Selections on Ψ-spaces

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Abstract. We show that if $\mathcal{A}$ is an uncountable AD (almost disjoint) family of subsets of ω then the space $\Psi(\mathcal{A})$ does not admit a continuous selection; moreover, if $\mathcal{A}$ is maximal then $\Psi(\mathcal{A})$ does not even admit a continuous selection on pairs, answering thus questions of T. Nogura.

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The program of studying continuous selections on topological spaces was initiated by E. Michael in an influential series of papers in the 1950's (see [Mi]). Since then a number of both positive and negative results have been established and research in the area is blooming.

The concept of a Ψ-space, introduced independently by S. Mrówka and J. Isbell, provides an important class of examples in the theory of Fréchet spaces. Let us mention Mrówka's construction of a Ψ-space with a unique compactification ([Mr]) and P. Simon's example ([Si]) of two compact Fréchet spaces whose product is not Fréchet. The set-theoretic notation used here is standard and follows [Ku].

Recall that an infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* (*AD*) if every two distinct elements of $\mathcal{A}$ have only finite intersection. A family $\mathcal{A}$ is *MAD* if it is almost disjoint and maximal with this property. Given an almost disjoint family $\mathcal{A}$, $\mathcal{I}(\mathcal{A})$ denotes the ideal of those subsets of ω which can be almost covered by finitely many elements of $\mathcal{A}$, $\mathcal{I}^*(\mathcal{A})$ denotes the dual filter and $\mathcal{I}^+(\mathcal{A}) = \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$ the coideal of large sets.

Definition 0.1. Let $\mathcal{A}$ be an AD family. Define the space $\Psi(\mathcal{A})$ as follows: The underlying set is $\omega \cup \mathcal{A}$, all elements of ω are isolated and basic neighborhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup (A \setminus F)$ for some finite set F.

It follows immediately from the definition that $\Psi(\mathcal{A})$ is a first countable, locally compact space. It is hardly surprising that there is a close relationship between topological properties of the space $\Psi(\mathcal{A})$ and combinatorial properties of the almost disjoint family $\mathcal{A}$. If $\mathcal{A}$ is infinite then $\Psi(\mathcal{A})$ is not countably compact and

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$\Psi(\mathcal{A})$ is pseudocompact (contains no infinite discrete family of open subsets) if and only if $\mathcal{A}$ is a MAD family.

The hyperspace of a space X (denoted by $\exp(X)$) consists of all closed non-empty subsets of X. There are many ways to define a topology on $\exp(X)$ the standard (and most useful) being the *Vietoris topology* generated by sets of the form:

$$\langle U_0, \dots, U_{n-1} \rangle = \{F \in \exp(X) : F \subseteq \bigcup_{i<n} U_i \text{ and } F \cap U_i \neq \emptyset \text{ for every } i < n\}$$

where $U_0, \dots, U_{n-1}$ are nonempty open subsets of X. Let $[X]^2$ denote the set of (unordered) pairs of elements of X. If X is a T_1-space then we consider $[X]^2$ as a subspace of $\exp(X)$ equipped with the Vietoris topology.

Definition 0.2. *A space X* admits a selection *if there exists a continuous $\phi : \exp(X) \longrightarrow X$ such that $\phi(F) \in F$ for every $F \in \exp(X)$. Similarly, X has a* weak selection *if there exists a continuous $\phi : [X]^2 \longrightarrow X$ such that $\phi(\{x, y\}) \in \{x, y\}$ for every pair $\{x, y\}$ of elements of X.*

Note that the existence of a weak selection is equivalent to the existence of a continuous function $\varphi : X^2 \longrightarrow X$ such that $\varphi((x, y)) = \varphi((y, x)) \in \{x, y\}$, where X^2 is given the product topology.

T. Nogura has asked the natural question whether $\Psi(\mathcal{A})$ admits a selection for some (any) MAD family $\mathcal{A}$. We answer this question in the negative by proving:

Theorem 0.3. *The space $\Psi(\mathcal{A})$ does not have a weak selection for any maximal almost disjoint family $\mathcal{A}$.*

It should be mentioned here that this theorem was proved independently by G. Artico, U. Marconi, J. Pelant, L. Rotter and M. Tkachenko in [A&al]. In fact, it follows directly from a much stronger theorem proved in [A&al].

Here we also show that

Theorem 0.4. *If X is regular, separable and contains an uncountable closed discrete set, then X does not admit a continuous selection.*

from which it directly follows that $\Psi(\mathcal{A})$ does not admit a continuous selection for any uncountable almost disjoint family $\mathcal{A}$.

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I. Proofs of the main theorems

Our proof of Theorem 0.3 is based on a Ramsey theoretic property of the coideal $\mathcal{I}^+(\mathcal{A})$. Recall that if $f : [\omega]^2 \longrightarrow 2$ is a coloring of pairs into two colors, then a set $A \subseteq \omega$ is *f-homogeneous* if $|f([A]^2)| = 1$, in other words, if all pairs of elements of A are colored by the same color. The famous Ramsey Theorem states that for any coloring f there is an infinite f-homogeneous set. The following crucial lemma is well known in set-theoretic circles (see also [BDS]):

Lemma I.1 ([Ma]). *For every MAD family $\mathcal{A}$ and every decreasing sequence $\{X_i : i \in \omega\} \subseteq \mathcal{I}^+(\mathcal{A})$ there is an $X \in \mathcal{I}^+(\mathcal{A})$ such that $X \setminus i \subseteq \bigcap_{j<i} X_j$ for every $i \in X$.*

Lemma I.2. *Let $\mathcal{A}$ be a MAD family and let $f : [\omega]^2 \longrightarrow 2$. Then there exists an f-homogeneous set B such that $B \in \mathcal{I}^+(\mathcal{A})$.*

Proof: Extend the filter $\mathcal{I}^*(\mathcal{A}) = \langle\{\omega \setminus A : A \in \mathcal{A}\}\rangle$ to an ultrafilter $\mathcal{U}$. We will construct an f-homogeneous set using this ultrafilter. Let $g : \omega \longrightarrow 2$ be such that $X_n = \{m \in \omega : f(\{n, m\}) = g(n)\} \in \mathcal{U}$. Note that $X_n \in \mathcal{I}^+(\mathcal{A})$. By previous lemma, there is an $X \in \mathcal{I}^+(\mathcal{A})$ such that $X \setminus n \subseteq \bigcap_{i<n} X_i$, for every $n \in X$. Let $B(i) = \{n \in X : g(n) = i\}$ for $i \in 2$. As $X = B(0) \cup B(1)$, there exists $i \in 2$ such that $B(i) \in \mathcal{I}^+(\mathcal{A})$. The set $B = B(i)$ is the desired f-homogeneous subset. □

Proof of Theorem 0.3: The proof proceeds by contradiction. Assume that $\phi : [\Psi(\mathcal{A})]^2 \longrightarrow \Psi(\mathcal{A})$ is a weak selection. Consider $\phi \restriction [\omega]^2$ and define $f : [\omega]^2 \longrightarrow 2$ by:

$$f(\{n, m\}) = 0 \text{ if and only if } \phi(\{n, m\}) = \min\{n, m\}.$$

By Lemma I.2 there is a $B \in \mathcal{I}^+(\mathcal{A})$ which is f-homogeneous. Let $A_0,\ A_1$ be distinct elements of $\mathcal{A}$ such that $B \cap A_i$ is infinite for both $i < 2$. We will show that ϕ is not continuous at $\{A_0, A_1\}$. Assume that $\phi(\{A_0, A_1\}) = A_0$. It suffices to show that the image of any open neighborhood of $\{A_0, A_1\}$ is not contained in $\{A_0\} \cup A_0$, a neighborhood of A_0.

Suppose U is a neighborhood of $\{A_0, A_1\}$. Then U contains $V = \langle\{A_0\} \cup (A_0 \setminus k), \{A_1\} \cup (A_1 \setminus k)\rangle$ for some $k \in \omega$.

Suppose that $f([B]^2) = 0$. Let $n > k$ be such that $n \in (A_1 \cap B) \setminus A_0$ and $m > n$ such that $m \in (A_0 \cap B) \setminus A_1$. Then $\{n, m\} \in V$ and $\phi(\{n, m\}) = n \notin A_0$. On the other hand, if $f([B]^2) = 1$, let $n > k$ be such that $n \in (A_0 \cap B) \setminus A_1$ and $m > n$ such that $m \in (A_1 \cap B) \setminus A_0$. Then $\{n, m\} \in V$ and $\phi(\{n, m\}) = m \notin A_0$.

Therefore, $\phi''U \nsubseteq \{A_0\} \cup A_0$. □

Proof of Theorem 0.4: Let X be a separable regular space and let A be an uncountable closed discrete subset of X, without loss of generality without isolated points. By way of contradiction assume that $\phi : \exp(X) \to X$ is a continuous selection. Define an enumeration

$$A = \{a_\alpha : \alpha < \lambda\}$$

by letting $a_0 = \phi(A)$ and $a_\alpha = \phi(A_\alpha)$ where

$$A_\alpha = A \setminus \{a_\beta : \beta < \alpha\}.$$

Fix open neighborhoods O_α of each a_α such that

$$\overline{O_\alpha} \cap A = \{a_\alpha\}.$$

By continuity, for each α, $\phi^{-1}(O_\alpha)$ is an open set in $\exp(X)$ containing A_α. So, by definition of the Vietoris topology on $\exp(X)$, there are $m_\alpha \in \omega$ and open sets U^n_α, $n < m_\alpha$, such that

$$A_\alpha \in \langle U^n_\alpha : n < m_\alpha \rangle \subseteq \phi^{-1}(O_\alpha).$$

Therefore, $A_\alpha \subseteq \bigcup_{n<m_\alpha} U^n_\alpha$ and $A_\alpha \cap U^n_\alpha \neq \emptyset$ for each $n < m_\alpha$.
By shrinking the U^n_α's we may assume that

(a) $U^0_\alpha \subseteq O_\alpha$ for each $\alpha < \lambda$.
(b) $\overline{O_\alpha} \cap \bigcup_{0<n<m_\alpha} U^n_\alpha = \emptyset$.

Therefore, as $\langle U^n_\alpha : n < m_\alpha \rangle \subseteq \phi^{-1}(O_\alpha)$, we have

(c) For each $F \in [X]^{<\aleph_0}$ if $F \in \langle U^n_\alpha : n < m_\alpha \rangle$ then $\phi(F) \in F \cap U^0_\alpha$.

Using that X is separable, fix D to be a countable dense subset of X.

Claim. *There is $F \in [D]^{<\aleph_0}$, and $\alpha < \beta < \lambda$ such that*

(d) $F \cap U^n_\alpha \neq \emptyset$ *for each* $n < m_\alpha$;
(e) $F \cap U^n_\beta \neq \emptyset$ *for each* $n < m_\beta$;
(f) $F \subseteq (\bigcup_{n<m_\alpha} U^n_\alpha) \cap (\bigcup_{n<m_\beta} U^n_\beta)$;
(g) $(F \cap U^0_\alpha) \cap (F \cap U^0_\beta) = \emptyset$.

First note that the Claim leads to a contradiction. Namely, by (b), $\phi(F) \in U^0_\alpha \cap U^0_\beta$ but by (g) this is impossible. Thus, proving the Claim will complete the proof of the theorem.

To this end let, for each α,

$$V_\alpha = \bigcup_{0<n<m_\alpha} U^n_\alpha.$$

Then $U^0_\alpha \cap V_\alpha = \emptyset$ by (a) and (b). As D is countable, there is an uncountable set $J \subset \omega_1$ and a finite set $G \subset D$ such that

$$\forall \alpha \in J \; \forall n,\ 0 < n < m_\alpha \ :\ G \cap U^n_\alpha \neq \emptyset \ \&\ G \subset V_\alpha.$$

Let $\{\delta_\alpha : \alpha \in \omega_1\}$ be an increasing enumeration of J.

For each $\alpha \in J$ let

$$D_{\alpha+1} = D \cap U^0_{\delta_{\alpha+1}} \cap V_{\delta_\alpha}.$$

Note that each $D_{\alpha+1}$ is a nonempty subset of D ($a_{\delta_{\alpha+1}} \in U^0_{\delta_{\alpha+1}} \cap V_{\delta_\alpha}$ and $a_{\delta_{\alpha+1}}$ is not isolated). Therefore $\{D_{\alpha+1} : \alpha < \omega_1\}$ is not pairwise disjoint. So we may fix successor ordinals $\alpha < \beta < \omega_1$ such that

$$U^0_{\delta_\alpha} \cap V_{\delta_\beta} \neq \emptyset.$$

Let $k_0 \in D \cap U^0_{\delta_\alpha} \cap V_{\delta_\beta}$. As $D \cap U^0_{\delta_\beta} \cap V_{\delta_\alpha} \neq \emptyset$ (recall that $a_{\delta_\beta} \in V_{\delta_\alpha}$ as V_{δ_α} is an open set containing $A_{\delta_\alpha+1}$ and $a_{\delta_\beta} \in A_{\delta_\alpha+1}$), we may choose $k_1 \in D \cap U^0_{\delta_\beta} \cap V_{\delta_\alpha}$. Now define $F = G \cup \{k_0, k_1\}$.

Notice that $F \cap U^0_{\delta_\alpha} = \{k_0\}$ and $F \cap U^0_{\delta_\beta} = \{k_1\}$, thus F satisfies (g). It is clear that F satisfies the other conclusions of the Claim. □

II. Concluding remarks

The proof of Theorem 0.3 is similar to the proof of the following proposition due to E. van Douwen ([vD1]).

Proposition II.1 (van Douwen). *If X is a countably compact, not sequentially compact space, then X does not have a weak selection. In particular, it does not admit a continuous selection.*

A natural question arises as to for which almost disjoint families $\Psi(\mathcal{A})$ admits a weak selection. Obviously, if $\mathcal{A}$ is a countable almost disjoint family, then $\Psi(\mathcal{A})$ is homeomorphic to an ordinal hence admits a continuous selection. For the proof of Theorem 0.3 we, in fact, only needed that $\mathcal{A}$ is *somewhere MAD*, i.e. there is an $X \in \mathcal{I}^+(\mathcal{A})$ such that for every infinite $Y \subseteq X$ there is an $A \in \mathcal{A}$ intersecting Y in an infinite set. If an AD family $\mathcal{A}$ is not somewhere MAD we say that $\mathcal{A}$ is *nowhere MAD*. Note that the one-point compactification of the locally compact space $\Psi(\mathcal{A})$ is Fréchet if and only if $\mathcal{A}$ is nowhere MAD (see e.g. [vD2]).

We will show that for some, but not all, uncountable nowhere MAD families $\mathcal{A}$, $\Psi(\mathcal{A})$ does admit a weak selection.

Example II.2. *There is an uncountable almost disjoint family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ admits a weak selection.*

Proof: Identify ω with $2^{<\omega}$ — the set of all finite sequences of 0's and 1's. For every $f \in 2^\omega$ let $A_f = \{f \restriction n : n \in \omega\}$. Let $\mathcal{A} = \{A_f : f \in 2^\omega\}$. For $s, t \in 2^{<\omega} \cup 2^\omega$ let $\Delta_{s,t} = \min\{n \in \omega : s(n) \neq t(n)\}$. Of course, $\Delta_{s,t}$ is not well-defined if $s \subseteq t$

or $t \subseteq s$. Define an ordering on $\Psi(\mathcal{A})$ by:

$$x \leq y \text{ if } \begin{cases} x, y \in 2^{<\omega} \text{ and } (x \subseteq y \text{ or } x(\Delta_{x,y}) < y(\Delta_{x,y})), \\ x \in 2^{<\omega}, y = A_f \text{ and } (x \subseteq f \text{ or } x(\Delta_{x,f}) < f(\Delta_{x,f})), \\ x = A_f, y \in 2^{<\omega} \text{ and } f(\Delta_{y,f}) < y(\Delta_{y,f}), \\ x = A_f, y = A_g \text{ and } (f = g \text{ or } f(\Delta_{f,g}) < g(\Delta_{f,g})). \end{cases}$$

The ordering $\leq$ is a linear order on $\Psi(\mathcal{A})$ and the usual topology on $\Psi(\mathcal{A})$ is finer than the interval topology induced by $\leq$. It is easy to verify that putting

$$\phi(\{x, y\}) = x \text{ if and only if } x \leq y$$

defines a continuous weak selection for $\Psi(\mathcal{A})$. □

On the other hand:

Proposition II.3. *There are nowhere MAD families whose Ψ-spaces do not have a weak selection.*

PROOF: Let $\mathcal{A}$ be the almost disjoint family $\mathcal{A}$ from Example II.2. Note that $\mathcal{A}$ is a nowhere MAD family of size $\mathfrak{c}$.

Enumerate all $f : [\omega]^2 \longrightarrow 2$ as $\{f_\alpha : \alpha < \mathfrak{c}\}$ and enumerate $\mathcal{A}$ as $\{A_\alpha : \alpha \in \mathfrak{c}\}$.

For every $\alpha < \mathfrak{c}$, find an infinite f_α-homogeneous subset C_α of A_α and split it into two infinite pieces C^0_α and C^1_α. Let $A^0_\alpha = C^0_\alpha$ and $A^1_\alpha = A_\alpha \setminus C^0_\alpha$. Let $\mathcal{B} = \{A^0_\alpha, A^1_\alpha : \alpha < \mathfrak{c}\}$. Now, the proof of Theorem 0.3 goes through, so $\Psi(\mathcal{B})$ does not have a weak selection, and $\mathcal{I}(\mathcal{B}) = \mathcal{I}(\mathcal{A})$, so $\mathcal{B}$ is nowhere MAD. □

Corollary II.4. *There is a separable scattered compact Fréchet space without a weak selection.*

PROOF: Let X be a one-point compactification of $\Psi(\mathcal{A})$ without a weak selection, where $\mathcal{A}$ is nowhere MAD. Then X is compact, Fréchet and scattered, and does not have a weak selection since $\Psi(\mathcal{A})$ does not admit one. □

As pointed out by the referee this follows directly from a result of J. van Mill and E. Wattel (see [vMW]) where they proved that *a compact space admits a weak selection if and only if it is orderable*.

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