of residuated finitary real functions. When discreteness is lacking, finite termination can still occur fortuitously, and then the results of earlier sections apply in the obvious way. Otherwise, an infinite non-increasing sequence of elements of $N(u, v)$ is generated and convergence will occur if the sequence is lower-bounded. Since ideals are now closed sets, it is clear that the limit also lies in $N(u, v)$.

To ensure that such sequences will converge to solutions, we shall need

Assumption 4. F, G are lower-semicontinuous.

Theorem 8.1. *In the absence of finite termination, the sequence $\{(x(k), y(k))\}$ converges iff $M(u, v)$ is non-empty; and then $M(u, v)$ has a greatest element (ξ, η), and $(x(k), y(k)) \downarrow (\xi, \eta)$.*

Proof: Suppose $\{(x(k), y(k))\}$ converges. Let $\lim(x(k), y(k)) = (\alpha, \beta)$. Since $\{(x(k), y(k))\}$ is monotonically non-increasing and F, G are lower-semicontinuous, the function-value sequence $\{(F(x(k)), (G(y(k)))\}$ is also convergent, with limit $(F(\alpha), G(\beta))$. Now, using isotonocity and (ii) of Definition 2.1,

$$F(x(k + 1)) = F(\pi(y(k)) \wedge x(k)) \leq F(\pi(y(k)) = (F \circ F^*)(G(y(k))) \leq G(y(k)).$$

Hence, in the limit, $F(\alpha) \leq G(\beta)$ and similarly $G(\beta) \leq F(\alpha)$. Thus, $F(\alpha) = G(\beta)$, i.e. $(\alpha, \beta) \in M$.

However, since all the sets $M(x(k), y(k))$, $k = 0, 1, \ldots$ are identical, every $(x(k), y(k))$ upper-bounds $M(u, v)$ and therefore in the limit so does (α, β), so $M(u, v)$ has a greatest element $(\xi, \eta) = (\alpha, \beta)$.

Conversely, suppose $M(u, v)$ is non-empty — say $(\alpha, \beta) \in M(u, v)$. Every $(x(k), y(k))$ upper-bounds $M(u, v)$, so (α, β) lower-bounds the monotonically non-increasing sequence $\{(x(k), y(k))\}$, which is therefore convergent.

□

Corollary 8.1. *If M is non-empty and pervasive, then under $TC1$ or $TC2$, for any choice of (u, v), the sequence terminates finitely at, or converges to, a solution.* □

Corollary 8.2. *In the constrained case, under $TC3$, the sequence $\{(x(k), y(k))\}$ either terminates finitely at a solution in the interval of constraint, or converges to a solution in the interval of constraint, or else terminates finitely at $(x(r), y(r)) \notin \Delta[(s, t), (u, v)]$ showing that there is no solution in the interval of constraint.* □

Since we may elect to start the algorithm under e.g. $TC1$ at any given pair (u, v), Theorems 4.1 and 8.1 together imply the following general structural result.

Corollary 8.3. *Every non-empty $M(u, v)$ has a greatest element.* □

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