Structure of the kernel of higher spin Dirac operators

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Abstract. Polynomials on $\mathbb{R}^n$ with values in an irreducible Spin_n-module form a natural representation space for the group Spin_n. These representations are completely reducible. In the paper, we give a complete description of their decompositions into irreducible components for polynomials with values in a certain range of irreducible modules. The results are used to describe the structure of kernels of conformally invariant elliptic first order systems acting on maps on $\mathbb{R}^n$ with values in these modules.

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Classification: 53A30, 53A55, 32A50, 43A65

0. Introduction

The spaces of solutions of invariant differential (systems of) equations are in a natural way modules for the corresponding symmetry group. The detailed study of conformally invariant equations, see e.g. [11], [20], [3] and references therein (or more generally invariant operators with respect to a chosen parabolic geometry, see [21], [10]) are being typical examples. It can be expected in general that the representation theory will play an important role in the study of the corresponding spaces of solutions.

The classical result of that type for the Laplace operator is a description of the spaces of spherical harmonics on $\mathbb{R}^n$. The space $\mathcal{H}_r$ of harmonic homogeneous polynomials of order r is, under the induced action of the Spin_n group, an irreducible representation with the highest weight $(r, 0, \dots, 0)$. The well known classical theorem is then saying that the space $\mathcal{P}_r$ of all homogeneous polynomials of order r is the sum of irreducible modules of Spin_n with highest weights $(p, 0, \dots, 0)$, where $p \in \{r, r-2, r-4, \dots, \}$, $p \geq 0$. Each module is appearing in the decomposition with multiplicity one.

Similarly, it is well known that the so called spherical monogenics, i.e. homogeneous solutions of the Dirac equation for spinor valued polynomials on $\mathbb{R}^n$ of order r are again irreducible modules under the induced action of Spin_n with the highest weight $(\frac{2r+1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ (see e.g. [22]).

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Recently, invariant first order differential systems of equations for maps on $\mathbb{R}^n$ with values in more complicated representations of the Spin_n group were considered. The first example was the case of the so called Rarita-Schwinger operator. Its homogeneous polynomial solutions were described in [5]. The spaces of such homogeneous solutions are no more irreducible but it is possible to understand how they decompose into a sum of irreducible components and to find highest weights of all irreducible components. This basic example was then generalized in several various directions ([2], [4], [5], [6], [13], [14]). The methods used for such a description were typically coming from the Clifford analysis ([9]), sometimes combined with some geometrical tools ([7], [8]).

In the presented paper, we are going to extend the class of equations for which the understanding of the structure of the space of polynomial solutions is possible. Tools used for that are much simpler than those used in previous papers. All results are deduced just using recent result by P. Littelmann [16] on the decomposition of certain tensor products of irreducible modules into irreducible components (the so called Littlewood-Richardson rules) together with the classical surjectivity result for elliptic operators.

In the first part of the paper, we define a certain class of elliptic invariant first order operators on $\mathbb{R}^n$ and we recall the classical surjectivity result. The Littlewood-Richardson rule is described in Section 2. In the third part, this rule is used for the decomposition of the space of spherical harmonics with values in irreducible modules with highest weights $(\frac{2k+1}{2},\dots,\frac{2k+1}{2},\frac{1}{2},\dots,\frac{1}{2})$.

Up to know, the structure of the kernel of an elliptic invariant operator acting on maps on $\mathbb{R}^n$ with values in irreducible modules was known for modules with highest weights of types $(\frac{2k+1}{2},\frac{1}{2},\dots,\frac{1}{2})$ or $(\frac{3}{2},\dots,\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2})$. These results are reproduced in Section 4 (together with the full treatment of the cases of small homogeneities) and extended to cover the case of values in any irreducible modules with highest weight of type $(\frac{2k+1}{2},\dots,\frac{2k+1}{2},\frac{1}{2},\dots,\frac{1}{2})$.

1. Higher spin Dirac operators

The operators of our interest will be the so called *higher spin Dirac operators*. We state here only the basic properties that we shall use in the last section when deriving the decomposition of their kernel. More detailed information can be found e.g. in [4], [8]. Also, as we are going to make our definitions simple, we will formulate them directly on $\mathbb{R}^n$ instead of on a general Riemannian spin-manifold.

Definition 1.1. Let $\mathbb{S}$ be a basic spinor representation of Spin_n with the highest weight $\lambda=(\frac{1}{2},\dots,\frac{1}{2})$ for n odd, or $\mathbb{S}=\mathbb{S}^+\oplus\mathbb{S}^-$ with $\lambda^\pm=(\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})$ in the even dimensional case. On the space of spinor-valued polynomials $\mathcal{P}(\mathbb{R}^n)\otimes\mathbb{S}$, we can consider the standard Dirac operator $\mathcal{D}$. Let $\boldsymbol{V}_\mu$ be a representation of Spin_n with the highest weight μ, and let $\{\boldsymbol{v}_i\}$ be a basis of this vector space. The *twisted Dirac operator* is the operator $\mathcal{D}_T$ acting on the polynomials $\mathcal{P}(\mathbb{R}^n)\otimes\boldsymbol{V}_\mu\otimes\mathbb{S}$, with values in the representation space $\boldsymbol{V}_\mu \otimes \mathbb{S}$, which is defined by the formula

$$\mathcal{D}_T : \sum_i \boldsymbol{v}_i \otimes s_i(x) \longrightarrow \sum_i \boldsymbol{v}_i \otimes \mathcal{D}s_i(x) \tag{1}$$

where $s_i(x)$ are $\mathbb{S}$-valued polynomials.

Similarly as the Dirac operator, the twisted Dirac operator is a Spin_n-invariant first-order differential operator with constant coefficients.

Definition 1.2. We know that the decomposition of the product $\boldsymbol{V}_\mu \otimes \mathbb{S}$ into irreducible components contains the representation $\boldsymbol{V}_{\lambda+\mu}$[1], with the highest weight $\lambda + \mu$, with multiplicity one. Thus, we can write

$$\mathcal{P}(\mathbb{R}^n) \otimes (\boldsymbol{V}_\mu \otimes \mathbb{S}) = \mathcal{P}(\mathbb{R}^n) \otimes \boldsymbol{V}_{\lambda+\mu} \oplus \textit{something}.$$

By restricting the twisted Dirac operator $\mathcal{D}_T$ to $\mathcal{P}(\mathbb{R}^n) \otimes \boldsymbol{V}_{\lambda+\mu}$ and by projecting its range onto the same space we get a new operator $\mathcal{D}$ called the *higher spin Dirac operator*:

$$\mathcal{D} : \mathcal{P}(\mathbb{R}^n) \otimes \boldsymbol{V}_{\lambda+\mu} \longrightarrow \mathcal{P}(\mathbb{R}^n) \otimes \boldsymbol{V}_{\lambda+\mu}. \tag{2}$$

The space $\boldsymbol{V}_{\lambda+\mu}$ will be called a *higher spin space* or simply a *spinor*(*-like*) *space*.

Also, the higher spin Dirac operator is a Spin_n-invariant first-order differential operator and thus decreases the degree of higher-spin-valued polynomials by one. It is known that the higher spin Dirac operator is elliptic (i.e. its symbol $\sigma(\mathcal{D}, \xi)$ is an isomorphism for every vector $\xi \neq 0$), see [1]. A classical theorem of the theory of differential equations states that any linear elliptic differential operator is surjective in smooth category. In other words, the equation

$$\mathcal{D}f = g \tag{3}$$

has a solution for any $\mathcal{C}^\infty$ right-hand side. Using the result in [17], we know that for any polynomial right-hand side the corresponding solution is also analytic, hence we can write its decomposition into the Taylor series on $\mathbb{R}^n$. Using the locally uniform convergence of the Taylor series and applying the Weierstrass theorem, we get the following proposition:

Proposition 1.1. *For any higher spin Dirac operator $\mathcal{D}$, the equation*

$$\mathcal{D}f = g \tag{4}$$

has a polynomial solution for every polynomial right-hand side.

[1] We shall denote $\boldsymbol{V}_{\lambda+\mu} = \boldsymbol{V}_{\lambda^+ + \mu} \oplus \boldsymbol{V}_{\lambda^- + \mu}$ in the even dimensional case.

2. Young tableaux

We are looking for a decomposition of the products of the Spin_n representations with the highest weights $(r, 0, \dots, 0)$ with a spinor-like representation $\mathbb{S}$. The approach that proved to be most powerful is the so called *Littlewood-Richardson rule*. This method is based on the notion of a "standard Young tableau". The following definitions and theorems are adapted from [16].

2.1 Spin-standard Young tableaux.

Definition 2.1. Let $\mathbf{p} = (p_1, \dots, p_m)$ be a partition of a natural number n, that is $p_1 \geq \cdots \geq p_m \geq 0$ are integers and $n = \sum_{i=1}^{m} p_i$. With $\mathbf{p}$ we associate its *Young diagram*, a figure consisting of m left justified rows of boxes, p_i boxes in the ith row from the top. By a *Young tableau $\mathcal{T}$ of shape* $\mathbf{p}$ we mean a filling of the boxes with positive integers. A Young tableau is *standard* if the numbers in the boxes are non-decreasing in the rows and strictly increasing in the columns from the top to the bottom.

If we omit any column from a standard Young tableau $\mathcal{T}$, we get a standard Young tableau again. Let us number columns of Young tableaux from the right hand side. It will be convenient to denote by $\mathcal{T}(k, l)$, $k \geq l$, the Young tableau consisting of the lth up to the kth column of a Young tableau $\mathcal{T}$, and by $\mathcal{T}(k)$ the tableau $\mathcal{T}(k, 1)$. If i is a positive integer, we define $c_{\mathcal{T}}(i)$ to be the number of boxes in $\mathcal{T}$ containing the number i.

To the dominant weight μ of a representation $\boldsymbol{V}_\mu$ of a Lie algebra $\mathfrak{g}$ we associate a partition $\mathbf{p}(\mu) = (p_1, \dots, p_m)$. The exact relation depends on the type of the Lie algebra $\mathfrak{g}$, here we shall restrict ourselves only to the formulas for Lie algebras of type B_m and D_m. The formulas for other types of Lie algebras can be found in [16].

Definition 2.2. Let ω_i be the ith fundamental weight of a Lie algebra $\mathfrak{g}$, and let $\mu = \sum_{i=1}^{m} a_i \omega_i$ be the decomposition of the dominant weight μ of a representation $\boldsymbol{V}_\mu$ of $\mathfrak{g}$. Then we associate to μ the partition $\mathbf{p}(\mu) = (p_1, \dots, p_m)$ with $p_i = \sum_{j=i}^{m-1} 2a_j + a_m$ for $\mathfrak{g}$ of type B_m, and $p_i = \sum_{j=i}^{m-2} 2a_j + a_{m-1} + a_m$ for $\mathfrak{g}$ of type D_m (void sums give 0).

Definition 2.3. Let $\mathbf{h}$ be a column of a standard Young tableau such that it does not contain numbers i and $2m + 1 - i$ together. For $i = 1, \dots, m$ we denote by $s_i(\mathbf{h})$ the columns defined as follows:

> If $i < m$ and both $i + 1$ and $2m + 1 - i$ are entries of the column $\mathbf{h}$, then $s_i(\mathbf{h})$ is the column obtained from $\mathbf{h}$ by replacing the entry $i + 1$ by i and $2m + 1 - i$ by $2m - i$. If $i = m$, $\mathfrak{g}$ is of type B_m and $\mathbf{h}$ contains an entry with value $m + 1$, then $s_i(\mathbf{h})$ is the column obtained from $\mathbf{h}$ by replacing $m+1$ by m. If $i = m$, $\mathfrak{g}$ is of type D_m and both $m+1$ and $m+2$ are entries of the column $\mathbf{h}$, then $s_i(\mathbf{h})$ is the column obtained from $\mathbf{h}$ by replacing $m + 1$ by $m - 1$ and $m + 2$ by m. In all other cases we set $s_i(\mathbf{h}) = \mathbf{h}$.

We say that a pair of columns $(\mathbf{h},\mathbf{h}')$ is *admissible*, if there exists a sequence of different columns $(\mathbf{h}_0,\dots,\mathbf{h}_k)$, $k\geq 0$, such that

$$\begin{aligned}&\mathbf{h}=\mathbf{h}_0,\ \mathbf{h}'=\mathbf{h}_k,\\ &s_{i_j}(\mathbf{h}_{j-1})=\mathbf{h}_j \qquad \text{for } j=1,\dots,k \text{ and some integers } 1\leq i_j\leq m.\end{aligned}$$

Definition 2.4. Let $\mathcal{T}$ be a Young tableau of shape $\mathbf{p}(\mu)$ that contains only positive integers smaller or equal to $2m$ and that does not contain integers i and $2m+1-i$ in the same column together. Denote $\bar{p_1}=p_1-a_m$ for $\mathfrak{g}$ of type B_m, and $\bar{p_1}=p_1-a_m-a_{m-1}$ for $\mathfrak{g}$ of type D_m. $\mathcal{T}$ is called *Spin-standard* if all of the following holds:

1. If $\mathfrak{g}$ is of type B_m then $\mathcal{T}$ is standard. If $\mathfrak{g}$ is of type D_m then we divide $\mathcal{T}$ into three tableaux: $\mathcal{T}_1:=\mathcal{T}(\bar{p_1})$, $\mathcal{T}_2:=\mathcal{T}(p_1-a_m,\bar{p_1}+1)$, and $\mathcal{T}_3:=\mathcal{T}(p_1,p_1-a_m+1)$. Then each of the tableaux $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ is standard.
2. Let $\mathbf{t}_i$ be the ith column of $\mathcal{T}$. The pair of columns $(\mathbf{t}_{2i-1},\mathbf{t}_{2i})$ is admissible for every $i=1,\dots,\bar{p_1}/2$.

For $\mathfrak{g}$ of type D_m there must be two further conditions satisfied:

3. In $\mathcal{T}_2$ (resp. $\mathcal{T}_3$) the number of integers in a column greater than m is odd (resp. even). The condition for $\mathcal{T}_1$ reads as follows: Let $1\leq i\leq \bar{p_1}/2-1$ and let the $2i$th column $\mathbf{t}_{2i}$ of $\mathcal{T}_1$ consists of entries $(h_1,\dots,h_k)$ and the $(2i+1)$st column $\mathbf{t}_{2i+1}=(j_1,\dots,j_l)$, $k\leq l$. For any sequence of integers $1\leq i_1<\cdots<i_q\leq k$ such that
$$\begin{aligned}m+1-q\leq h_{i_1}<\cdots<h_{i_q}\leq m+q\\ m+1-q\leq j_{i_1}<\cdots<j_{i_q}\leq m+q\end{aligned}$$
there holds $h_{i_1}+\cdots+h_{i_q}\equiv j_{i_1}+\cdots+j_{i_q} \mod 2$.
4. The last condition is needed only if either $a_{m-1}>0$ and $a_m>0$, or $p_1>a_{m-1}+a_m>0$. For a column $\mathbf{h}=(h_1,\dots,h_l)$ denote by $\mathcal{H}(\mathbf{h})$ the set of mutually different positive integers $\{h_1,\dots,h_l,l_1,\dots,l_{m-l}\}$ such that $2m\geq l_j>m$ for $1\leq j\leq m-l$ and numbers i and $2m+1-i$ are not in the set together for any i. Further, for any set $\mathcal{H}$ of mutually different positive integers that are smaller or equal to $2m$ denote by $\mathbf{h}(\mathcal{H})$ the column consisting of the same integers as in $\mathcal{H}$ in increasing order (from the top to the bottom).

   Let $\mathbf{h}$ be the very left (the $\bar{p_1}$th) column of $\mathcal{T}_1$. Let us define by induction an auxiliary collection of sets $\mathcal{H}_i$, $i\geq 0$. Set $\mathcal{H}_0:=\mathcal{H}(\mathbf{h})$. If $\mathcal{H}_{i-1}$ is already defined, find the least number x among all its elements bigger than m and replace it with its "mirror" $2m+1-x$. Denote the new set by $\mathcal{H}_i$.

Add one of the columns $\mathbf{h}(\mathcal{H}_0)$ or $\mathbf{h}(\mathcal{H}_1)$ as the very right (the 0th) column to the tableau $\mathcal{T}_2$, according to which of them has odd number of elements that are bigger than m, and denote the new tableau by $\mathcal{T}_2'$. Then $\mathcal{T}_2'$ is standard.

Similarly, denote by $\mathcal{T}_3'$ the tableau obtained from $\mathcal{T}_3$ by adding a new 0th column. The new column is equal to $\mathbf{h}(\mathcal{H}_{a_{m-1}})$ if $\mathcal{H}_0$ contains even number of elements that are greater than m, and equal to $\mathbf{h}(\mathcal{H}_{a_{m-1}+1})$ otherwise. Then $\mathcal{T}_3'$ is standard.

2.2 The Littlewood-Richardson rule.

Definition 2.5. Let μ be a dominant weight of a representation of $\mathfrak{g}$ and let $\mathcal{T}$ be a Spin-standard Young tableau of shape $\mathbf{p}(\mu) = (p_1, \ldots, p_m)$. Define the *weight of* $\mathcal{T}$ as

$$\nu(\mathcal{T}) := \frac{1}{2}\left[(c_{\mathcal{T}}(1) - c_{\mathcal{T}}(2m))\varepsilon_1 + \cdots + (c_{\mathcal{T}}(m) - c_{\mathcal{T}}(m+1))\varepsilon_m\right] \tag{5}$$

where $(\varepsilon_1, \ldots, \varepsilon_m)$ is the standard weight basis of $\mathfrak{g}$.[2] For $1 \le l \le p_1$ denote by $\nu_l(\mathcal{T})$ the weight $2\nu(\mathcal{T}(l))$.

If λ is a dominant weight for $\mathfrak{g}$, then a Spin-standard Young tableau $\mathcal{T}$ of shape $\mathbf{p}(\mu)$ is called λ-*dominant* if all the weights $2\lambda + \nu_1(\mathcal{T}), \ldots, 2\lambda + \nu_{p_1}(\mathcal{T})$ are contained in the dominant Weyl chamber of $\mathfrak{g}$.

Theorem 2.1. *The decomposition of the tensor product $\boldsymbol{V}_\lambda \otimes \boldsymbol{V}_\mu$ into a sum of irreducible representations of $\mathfrak{g}$ is given by the formula*

$$\boldsymbol{V}_\lambda \otimes \boldsymbol{V}_\mu = \bigoplus_{\mathcal{T}} \boldsymbol{V}_{\lambda+\nu(\mathcal{T})} \tag{6}$$

where $\mathcal{T}$ runs over all λ-dominant Spin-standard Young tableaux of shape $\mathbf{p}(\mu)$.

3. Spinor-valued polynomials

Spinor-valued polynomials are Spin-group representations of the form $\mathcal{P}\otimes\mathbb{S}$, where $\mathcal{P} \equiv \mathcal{P}(\mathbb{R}^n)$ are the ordinary scalar valued polynomials and $\mathbb{S}$ is a "spinor-like space" — $(\frac{2s+1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ or a similarly simple representation of Spin_n. We know that the space of polynomials $\mathcal{P}$ decomposes into irreducible components with the highest weights $(r, 0, \ldots, 0)$. Hence it is sufficient to consider products of $\mathbb{S}$ with the irreducible representations $(r, 0, \ldots, 0)$ only. Here and below, the highest

[2] We suppose that the relation between the fundamental and the standard weight basis is the following: In the case of B_m we have $\varepsilon_1 = \omega_1$, $\varepsilon_i = \omega_i - \omega_{i-1}$ for $i = 2, \ldots, m-1$ and $\varepsilon_m = 2\omega_m - \omega_{m-1}$; In the case of D_m we have $\varepsilon_1 = \omega_1$, $\varepsilon_i = \omega_i - \omega_{i-1}$ for $i = 2, \ldots, m-2, m$ and $\varepsilon_{m-1} = \omega_m + \omega_{m-1} - \omega_{m-2}$.

weight will be used to denote at the same time the corresponding irreducible module.

In the foregoing section, the representations $(r, 0, \dots, 0)$ will play the role of $\boldsymbol{V}_\mu$ from Theorem 2.1. It follows that all our Spin-standard Young tableaux will have only one row, of length $2r$. Our task (of finding all Spin-standard Young tableaux) will be simplified by this fact a lot. Also, most of the conditions for the standard Young tableaux either will not apply, or will be fulfilled automatically.

Let us rephrase the previous definitions for this much simpler case:

Proposition 3.1. *Let $\mu = (r, 0, \dots, 0)$. Then all Young tableaux of shape $\mathbf{p}(\mu) = (2r, 0, \dots, 0)$ have only one row that is of length $2r$. Such a Young tableau $\mathcal{T}$ is Spin-standard iff all of the following holds:*

1. *$\mathcal{T}$ is standard and contains integers between 1 and $2m$ only.*
2. *If we write $\mathcal{T} = \boxed{t_{2r}, \dots, t_1}$ then $t_{2i} = t_{2i-1}$ for all $1 \le i \le r$, or if $\mathfrak{g}$ is of type B_m, there can also be $t_{2i} = m$, $t_{2i-1} = m + 1$ for some i.*
3. *For $\mathfrak{g}$ of type D_m, $\mathcal{T}$ does not contain both m and $m + 1$.*

Proof: We have to compare Definition 2.4. (1) only rephrases condition 1 from the definition. Note that, in case $\mathfrak{g}$ being of type D_m, $\mathcal{T}_1 = \mathcal{T}$, and $\mathcal{T}_2$, $\mathcal{T}_3$ are empty. (2) expresses the "admissibility" condition. Finally, (3) corresponds to condition 3 from the definition. Condition 4 from the definition is void in our case. □

Turn now to the question of decomposing the tensor product of two very special representations — a polynomial one with a "spinor-like" one. In the following, it will be easier to write the weights in the coordinates of the fundamental weight basis.[3] Also, for convenience, we will not distinguish in notation between a representation and its highest weight.

We will see that there is actually a very little difference between the B_m and D_m cases. In the next paragraphs we will develop a notation that will enable us to formulate the basic results in a common language for both cases.

3.1 The product $[r, 0, \dots, 0] \otimes [s, 0, \dots, 0, 1]$.

3.1.1 The case $\mathfrak{g}$ of type D_m.

First, let us examine the case $[r, 0, \dots, 0] \otimes [s, 0, \dots, 0, 1]$, $\mu = [r, 0, \dots, 0]$ and $\lambda = [s, 0, \dots, 0, 1]$.[4] The twisted case $[r, 0, \dots, 0] \otimes [s, 0, \dots, 0, 1, 0]$ is similar.

[3] Square brackets will suggest that the coordinates are relative to the fundamental weight basis. We shall use ordinary round brackets for weights in standard notation.
A weight lies in the dominant Weyl chamber iff all its coordinates relatively to the fundamental basis are nonnegative.

[4] In agreement with the footnote 2 we suppose that $[1, 0, \dots, 0] = (1, 0, \dots, 0)$, $\dots$, $[0, \dots, 0, 1, 0, 0] = (1, \dots, 1, 0, 0)$, $[0, \dots, 0, 1, 0] = (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$, $[0, \dots, 0, 1] = (\frac{1}{2}, \dots, \frac{1}{2})$. The last two weights correspond to the spinor representations $\mathbb{S}^-_{1/2}$ and $\mathbb{S}^+_{1/2}$ respectively.

According to Proposition 3.1, all Spin-standard tableaux are composed of pairs of equal numbers, *segments*. Looking at the formula (5) we see that each of the segments in $\mathcal{T}$ adds or removes from $\nu(\mathcal{T})$ exactly one of the ε_i's.

The decomposition theorem 2.1 requires not only the dominance of the final $\lambda + \nu(\mathcal{T})$, but also the dominance of the "partial" weights $2\lambda + \nu_l(\mathcal{T})$. In other words, we must not get out of the dominant Weyl chamber with $2\lambda + \nu_l(\mathcal{T})$ when changing l. This condition determines the set of those segments that $\mathcal{T}$ can consist of in order to be λ-dominant. We shall assign to them letters A, C, D, E, the corresponding weights are those with which the segments contribute to $\nu(\mathcal{T})$:

$$\begin{array}{lll} \boxed{1,1} & [1,0,\dots,0] & A \\ \boxed{2,2} & [-1,1,0,\dots,0] & C \\ \boxed{m+1,m+1} & [0,\dots,0,1,-1] & D \\ \boxed{2m,2m} & [-1,0,\dots,0] & E \end{array} \tag{7}$$

From the same reason as above it follows that the number of the segments $\boxed{2,2}$ and $\boxed{2m,2m}$ altogether cannot exceed the number s — otherwise $\lambda + \nu(T)$ would not lie in the dominant Weyl chamber. Note, that it is important how the segments are ordered in $\mathcal{T}$, and that it does not suffice to require the dominance of $\lambda + \nu(\mathcal{T})$ alone.

Note also, that the weights corresponding to the four segments (7) are linearly independent, hence to different λ-dominant tableaux there correspond different weights $\nu(\mathcal{T})$. Therefore all components in the decomposition of the product $[r,0,\dots,0] \otimes [s,0,\dots,0,1]$ occur with multiplicity one.

Let us summarize.

Lemma 3.2. *It holds*

$$[r,0,\dots,0] \otimes [s,0,\dots,0,1] = \bigoplus_{\mathcal{T}} [s,0,\dots,0,1] + \nu(\mathcal{T}), \tag{8}$$

where $\mathcal{T}$ runs over all standard Young tableaux of the form $\mathcal{T} = \boxed{i_r, i_r, \dots, i_2, i_2, i_1, i_1}$, $i_k = 1, 2, m+1$ *or* $2m$, *that contain at most* $2s$ *elements with value either* 2 *or* $2m$, *and at most two elements with value* $m+1$.

3.1.2 The case $\mathfrak{g}$ of type B_m.

Set $\mu = [r,0,\dots,0]$, $\lambda = [s,0,\dots,0,1]$.[5] According to Proposition 3.1, all Spin-standard tableaux are composed of pairs (*segments*) of equal numbers and

[5] For $\mathfrak{g}$ of type B_m we have $[1,0,\dots,0] = (1,0,\dots,0)$, $\dots$, $[0,\dots,0,1,0] = (1,\dots,1,0)$, $[0,\dots,0,1] = (\frac{1}{2},\dots,\frac{1}{2})$, according to the footnote 2.

of the segment $\boxed{m, m+1}$. In the same way as in the case of D_m, it can be shown that a λ-dominant standard Young-tableau can contain only the following segments:

$$
\begin{array}{llll}
 & \boxed{1,1} & [1,0,\dots,0] & A \\
 & \boxed{2,2} & \left\{\begin{array}{ll} [-1,1,0,\dots,0] & \text{if } m>2 \\ [-1,2] & \text{if } m=2 \end{array}\right\} & C \\
(9) & \boxed{m,m+1} & [0,\dots,0] & D \\
 & \boxed{2m,2m} & [-1,0,\dots,0] & E
\end{array}
$$

As before, we assigned to each segment a letter, but note that this time the segments and the weights differ a little from those in the case of D_m. In particular, though the segment $\boxed{m, m+1}$ contributes to $\nu(\mathcal{T})$ with the zero weight, we have to check that $\lambda + \nu(\mathcal{T}(l)) + [0,\dots,0,-1]$ lies in the dominant Weyl chamber whenever $\boxed{m, m+1}$ is at the end (i.e. on the left hand side) of $\mathcal{T}(l)$.

3.1.3 Reformulation in the language of letters.

Instead of stating an analogue of Lemma 3.2 for the B_m case in terms of elements of Young-tableaux, we rephrase both versions of the lemma in the language of the letters we have assigned to the segments.

Proposition 3.3. *It holds*

$$
[r,0,\dots,0] \otimes [s,0,\dots,0,1] = \bigoplus_{\mathcal{T}} [s,0,\dots,0,1] + \nu(\mathcal{T}), \tag{10}
$$

where $\mathcal{T}$ runs over all non-decreasing sequences of length r of letters A, C, D, E that contain at most s letters C and E altogether, and at most one letter D. For simplicity we will express these conditions as $C+E \le s$ and $D \le 1$.

Proof: According to Lemma 3.2 and its analogue for the B_m case, the contributing tableaux are glued from segments of the form (7) and (9). The condition $C+E \le s$ was derived for the D_m case before Lemma 3.2, the case of B_m is quite similar. The segment D cannot occur in $\mathcal{T}$ more than once because: (D_m case) otherwise $\lambda + \nu(\mathcal{T})$ would not lie in the dominant Weyl chamber; (B_m case) the numbers in any row of a standard Young-tableau $\mathcal{T}$ cannot decrease. □

It is easy now to count the number of components into which the product (10) decomposes.

Proposition 3.4. *The product* (10) *decomposes into*

$$
\begin{array}{ll}
(r+1)^2 & \text{for } 0 \le r \le s \\
(s+1)(s+2) & \text{for } r > s \ge 0
\end{array} \tag{11}
$$

irreducible components, all with multiplicity 1.

PROOF: The proof is a simple exercise in basic combinatorics. □

Example 3.1. Let us decompose the product $[r,0,\dots,0]\otimes[0,\dots,0,1]$ using the technique described in Proposition 3.3. We consider all non-decreasing sequences of letters A, C, D, E of length r with $D\le 1$ and $C+E\le 0$. There are only two such sequences, $A\dots AA$ and $A\dots AD$, for $r>0$. The corresponding $\nu(\mathcal{T})$'s are then $[r,0,\dots,0]$ and $[r-1,0,\dots,0]$ in the B_m case, and $[r,0,\dots,0]$ and $[r-1,0,\dots,1,-1]$ in the D_m case. So,

$$[r,0,\dots,0]\otimes[0,\dots,0,1]=\begin{cases}[r,0,\dots,0,1]\oplus[r-1,0,\dots,0,1] & \text{for } B_m \text{ case}\\ [r,0,\dots,0,1]\oplus[r-1,0,\dots,1,0] & \text{for } D_m \text{ case}\end{cases}$$

Example 3.2. Case B_m. There are 6 sequences contributing to $[r,0,\dots,0]\otimes[1,0,\dots,0,1]$ for $r>1$:

$$\begin{array}{ll} A\dots AAA & [r,0,\dots,0]\\ A\dots AAC & [r-2,1,0,\dots,0]\\ A\dots AAD & [r-1,0,\dots,0]\\ A\dots AAE & [r-2,0,\dots,0]\\ A\dots ACD & [r-3,1,0,\dots,0]\\ A\dots ADE & [r-3,0,\dots,0]\end{array}$$

The corresponding weights $\nu(\mathcal{T})$ are for $m>2$. Therefore, for $r>1$, $m>2$,

$$\begin{aligned}[r,0,\dots,0]\otimes[1,0,\dots,0,1]=\;&[r+1,0,\dots,0,1]\oplus[r,0,\dots,0,1]\oplus\\ &[r-1,1,0,\dots,0,1]\oplus[r-1,0,\dots,0,1]\oplus\\ &[r-2,1,0,\dots,0,1]\oplus[r-2,0,\dots,0,1].\end{aligned}$$

Example 3.3. Case D_m. When trying to decompose the same product as above, $[r,0,\dots,0]\otimes[1,\dots,0,1]$, we get the same contributing sequences of letters, but different corresponding weights (and therefore a different decomposition; $r>1$):

$$\begin{array}{ll} A\dots AAA & [r,0,\dots,0]\\ A\dots AAC & [r-2,1,0,\dots,0]\\ A\dots AAD & [r-1,0,\dots,1,-1]\\ A\dots AAE & [r-2,0,\dots,0]\\ A\dots ACD & [r-3,1,0,\dots,1,-1]\\ A\dots ADE & [r-3,0,\dots,1,-1]\end{array}$$

Hence, for $r > 1$,

$$
\begin{aligned}
[r,0,\ldots,0]\otimes[1,\ldots,0,1] = {} & [r+1,0,\ldots,0,1]\oplus[r,0,\ldots,1,0]\oplus \\
& [r-1,1,0,\ldots,0,1]\oplus[r-1,0,\ldots,0,1]\oplus \\
& [r-2,1,0,\ldots,1,0]\oplus[r-2,0,\ldots,1,0]
\end{aligned}
$$

3.2 The product $[r,0,\ldots,0]\otimes[0,\ldots,0,s,0,\ldots,0,1]$.

Our examination will continue with the product $[r,0,\ldots,0]\otimes[0,\ldots,0,{}^{j}s,0,\ldots,0,1]$, s being on the jth place, $j>1$. Instead of stating analogs of Lemma 3.2, we shall focus directly on formulation of the results in the language of letters. As all the calculations here go in the way indicated in Section 3.1, no further comments should be needed.

3.2.1 The case $\mathfrak{g}$ of type D_m.

Similarly as in the paragraph 3.1.1, all the Spin-standard Young tableaux contributing to the sum (6) can be decomposed into pairs of equal numbers — we shall call them *segments* again. There are five possible segments — to each one we assign a letter and quote the weight with which it contributes to $\nu(\mathcal{T})$ in (5).

$$
\begin{array}{llll}
 & \boxed{1,1} & [1,0,\ldots,0] & A \\
 & \boxed{j,j} & [0,\ldots,-1,{}^{j}1,0,\ldots,0] & B \\
(12) & \boxed{j+1,j+1} & \begin{cases} [0,\ldots,0,{}^{j}{-1},1,0,\ldots,0] & \text{if } j<m-2 \\ [0,\ldots,0,-1,1,1] & \text{if } j=m-2 \end{cases} & C \\
 & \boxed{m+1,m+1} & [0,\ldots,0,1,-1] & D \\
 & \boxed{2m+1-j,2m+1-j} & [0,\ldots,1,{}^{j}{-1},0,\ldots,0] & E
\end{array}
$$

3.2.2 The case $\mathfrak{g}$ of type B_m.

In the B_m-case, all the segments that contribute to a λ-dominant Spin-standard Young tableau are listed below:

$$
\begin{array}{llll}
 & \boxed{1,1} & [1,0,\ldots,0] & A \\
 & \boxed{j,j} & [0,\ldots,-1,{}^{j}1,0,\ldots,0] & B \\
(13) & \boxed{j+1,j+1} & \begin{cases} [0,\ldots,0,{}^{j}{-1},1,0,\ldots,0] & \text{if } j<m-1 \\ [0,\ldots,0,-1,2] & \text{if } j=m-1 \end{cases} & C \\
 & \boxed{m,m+1} & [0,\ldots,0] & D \\
 & \boxed{2m+1-j,2m+1-j} & [0,\ldots,1,{}^{j}{-1},0,\ldots,0] & E
\end{array}
$$

Similarly as in Section 3.1.2: the segment $\boxed{m, m+1}$ contributes to $\nu(\mathcal{T})$ with the zero weight, but in $2\lambda + \nu_l(\mathcal{T})$ it is treated as if it corresponded to the weight $[0,\dots,0,-1]$. Because a standard Young tableau can contain only E-segments on the right hand side of a D-segment, and because $\lambda = [0,\dots,0,s,0,\dots,0,1]$ is such a special weight, it follows that a D-segment can occur in a standard Young tableaux at most once. A different argument for the inequality $D \le 1$ is in the proof of Proposition 3.3.

3.2.3 Results formulated in the language of letters.

Proposition 3.5. *It holds*

$$[r,0,\dots,0] \otimes [0,\dots,0,s,0,\dots,0,1] = \bigoplus_{\mathcal{T}} [0,\dots,0,s,0,\dots,0,1] + \nu(\mathcal{T}), \tag{14}$$

where $\mathcal{T}$ runs over all non-decreasing sequences of length r of letters A to E that satisfy inequalities $B \le E$, $C + E \le s$ and $D \le 1$.

Proof: The proof of Proposition 3.5 is similar to the proofs of Lemma 3.2 and of Proposition 3.3. □

Proposition 3.6. *The product* (14) *decomposes into*

$$\begin{aligned}
&\frac{1}{6}(r+1)(r+2)(r+3) && \text{for } 0 \le r \le s \\
&\frac{1}{3}(s+1)(s+2)(s+3) - \frac{1}{6}(2s+1-r)(2s+2-r)(2s+3-r) && \\
& && \text{for } 2s \ge r \ge s \ge 0 \\
&\frac{1}{3}(s+1)(s+2)(s+3) && \text{for } r > 2s \ge 0
\end{aligned} \tag{15}$$

irreducible components, all with multiplicity 1.

Proof: The proof is only a combinatorial calculation. □

4. Decomposition of $\ker \mathcal{D}$

Now we have come to the point, where we are able to describe the structure of the kernel of the higher spin Dirac operator $\mathcal{D}$.

Theorem 4.1. *As earlier, denote the highest weight of the spinor-like representation $\mathbb{S}$ by $\lambda = [s,0,\dots,0,1]$, resp. $\lambda = [0,\dots,0,s,0,\dots,0,1]$. Then*[6]

$$\ker \mathcal{D}|_{\mathcal{P}_r} = \bigoplus_{\mathcal{T}} \lambda + \nu(\mathcal{T}) \tag{16}$$

[6] Here $\mathcal{D}|_{\mathcal{P}_r}$ means restriction of $\mathcal{D}$ to $\mathcal{P}_r \otimes \mathbb{S}$.

where $\mathcal{T}$ runs over all those sequences of letters from formulas (10)*, resp.* (14) *that do not contain letter D. In more detail,*

$$\ker \mathcal{D} = \bigoplus_{\mathcal{T}} \lambda + \nu(\mathcal{T}) \tag{17}$$

where $\mathcal{T}$ runs over all finite non-decreasing sequences of letters A, C, E that satisfy $C + E \leq s$, resp. over all finite non-decreasing sequences of letters A, B, C and E that satisfy $C + E \leq s$ and $B \leq E$.

Proof: The proof is based on the observation that omitting the letter D from a sequence that contains it, or adding it to a sequence that does not contain it, either does not change the corresponding weight at all (in the odd dimension, the B_m case), or changes only its "polarity" (i.e. the sign of the last coordinate in the standard weight basis; in the even dimension, D_m case). In this way we establish a natural correspondence between sequences contributing to the decomposition of $\mathcal{P}_r \otimes \mathbb{S}$ and sequences contributing to $\mathcal{P}_{r+1} \otimes \mathbb{S}$. To each sequence from $\mathcal{P}_r \otimes \mathbb{S} = \bigoplus_M [i, 0, \dots, 0] \otimes \mathbb{S}$, M containing every other number from r down to 0, we can assign a unique sequence from $\mathcal{P}_{r+1} \otimes \mathbb{S}$. What remains unassigned in $\mathcal{P}_{r+1} \otimes \mathbb{S}$ are all the sequences of length $r+1$ that do not contain the letter D. We have to proof that this assignment somehow reflects the behaviour of the operator $\mathcal{D}$.

In order to simplify the notation in the rest of the proof, let us denote $S^{\pm} = \mathbb{S}$ in the odd dimension, and $\mathbb{S}^+ = \mathbb{S}$, $\mathbb{S}^-$ corresponding to the "twisted" weight λ in the even case.

Clearly, as $\mathcal{D}$ is a first-order differential operator, it maps $\mathcal{P}_{r+1} \otimes \mathbb{S}^+$ into $\mathcal{P}_r \otimes \mathbb{S}^-$. Because $\mathcal{D}$ is invariant, any component in the decomposition of $\mathcal{P}_{r+1} \otimes \mathbb{S}^+$ is mapped either to a component with the same highest weight in the decomposition of $\mathcal{P}_r \otimes \mathbb{S}^-$, or to zero. From Proposition 1.1 we know that $\mathcal{D}$ maps $\mathcal{P}_{r+1} \otimes \mathbb{S}^+$ *onto* $\mathcal{P}_r \otimes \mathbb{S}^-$. Hence, the complement of the kernel of $\mathcal{D}|_{\mathcal{P}_{r+1}}$ must consist exactly of those components that contribute to the decomposition $\mathcal{P}_r \otimes \mathbb{S}^-$ — we have identified these components using the sequence-of-letters assignment. Thus the components corresponding to the remaining sequences must form the kernel of the operator $\mathcal{D}$. □

Remark 4.1. Note that in order to derive the formula (17) we did not need any special knowledge about the nature of the higher spin Dirac operator — the information about its conformal invariance and its ellipticity was completely sufficient. The result was deduced purely by representation theoretical methods without using other geometrical or analytical properties of solutions, as it was in the approach to solving certain special cases in [8], [9].

Remark 4.2. Let again $\lambda = [s, 0, \dots, 0, 1]$, resp. $\lambda = [0, \dots, 0, s, 0, \dots, 0, 1]$. To every component κ of the decomposition of $\ker \mathcal{D}$ there corresponds a unique sequence of letters π that does not contain the letters A and E, resp. letters

B and E together. Denote by $C(\pi)$ and $E(\pi)$ the number of letters C and E contained in the sequence π. Then, clearly, the multiplicity of the component κ in the decomposition of $\ker \mathcal{D}$ is equal to

$$s + 1 - (C(\pi) + E(\pi)). \tag{18}$$

Note that the other sequences contributing with the same component can be obtained from π by gradually adding the couples of letters A, E, resp. B, E to it while the inequality $C + E \leq s$ is satisfied.

Example 4.1. As an example let us decompose $\ker \mathcal{D}$ for the case $\lambda = (\frac{5}{2}, \frac{1}{2}, \dots, \frac{1}{2}) = [2, 0, \dots, 0, 1]$ in odd dimension (the B_m case). The following sequences contribute to (17) with the corresponding $\nu(\mathcal{T})$s ($r \geq 2$ is the length of the sequence, when used):

empty sequence	$[0, \dots, 0]$
A	$[1, 0, \dots, 0]$
C	$[-1, 1, 0, \dots, 0]$
E	$[-1, 0, \dots, 0]$
$A \dots AAA$	$[r, 0, \dots, 0]$
$A \dots AAC$	$[r-2, 1, 0, \dots, 0]$
$A \dots AAE$	$[r-2, 0, \dots, 0]$
$A \dots ACC$	$[r-4, 2, 0, \dots, 0]$
$A \dots ACE$	$[r-4, 1, 0, \dots, 0]$
$A \dots AEE$	$[r-4, 0, \dots, 0]$

Hence, using the more common standard weight notation:

$$\begin{aligned} \ker \mathcal{D}|_{\mathcal{P}_0} &= (\tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \\ \ker \mathcal{D}|_{\mathcal{P}_1} &= (\tfrac{7}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \\ \ker \mathcal{D}|_{\mathcal{P}_{r \geq 2}} &= (\tfrac{2r+5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{2r+3}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{2r+1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus \\ &\quad (\tfrac{2r+1}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{2r-1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{2r-3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \end{aligned} \tag{19}$$

Note, that there are no multiplicities in $\ker \mathcal{D}|_{\mathcal{P}_r}$, while the complete kernel has many:

$$\begin{aligned} \ker \mathcal{D} =& [0, \dots, 0, 1] \oplus [0, 1, 0, \dots, 0, 1] \oplus \bigoplus_{i=0}^{\infty} [i, 2, 0, \dots, 0, 1] \\ & \oplus 2 \cdot [1, 0, \dots, 0, 1] \oplus \bigoplus_{i=1}^{\infty} 2 \cdot [i, 1, 0, \dots, 0, 1] \\ & \oplus \bigoplus_{i=2}^{\infty} 3 \cdot [i, 0, \dots, 0, 1] \end{aligned} \tag{20}$$

Example 4.2. Let $\lambda = (\frac{5}{2}, \frac{5}{2}, \frac{1}{2}, \dots, \frac{1}{2}) = [0, 2, 0, \dots, 0, 1]$ in even dimension $2m \geq 10$ (the D_m case). The following sequences contribute to (17):

$$\begin{aligned}
&\text{empty sequence},\\
&A, C, E,\\
&AA, AC, AE, BE, CC, CE, EE,\\
&AAA, AAC, AAE, ABE, ACC, ACE, AEE, BCE, BEE,\\
&A\dots AAAA, A\dots AAAC, A\dots AAAE, A\dots AABE, A\dots AACC,\\
&A\dots AACE, A\dots AAEE, A\dots ABCE, A\dots ABEE, A\dots BBEE.
\end{aligned}$$

Thus the kernel of the higher spin Dirac operator consists of the following parts:

$$\begin{aligned}
\ker \mathcal{D}|_{\mathcal{P}_0} &= (\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}),\\
\ker \mathcal{D}|_{\mathcal{P}_1} &= (\tfrac{7}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}),\\
\ker \mathcal{D}|_{\mathcal{P}_2} &= (\tfrac{9}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{7}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{7}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2})\\
&\quad \oplus (\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2})\\
&\quad \oplus (\tfrac{5}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}),\\
\ker \mathcal{D}|_{\mathcal{P}_3} &= (\tfrac{11}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{9}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{9}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2})\\
&\quad \oplus (\tfrac{7}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{7}{2}, \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{7}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2})\\
&\quad \oplus (\tfrac{7}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}),\\
\ker \mathcal{D}|_{\mathcal{P}_{r\geq 4}} &= (\tfrac{2r+5}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{2r+3}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2})\\
&\quad \oplus (\tfrac{2r+3}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{2r+1}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2})\\
&\quad \oplus (\tfrac{2r+1}{2}, \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{2r+1}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2})\\
&\quad \oplus (\tfrac{2r+1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{2r-1}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2})\\
&\quad \oplus (\tfrac{2r-1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \oplus (\tfrac{2r-3}{2}, \tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}).
\end{aligned} \tag{21}$$

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