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Lipschitz-quotients and the Kunen-Martin Theorem

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Abstract. We show that there is a universal control on the Szlenk index of a Lipschitz-quotient of a Banach space with countable Szlenk index. It is in particular the case when two Banach spaces are Lipschitz-homeomorphic. This provides information on the Cantor index of scattered compact sets K and L such that $C(L)$ is a Lipschitz-quotient of $C(K)$ (that is the case in particular when these two spaces are Lipschitz-homeomorphic). The proof requires tools of descriptive set theory.

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In the non-linear classification of Banach spaces, it is an open problem to know whether two separable Lipschitz-homeomorphic Banach spaces are isomorphic. Several partial results appeared recently on the subject. We refer to [10] (especially Chapters 7 and 11) for an up-to-date account of the theory. In Theorem 3.18 of [2], it is shown that the class of Asplund spaces is stable under Lipschitz-quotient (this is false under uniform homeomorphism; see Theorem 1 in [12]). The aim of this paper is to precise this result: we show that there exists a universal control on the Szlenk index of a Lipschitz-quotient of a Banach space X, provided X has a countable Szlenk index. For that, we need to estimate the topological complexity of the relation of Lipschitz-quotient and apply the Kunen-Martin theorem.

1. Analyticity of the relation of Lipschitz quotient

The aim of this section is to prove that the relation of Lipschitz-quotient (see Definitions 3.1 and 3.2 in [2]) is analytic in a sense which will be made precise later. First, let us introduce some notation:

Notation. • E will denote the space $C(2^\omega)$ of all continuous functions on the Cantor set. Let us recall that E is universal for all separable Banach spaces.

- $\mathcal{S}$ will denote the set of all closed subspaces of E. It is shown in Proposition 2.1 of [3] (see also pages 15 and 16) that the restriction of the Effros Borel structure on the closed subsets of E makes it into a standard Borel set.
- If X and Y are two Banach spaces, the fact that Y is a Lipschitz-quotient of X will be written $X \twoheadrightarrow_\ell Y$.

When we say that the relation of Lipschitz-quotient is analytic, we mean that the set $\{(X,Y)\in\mathcal{S}^2;\ X \twoheadrightarrow_\ell Y\}$ is analytic in the standard Borel structure of $\mathcal{S}$ (see Definition 0.4, page 9 in [8]).

We will show the following crucial technical proposition:

Proposition 1. *$\twoheadrightarrow_\ell$ is analytic.*

Let us introduce some more notation:

Notation.
- The sequence of the vectors x_n will be denoted by $\mathbf{x}$.
- When the sequence $\mathbf{x}$ is dense in X, we write $X=\overline{\mathbf{x}}$.
- $\mathbf{x}$ and $\mathbf{y}$ being two sequences of vectors, we will write $\mathbf{x}\twoheadrightarrow_\ell \mathbf{y}$ to mean that there exist two constants L and C in ω such that

$$\forall\, k,l\in\omega,\quad \|y_k-y_l\|\le L\,\|x_k-x_l\|$$

and such that, for any $n,p\in\omega$ and any $r\in\mathbb{Q}_+^*$ such that $\|y_p-y_n\|\le r/C$, there exists a convergent subsequence $\mathbf{x}_\varphi=(x_{\varphi(m)})_{m\in\omega}$ verifying:

$$\mathbf{x}_\varphi\in B_X(x_n,r)^\omega\quad\text{and}\quad \mathbf{y}_\varphi\to y_p.$$

The link between $\twoheadrightarrow_\ell$ for spaces and $\twoheadrightarrow_\ell$ for sequences is given by the following lemma:

Lemma 2. *Let X and Y be two separable Banach spaces. Then $X\twoheadrightarrow_\ell Y$ if and only if there exist two sequences $\mathbf{x}$ and $\mathbf{y}$ such that $X=\overline{\mathbf{x}}$, $Y=\overline{\mathbf{y}}$ and $\mathbf{x}\twoheadrightarrow_\ell\mathbf{y}$.*

Proof: If there exists a L-Lipschitz and C-co-Lipschitz map f from X to Y then, taking any dense sequence $\mathbf{x}$ and defining $\mathbf{y}$ as the image of $\mathbf{x}$ by f, we clearly have

$$\forall\, k,l\in\omega,\quad \|y_k-y_l\|\le L\,\|x_k-x_l\|\,.$$

Moreover, let $n,p\in\omega$ and $r\in\mathbb{Q}_+^*$ be such that $\|y_p-y_n\|\le r/C$. Then, $y_p\in f(B_X(x_n,r))$. Since there is a preimage x of y_p in $B_X(x_n,r)$, there exists a subsequence $\mathbf{x}_\varphi$ of $\mathbf{x}$ in the open ball such that $\mathbf{x}_\varphi\to x$. Then $\mathbf{y}_\varphi\to f(x)=y_p$.

Conversely, let us suppose that $X=\overline{\mathbf{x}}$, $Y=\overline{\mathbf{y}}$ and $\mathbf{x}\twoheadrightarrow_\ell\mathbf{y}$ with constants L and C. We can define $f:X\to Y$ by $f(x_n)=y_n$ for all $n\in\omega$ and f is L-Lipschitz. Moreover f clearly satisfies:

$$(1)\qquad \forall\, n,p\in\omega,\ \forall\, r\in\mathbb{Q}_+^*,\ \|y_p-y_n\|\le\frac{r}{C},\quad \exists\, x\in B_X(x_n,r),\ f(x)=y_p.$$

Let us state and prove some facts:

Fact 1. *For every $x \in X$, $p \in \omega$, $r \in \mathbb{Q}_+^*$ and $C' > C$ such that the inequality $\left\|y_p - f(x)\right\| \le r/C'$ holds, there exists $z \in B_X(x,r)$ such that $f(z) = y_p$.*

Let $\mathbf{x}_\varphi$ be a subsequence of $\mathbf{x}$ converging to x and verifying, for all $n \in \omega$, $\left\|x - x_{\varphi(n)}\right\| \le k$, $k > 0$ being chosen such that $Lk + r/C' \le r/C''$, with $C'' > C$ and $C''/C \in \mathbb{Q}$. Then we have $\left\|y_p - f(x_{\varphi(n)})\right\| \le r/C''$. By (1), there exists $z_n \in B_X(x_{\varphi(n)}, Cr/C'')$ such that $f(z_n) = y_p$. Since $\mathbf{x}_\varphi \to x$, for n large enough, $z_n \in B(x,r)$. Taking $z = z_n$ for such an n gives the result.

Fact 2. *f is surjective.*

Let $y \in Y$ and let $\mathbf{y}_\varphi$ be a subsequence such that $\left\|y - y_{\varphi(n)}\right\| \le 2^{-n-1}/C'$ ($C' > C$), for all $n \in \omega$. Applying Fact 1, one can define by induction a sequence $\mathbf{z}$ such that $z_0 = x_{\varphi(0)}$, $\|z_{k+1} - z_k\| \le 2^{-k}$ and $f(z_k) = y_{\varphi(k)}$ for all $k \in \omega$. The limit z of $\mathbf{z}$ satisfies $f(z) = y$.

Fact 3. *For every $C' > C$, f is C'-co-Lipschitz.*

The proof is similar to the proof of Fact 2 and will be omitted.

Finally, f is a Lipschitz-quotient map from X to Y and $X \twoheadrightarrow_\ell Y$. □

We now give a characterization of the condition $\mathbf{x} \twoheadrightarrow_\ell \mathbf{y}$ which is useful for our purpose. We denote by G the set of all infinite subsets of ω. As a G_δ set of a compact set, it is a Polish space. Let us also define

$$\mathcal{G} = G^{\omega \times \omega \times \mathbb{Q}_+^*}.$$

Lemma 3. *Let $\mathbf{x}$ and $\mathbf{y}$ be two sequences of vectors. The condition $\mathbf{x} \twoheadrightarrow_\ell \mathbf{y}$ is equivalent to the existence of $P \in \mathcal{G}$ such that the conjunction of the following two conditions holds:*

1. *There exists $L \in \omega$ such that*

$$\forall\, k, l \in \omega, \quad \|y_k - y_l\| \le L\, \|x_k - x_l\|.$$

This first condition will be denoted by $L(\mathbf{x}, \mathbf{y})$.

2. *There exists $C \in \omega$ that satisfies: for any $n, p \in \omega$ and $r \in \mathbb{Q}_+^*$ such that $\left\|y_p - y_n\right\| \le r/C$, we have $\|x_m - x_n\| \le r$ for all $m \in P_{n,p,r}$ and*

$$\forall\, q \in \omega,\ \exists\, Q \in 2^{<\omega};\ \forall\, m', m \in P_{n,p,r} \backslash Q, \quad \|x_{m'} - x_m\| + \left\|y_m - y_p\right\| \le 1/q.$$

This second condition will be denoted by $C(\mathbf{x}, \mathbf{y}, P)$.

Proof: It is an easy reformulation of the condition $\mathbf{x} \twoheadrightarrow_\ell \mathbf{y}$: for a given (n,p,r), $P_{n,p,r}$ is the set $\{\varphi(m);\ m \in \omega\}$ where $\mathbf{x}_\varphi$ is the subsequence of the definition of $\mathbf{x} \twoheadrightarrow_\ell \mathbf{y}$. □

Lemma 4. *Let $\mathcal{A}$ be the set*

$$\Big\{(X,Y,\mathbf{x},\mathbf{y},P) \in \mathcal{S}^2 \times (E^\omega)^2 \times \mathcal{G};\ X = \overline{\mathbf{x}},\ Y = \overline{\mathbf{y}},\ L(\mathbf{x},\mathbf{y}),\ C(\mathbf{x},\mathbf{y},P)\Big\}.$$

Then $\mathcal{A}$ is a Borel set.

Proof: It is enough to see that the sets

$$\mathcal{B} = \{(X,\mathbf{x}) \in E \times E^\omega;\ X = \overline{\mathbf{x}}\}, \quad \mathcal{C} = \{(\mathbf{x},\mathbf{y}) \in (E^\omega)^2;\ L(\mathbf{x},\mathbf{y})\}$$
$$\text{and} \quad \mathcal{D} = \{(\mathbf{x},\mathbf{y},P) \in (E^\omega)^2 \times \mathcal{G};\ C(\mathbf{x},\mathbf{y},P)\}$$

are Borel sets.

It is easy to check that $\mathcal{C}$ is an F_σ.

Let us define $\mathcal{O}$ a countable basis of the topology of E. Recall that the Effros Borel structure on the closed subsets of E is generated by the basis:

$$\Big(\{F \subseteq E;\ O \cap F \neq \emptyset\}\Big)_{O \in \mathcal{O}}.$$

$X = \overline{\mathbf{x}}$ is equivalent to the two conditions:

(i) $x_n \in O$ implies $O \cap X \neq \emptyset$, for all $n \in \omega$ and all $O \in \mathcal{O}$.

(ii) For all $O \in \mathcal{O}$, $O \cap X \neq \emptyset$ implies that there exists $n \in \omega$ such that $x_n \in O$.

Then, it is easy to see that $\mathcal{B}$ is a Borel set.

$\mathcal{D}$ is the union over C of the intersection over n,p,r of:

$$\{\|y_n - y_p\| > r/C\} \cup \Big[\bigcap_{m\in\omega} \big(\{m \notin P_{n,p,r}\} \cup \{\|x_m - x_n\| \le r\}\big) \cap$$
$$\Big(\bigcap_{q\in\omega} \bigcup_{Q\in 2^{<\omega}} \bigcap_{m,m'\in\omega} \big[\{m \notin P_{n,p,r} \text{ or } m' \notin P_{n,p,r}\} \cup$$
$$\{\|x_{m'} - x_m\| + \|y_m - y_p\| \le 1/q\}\big]\Big)\Big].$$

Therefore, $\mathcal{D}$ is a Borel set. □

The set $\{(X,Y);\ X \twoheadrightarrow_\ell Y\}$ being the projection on the first two coordinates of the set $\mathcal{A}$, it is analytic. This concludes the proof of our technical proposition. □

Before investigating the consequences of Proposition 1, let us add some more details on the Lipschitz-homeomorphisms between Banach spaces. In Theorem 2.4 of [3], Benoît Bossard proved that the linear isomorphism relation is analytic and non Borel. It is therefore natural to ask whether the Lipschitz-homeomorphism relation is also non Borel.

Notation. Let X and Y be two subspaces of E. When X and Y are Lipschitz-homeomorphic, we write $X \sim_\ell Y$.

Proposition 5. *The relation $\sim_\ell$ is analytic and non Borel.*

Proof: The proof of the analyticity of $\sim_\ell$ is similar to (and technically simpler than) the proof of the analyticity of $\twoheadrightarrow_\ell$. It will thus be omitted.

Let us show that $\sim_\ell$ is non Borel. Let us introduce $\mathcal{C} = \omega^{<\omega}$ and the group $G = 2^{\mathcal{C}}$. G is isomorphic to the Cantor group. Let p be a real number greater than 1 and different from 2. It suffices for our purpose to show that the set $\mathcal{L} = \{X \in \mathcal{S};\ X \sim_\ell L_p(G)\}$ is non Borel.

The dual of G is the group $\widehat{G}$ of all finite subsets of $\mathcal{C}$ where we identify b, a finite subset of $\mathcal{C}$, and its Walsh function w_b. For any tree T on ω, let us define the set $FB(T)$ of all finite branches of T. The space L_p^T is the closed (for the L_p norm) linear span of the set $\{w_b;\ b \in FB(T)\}$. Theorem 4.34 in [7] shows that all the spaces L_p^T are complemented subspaces of $L_p(G)$. According to Theorem 4.35 in [7], $L_p(G)$ does not embed in L_p^T if T is well-founded (that we write $T \in WF$). Conversely, if T has an infinite branch, then obviously $L_p(G)$ is isomorphic to a complemented subspace of L_p^T. Pełczyński's decomposition method then implies that $L_p(G)$ is isomorphic to L_p^T if and only if $T \notin WF$. Now we need the following fact:

Fact 4. *The map θ defined on the set $\mathcal{T}$ of all trees on ω by $\theta(T) = L_p^T$ is Borel.*

Let O be an open set of E. It is enough to show that the set $\Omega = \{T \in \mathcal{T};\ \theta(T) \cap O \neq \emptyset\}$ is Borel. Since $\theta(T) = \overline{\operatorname{span}}\{w_b;\ b \in FB(T)\}$, we have, defining $\Lambda = \{(\lambda_b) \in \mathbb{Q}^{FB(\mathcal{C})};\ \sum_b \lambda_b w_b \in O\}$:

$$\Omega = \bigcup_{(\lambda_b)\in\Lambda}\ \bigcap_{\{b;\ \lambda_b \neq 0\}} \{T;\ b \subseteq T\}.$$

It is now clear that Ω is a Borel set, which ends the proof of Fact 4.

According to Corollary 2.9 in [6], $\mathcal{L} = \{X \in S;\ X \text{ isomorphic to } L_p(G)\}$. Thus, $\mathcal{L} = \theta(\mathcal{T}\backslash WF)$ is non Borel. Indeed, if it was Borel then, since $\mathcal{T}\backslash WF = \theta^{-1}(\mathcal{L})$ and θ is Borel, $\mathcal{T}\backslash WF$ would be Borel which is absurd. □

It would come as a very big surprise for us if the relation of Lipschitz-quotient is actually Borel.

2. Control on the Szlenk index of a Lipschitz quotient

Our main result is a consequence of Proposition 1:

Theorem 6. *There exists a universal function $\psi_1 : \omega_1 \to \omega_1$ such that, if X is a Banach space with countable Szlenk index and Y a Lipschitz-quotient of X, then* $\mathrm{Sz}\,(Y) \leq \psi_1(\mathrm{Sz}\,(X))$.

Proof: Let us recall that, for separable Banach spaces, having a countable Szlenk index is equivalent to having a separable dual (see Proposition 4.12 of [3] for example). Thus, we will show that the general case boils down to the separable case and then use Theorem 3.18 of [2] concerning Asplund spaces.

According to Corollary 3.17 in [2], if f is a Lipschitz-quotient from a Banach space X onto another Banach space Y, then for any separable subspaces X_0 and Y_0 in X and Y respectively, there exist X_1 and Y_1, separable subspaces of X and Y respectively such that $X_0 \subseteq X_1$, $Y_0 \subseteq Y_1$ and the restriction of f to X_1 is a Lipschitz quotient mapping from X_1 onto Y_1. Moreover, the Szlenk index of a Banach space, when countable, is the supremum of the Szlenk indices of its separable subspaces (Proposition 3.1 in [4]). Thus, it is enough to deal with separable Banach spaces in our proof. Since the Szlenk index is invariant under linear isomorphism and since E is universal for separable Banach spaces, we can restrict our study to subspaces of E. It is shown in Lemma 3.5 and Theorem 4.13 of [3] that the set of all separable Asplund subspaces of E is a co-analytic set and that the Szlenk index is a Π^1_1-rank on it (see page 140 of [5] for a definition of Π^1_1-rank). For any ordinal ξ, let us call S_ξ the set of all subspaces of E whose Szlenk index is less than or equal to ξ and P_ξ the set of all subspaces of E Lipschitz homeomorphic to some element of S_ξ. With this notation, S_{ω_1} is the co-analytic set of all Asplund subspaces of E. Let ξ be a countable ordinal. The set S_ξ is Borel. According to Proposition 1, the set $H = \{(X, Y);\ X \in S_\xi \text{ and } X \twoheadrightarrow_\ell Y\}$ is analytic. Since P_ξ is the projection of H on the second coordinate, it is also analytic. Theorem 3.18 in [2] shows that P_ξ is included in S_{ω_1}. Kunen-Martin's theorem (see Theorem 7 p. 148 in [5] for instance) then proves that P_ξ is included in S_ζ for some countable ordinal ζ. We can define ψ_1 by $\psi_1(\xi) = \zeta$. □

In the special case of Lipschitz-homeomorphisms, we obtain the following result:

Corollary 7. *There exists a universal function $\psi_2 : \omega_1 \to \omega_1$ such that, if X is a Banach space with a countable Szlenk index and Y is a Banach space which is Lipschitz-homeomorphic to X, then* $\mathrm{Sz}\,(Y) \leq \psi_2(\mathrm{Sz}\,(X))$.

Theorem 5.5 in [1] proves that, if X and Y are uniformly homeomorphic, then $\mathrm{Sz}\,(X) \leq \omega$ if and only if $\mathrm{Sz}\,(Y) \leq \omega$. Thus, if we consider the minimal choices for ψ_1 and ψ_2, we have $\psi_2(\omega) = \omega$. It is not clear to us whether $\psi_2(\omega^2)$ equals ω^2. We do not know either the value of $\psi_1(\omega)$. More generally, it could be possible that, in fact, ψ_1 and ψ_2 are simply the identity.

As a corollary of Theorem 6, we get the following theorem about the Cantor index of scattered compact sets:

Corollary 8. *There exists a universal function $\lambda : \omega_1 \to \omega_1$ such that, if K is a scattered compact with a countable derivative empty and if $C(L)$ is a Lipschitz-quotient of $C(K)$, then the Cantor index $i(L)$ of L is less than or equal to $\lambda(i(K))$.*

Proof: This corollary is a straightforward consequence of Theorem 6 and of Theorem 5.1 in [4]. □

Example 4.9 from [11] shows that there exist two non metrizable scattered compact sets K and L with a countable derivative empty such that $C(K)$ and $C(L)$ are Lipschitz-isomorphic but not isomorphic. Thus Corollaries 7 and 8 deal with a situation which is known not to be linear. In the linear case, the Bessaga-Pełczyński result (see Theorem 3 in [13]) gives a necessary and sufficient condition for two countable compact sets K and L to be such that $C(K)$ is isomorphic to $C(L)$ (namely, that $i(K) < i(L) \cdot \omega$ and conversely). Thus, in the context of countable compact sets, it would be natural to compare λ and the function $\xi \mapsto \xi \cdot \omega$.

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